

## Parties imaginaires des zéros non triviaux de zeta, fonction de Lambert, Denise Vella-Chemla, décembre 2024

On a la curiosité de simplifier encore le calcul des parties imaginaires des zéros non triviaux de la fonction  $\zeta$  de Riemann par la formule trouvée dans un article de França et LeClair, modifié en 2024, téléchargeable sur arxiv ici

<https://arxiv.org/pdf/1407.4358>,

qu'on simplifie au maximum : on utilise le programme suivant pour calculer la fonction

$$x \mapsto \frac{2\pi x}{W\left(\frac{x}{e}\right)}$$

jusqu'à 2 millions. L'erreur va diminuant jusque là, on ne sait pas si elle finira par croître, plus loin. Le fichier contenant les 2 millions de premiers zéros de  $\zeta$  a été téléchargé sur le site d'Andrew Odlyzko ici

[https://www-users.cse.umn.edu/~odlyzko/zeta\\_tables/index.html](https://www-users.cse.umn.edu/~odlyzko/zeta_tables/index.html).

*Programme python de test de l'obtention des parties imaginaires des zéros non triviaux de  $\zeta$  en utilisant la fonction de Lambert  $W(x)$*

```
import math
from math import log,exp,pi,e
import scipy
from scipy.special import lambertw

with open("zeros2millions", 'r') as f:
    lines = f.readlines()
    zeros = [float(line) for line in lines]

print('k |2pi k/Lambert(k/e) | zeta(k) | erreur |')
for k in range(1,2000000):
    res1 = 2*pi*k/lambertw(k/e)
    print(':5d :9.5f :14.5f :8.5f'.format(k,res1.real,zeros[k],
                                         abs(res1-zeros[k])/zeros[k]))
```

*Résultat du programme ci-dessus pour les 100 premiers zéros de  $\zeta$*

Python se plaint pour le premier zéro en écrivant : *RuntimeWarning: invalid value encountered in scalar divide*  $res1 = 2 * pi * k / lambertw(k/e)$ .

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
0	nan	14.13473	nan
1	22.56368	21.02204	0.07333
2	27.13794	25.01086	0.08505
3	31.23136	30.42488	0.02651
4	35.01238	32.93506	0.06307
5	38.56829	37.58618	0.02613
6	41.95147	40.91872	0.02524
7	45.19633	43.32707	0.04314
8	48.32696	48.00515	0.00670
9	51.36105	49.77383	0.03189
10	54.31201	52.97032	0.02533

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
11	57.19037	56.44625	0.01318
12	60.00451	59.34704	0.01108
13	62.76130	60.83178	0.03172
14	65.46641	65.11254	0.00543
15	68.12461	67.07981	0.01558
16	70.73997	69.54640	0.01716
17	73.31597	72.06716	0.01733
18	75.85564	75.70469	0.00199
19	78.36160	77.14484	0.01577
20	80.83619	79.33738	0.01889
21	83.28146	82.91038	0.00448
22	85.69926	84.73549	0.01137
23	88.09121	87.42527	0.00762
24	90.45882	88.80911	0.01858
25	92.80341	92.49190	0.00337
26	95.12620	94.65134	0.00502
27	97.42829	95.87063	0.01625
28	99.71071	98.83119	0.00890
29	101.97438	101.31785	0.00648
30	104.22016	103.72554	0.00477
31	106.44884	105.44662	0.00950
32	108.66114	107.16861	0.01393
33	110.85776	111.02954	0.00155
34	113.03931	111.87466	0.01041
35	115.20638	114.32022	0.00775
36	117.35953	116.22668	0.00975
37	119.49927	118.79078	0.00596
38	121.62608	121.37013	0.00211
39	123.74041	122.94683	0.00645
40	125.84268	124.25682	0.01276
41	127.93330	127.51668	0.00327
42	130.01265	129.57870	0.00335
43	132.08107	131.08769	0.00758
44	134.13892	133.49774	0.00480
45	136.18649	134.75651	0.01061
46	138.22411	138.11604	0.00078
47	140.25206	139.73621	0.00369
48	142.27061	141.12371	0.00813
49	144.28002	143.11185	0.00816
50	146.28053	146.00098	0.00191
51	148.27240	147.42277	0.00576
52	150.25583	150.05352	0.00135
53	152.23105	150.92526	0.00865

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
54	154.19826	153.02469	0.00767
55	156.15766	156.11291	0.00029
56	158.10944	157.59759	0.00325
57	160.05377	158.84999	0.00758
58	161.99084	161.18896	0.00497
59	163.92080	163.03071	0.00546
60	165.84382	165.53707	0.00185
61	167.76006	167.18444	0.00344
62	169.66965	169.09452	0.00340
63	171.57274	169.91198	0.00977
64	173.46948	173.41154	0.00033
65	175.35998	174.75419	0.00347
66	177.24438	176.44143	0.00455
67	179.12280	178.37741	0.00418
68	180.99535	179.91648	0.00600
69	182.86216	182.20708	0.00360
70	184.72334	184.87447	0.00082
71	186.57898	185.59878	0.00528
72	188.42920	187.22892	0.00641
73	190.27409	189.41616	0.00453
74	192.11374	192.02666	0.00045
75	193.94827	193.07973	0.00450
76	195.77774	195.26540	0.00262
77	197.60226	196.87648	0.00369
78	199.42191	198.01531	0.00710
79	201.23677	201.26475	0.00014
80	203.04692	202.49359	0.00273
81	204.85243	204.18967	0.00325
82	206.65339	205.39470	0.00613
83	208.44987	207.90626	0.00261
84	210.24194	209.57651	0.00318
85	212.02967	211.69086	0.00160
86	213.81312	213.34792	0.00218
87	215.59237	214.54704	0.00487
88	217.36747	216.16954	0.00554
89	219.13849	219.06760	0.00032
90	220.90549	220.71492	0.00086
91	222.66853	221.43071	0.00559
92	224.42767	224.00700	0.00188
93	226.18296	224.98332	0.00533
94	227.93445	227.42144	0.00226
95	229.68220	229.33741	0.00150
96	231.42627	231.25019	0.00076

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
97	233.16670	231.98724	0.00508
98	234.90354	233.69340	0.00518
99	236.63684	236.52423	0.00048
100	238.36666	237.76982	0.00251

*Résultat du programme ci-dessus en sautant de 10000 en 10000*

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
1	22.56368	21.02204	0.07333
10001	9879.65707	9879.03673	0.00006
20001	18048.41868	18047.99999	0.00002
30001	25756.64629	25756.39196	0.00001
40001	33191.72719	33191.34319	0.00001
50001	40435.32219	40434.98023	0.00001
60001	47532.86606	47532.50090	0.00001
70001	54513.11769	54512.56194	0.00001
80001	61395.82265	61395.85156	0.00000
90001	68195.32380	68195.19509	0.00000
100001	74922.48085	74922.59418	0.00000
110001	81585.78248	81585.87830	0.00000
120001	88192.03383	88192.10598	0.00000
130001	94746.80367	94746.68559	0.00000
140001	101254.72730	101254.70645	0.00000
150001	107719.71887	107719.30879	0.00000
160001	114145.12437	114144.89073	0.00000
170001	120533.83460	120533.66746	0.00000
180001	126888.37036	126888.50832	0.00000
190001	133210.94780	133210.65431	0.00000
200001	139503.52940	139503.24650	0.00000
210001	145767.86430	145767.64097	0.00000
220001	152005.52056	152005.45867	0.00000
230001	158217.91134	158217.97828	0.00000
240001	164406.31630	164406.12285	0.00000
250001	170571.89923	170571.85741	0.00000
260001	176715.72285	176715.35385	0.00000
270001	182838.76106	182838.87806	0.00000
280001	188941.90947	188941.72250	0.00000
290001	195025.99423	195025.50674	0.00000
300001	201091.77965	201091.34799	0.00000
310001	207139.97475	207140.01038	0.00000
320001	213171.23890	213170.84904	0.00000

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
330001	219186.18676	219186.18476	0.00000
340001	225185.39261	225185.04899	0.00000
350001	231169.39405	231169.02030	0.00000
360001	237138.69538	237138.26425	0.00000
370001	243093.77052	243093.40254	0.00000
380001	249035.06562	249034.78222	0.00000
390001	254963.00135	254962.81145	0.00000
400001	260877.97502	260877.77142	0.00000
410001	266780.36240	266780.38777	0.00000
420001	272670.51938	272670.58651	0.00000
430001	278548.78351	278548.54962	0.00000
440001	284415.47531	284415.40621	0.00000
450001	290270.89952	290270.67490	0.00000
460001	296115.34621	296115.00624	0.00000
470001	301949.09178	301948.75496	0.00000
480001	307772.39990	307771.85195	0.00000
490001	313585.52233	313585.33992	0.00000
500001	319388.69972	319388.65016	0.00000
510001	325182.16229	325182.11046	0.00000
520001	330966.13047	330965.78541	0.00000
530001	336740.81552	336740.48966	0.00000
540001	342506.42009	342506.07911	0.00000
550001	348263.13867	348262.78609	0.00000
560001	354011.15811	354010.84011	0.00000
570001	359750.65805	359750.51936	0.00000
580001	365481.81130	365481.98893	0.00000
590001	371204.78421	371204.57638	0.00000
600001	376919.73705	376919.43116	0.00000
610001	382626.82429	382626.49448	0.00000
620001	388326.19493	388326.15656	0.00000
630001	394017.99277	394017.68117	0.00000
640001	399702.35667	399702.23199	0.00000
650001	405379.42080	405379.13884	0.00000
660001	411049.31486	411049.17763	0.00000
670001	416712.16430	416711.82640	0.00000
680001	422368.09053	422367.53258	0.00000
690001	428017.21108	428016.90685	0.00000
700001	433659.63981	433659.44198	0.00000
710001	439295.48706	439295.51621	0.00000
720001	444924.85980	444924.82418	0.00000
730001	450547.86182	450547.75865	0.00000
740001	456164.59379	456164.48468	0.00000

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
750001	461775.15347	461774.88004	0.00000
760001	467379.63579	467379.65372	0.00000
770001	472978.13299	472977.63874	0.00000
780001	478570.73471	478570.15765	0.00000
790001	484157.52811	484157.40140	0.00000
800001	489738.59797	489738.18914	0.00000
810001	495314.02677	495313.65048	0.00000
820001	500883.89480	500883.83464	0.00000
830001	506448.28023	506448.22239	0.00000
840001	512007.25918	512006.90040	0.00000
850001	517560.90585	517561.03474	0.00000
860001	523109.29251	523109.28876	0.00000
870001	528652.48964	528652.37676	0.00000
880001	534190.56597	534190.15276	0.00000
890001	539723.58853	539723.73655	0.00000
900001	545251.62274	545251.18121	0.00000
910001	550774.73243	550774.48683	0.00000
920001	556292.97993	556292.75192	0.00000
930001	561806.42610	561806.31406	0.00000
940001	567315.13039	567314.78890	0.00000
950001	572819.15089	572819.08058	0.00000
960001	578318.54435	578318.07787	0.00000
970001	583813.36626	583812.89582	0.00000
980001	589303.67087	589303.24895	0.00000
990001	594789.51124	594789.13354	0.00000
1000001	600270.93927	600270.74787	0.00000
1010001	605748.00573	605747.54049	0.00000
1020001	611220.76031	611220.62865	0.00000
1030001	616689.25166	616688.72506	0.00000
1040001	622153.52740	622153.26585	0.00000
1050001	627613.63416	627613.41935	0.00000
1060001	633069.61761	633069.33843	0.00000
1070001	638521.52251	638521.70983	0.00000
1080001	643969.39268	643969.18514	0.00000
1090001	649413.27110	649413.09778	0.00000
1100001	654853.19988	654852.46934	0.00000
1110001	660289.22031	660289.22603	0.00000
1120001	665721.37288	665720.95905	0.00000
1130001	671149.69728	671149.49521	0.00000
1140001	676574.23248	676573.90360	0.00000
1150001	681995.01667	681994.76397	0.00000
1160001	687412.08736	687411.78506	0.00000

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
1170001	692825.48134	692825.32909	0.00000
1180001	698235.23472	698235.19704	0.00000
1190001	703641.38297	703641.24282	0.00000
1200001	709043.96090	709043.92301	0.00000
1210001	714443.00270	714442.50436	0.00000
1220001	719838.54194	719838.27397	0.00000
1230001	725230.61161	725230.18472	0.00000
1240001	730619.24413	730619.05953	0.00000
1250001	736004.47132	736004.41822	0.00000
1260001	741386.32450	741386.63848	0.00000
1270001	746764.83441	746764.54735	0.00000
1280001	752140.03129	752140.00995	0.00000
1290001	757511.94487	757511.86133	0.00000
1300001	762880.60437	762880.44858	0.00000
1310001	768246.03855	768246.09762	0.00000
1320001	773608.27567	773608.09124	0.00000
1330001	778967.34355	778967.01138	0.00000
1340001	784323.26953	784322.85604	0.00000
1350001	789676.08055	789675.84188	0.00000
1360001	795025.80309	795025.40398	0.00000
1370001	800372.46322	800372.30529	0.00000
1380001	805716.08659	805716.03043	0.00000
1390001	811056.69847	811056.46364	0.00000
1400001	816394.32372	816394.18397	0.00000
1410001	821728.98683	821728.59790	0.00000
1420001	827060.71190	827060.42527	0.00000
1430001	832389.52267	832389.20015	0.00000
1440001	837715.44254	837715.11634	0.00000
1450001	843038.49453	843038.31553	0.00000
1460001	848358.70134	848358.44305	0.00000
1470001	853676.08532	853675.80764	0.00000
1480001	858990.66852	858990.43584	0.00000
1490001	864302.47262	864302.15119	0.00000
1500001	869611.51904	869611.24708	0.00000
1510001	874917.82886	874917.47179	0.00000
1520001	880221.42286	880221.35373	0.00000
1530001	885522.32153	885522.12472	0.00000
1540001	890820.54509	890820.43951	0.00000
1550001	896116.11344	896116.04094	0.00000
1560001	901409.04624	901408.72559	0.00000
1570001	906699.36286	906698.86174	0.00000
1580001	911987.08240	911986.96392	0.00000
1590001	917272.22371	917272.07451	0.00000
1600001	922554.80538	922554.83112	0.00000

$k$	$2\pi k/W(k/e)$	$\zeta(k)$	erreur
1610001	927834.84574	927834.29171	0.00000
1620001	933112.36291	933112.10103	0.00000
1630001	938387.37472	938386.95239	0.00000
1640001	943659.89880	943659.67606	0.00000
1650001	948929.95253	948929.75814	0.00000
1660001	954197.55308	954197.13337	0.00000
1670001	959462.71738	959462.19987	0.00000
1680001	964725.46215	964725.24207	0.00000
1690001	969985.80390	969985.33118	0.00000
1700001	975243.75891	975243.30147	0.00000
1710001	980499.34328	980498.83436	0.00000
1720001	985752.57289	985752.62638	0.00000
1730001	991003.46343	991003.27164	0.00000
1740001	996252.03039	996252.01168	0.00000
1750001	1001498.28907	1001498.15467	0.00000
1760001	1006742.25457	1006742.06544	0.00000
1770001	1011983.94183	1011983.66595	0.00000
1780001	1017223.36559	1017223.14191	0.00000
1790001	1022460.54041	1022460.37458	0.00000
1800001	1027695.48069	1027695.67711	0.00000
1810001	1032928.20064	1032928.00372	0.00000
1820001	1038158.71432	1038158.58525	0.00000
1830001	1043387.03560	1043386.82934	0.00000
1840001	1048613.17821	1048612.93420	0.00000
1850001	1053837.15571	1053836.87447	0.00000
1860001	1059058.98149	1059058.60937	0.00000
1870001	1064278.66881	1064278.34762	0.00000
1880001	1069496.23075	1069496.33222	0.00000
1890001	1074711.68027	1074711.41306	0.00000
1900001	1079925.03016	1079924.78216	0.00000
1910001	1085136.29306	1085135.70400	0.00000
1920001	1090345.48150	1090345.37228	0.00000
1930001	1095552.60783	1095552.09859	0.00000
1940001	1100757.68429	1100757.75763	0.00000
1950001	1105960.72297	1105960.54403	0.00000
1960001	1111161.73584	1111161.70085	0.00000
1970001	1116360.73473	1116360.55604	0.00000
1980001	1121557.73133	1121557.25561	0.00000
1990001	1126752.73722	1126752.41390	0.00000

Le fichier résultat complet pour les 2 millions de premiers zéros est téléchargeable à cette adresse <https://denisevellachemla.eu/pix-par-W-de-x-2-millions.txt>.

On trouve dans cet article d'István Mezö <https://arxiv.org/pdf/2012.02480> une expression de la fonction de Lambert par une intégrale qu'on teste jusqu'à 2024, on fournit les fonctions de calcul

correspondantes, qu'on a ajoutées à notre programme pour effectuer la comparaison :

```
def cotan(x):
    return -tan(x+pi/2)

def lambertintegrale(x):
    return((1/pi)*quad(lambda t:log(1+(x*(sin(t)/t))*exp(t*cotan(t))),0,pi)[0])
```

$k$	$\frac{2\pi k}{W\left(\frac{k}{e}\right)}$	$\left(\frac{1}{\pi}\right) \int_0^\pi \log\left(1+k \frac{\sin(k)}{kt}\right)^t \exp(t \cot(t))$	$\zeta(k)$
1965	2480.11374	2480.11374	2479.97735
1966	2481.16472	2481.16472	2480.43361
1967	2482.21563	2482.21563	2481.35677
1968	2483.26646	2483.26646	2482.68341
1969	2484.31722	2484.31722	2484.50324
1970	2485.36790	2485.36790	2485.08759
1971	2486.41851	2486.41851	2485.78562
1972	2487.46905	2487.46905	2487.33384
1973	2488.51951	2488.51951	2488.11383
1974	2489.56990	2489.56990	2489.26116
1975	2490.62021	2490.62021	2490.05109
1976	2491.67045	2491.67045	2491.10552
1977	2492.72062	2492.72062	2491.90747
1978	2493.77071	2493.77071	2493.48930
1979	2494.82073	2494.82073	2494.86618
1980	2495.87067	2495.87067	2495.84684
1981	2496.92054	2496.92054	2496.48534
1982	2497.97034	2497.97034	2497.64847
1983	2499.02006	2499.02006	2497.96997
1984	2500.06971	2500.06971	2499.86204
1985	2501.11928	2501.11928	2500.56439
1986	2502.16879	2502.16879	2501.57774
1987	2503.21821	2503.21821	2503.18523
1988	2504.26757	2504.26757	2504.00893
1989	2505.31685	2505.31685	2504.77215
1990	2506.36606	2506.36606	2506.30535
1991	2507.41519	2507.41519	2507.15189
1992	2508.46425	2508.46425	2507.87104
1993	2509.51324	2509.51324	2508.76605
1994	2510.56216	2510.56216	2509.65300
1995	2511.61100	2511.61100	2511.66019
1996	2512.65977	2512.65977	2512.53822
1997	2513.70846	2513.70846	2513.57952
1998	2514.75709	2514.75709	2514.25986
1999	2515.80564	2515.80564	2515.28648

$k$	$\frac{2\pi k}{W\left(\frac{k}{e}\right)}$	$\left(\frac{1}{\pi}\right) \int_0^\pi \log\left(1 + k \frac{\sin(k)}{k}\right) \exp(t \cot(t))$	$\zeta(k)$
2000	2516.85411	2516.85411	2516.57105
2001	2517.90252	2517.90252	2517.14693
2002	2518.95085	2518.95085	2518.55285
2003	2519.99911	2519.99911	2519.63425
2004	2521.04729	2521.04729	2520.40232
2005	2522.09540	2522.09540	2521.88518
2006	2523.14344	2523.14344	2523.26348
2007	2524.19141	2524.19141	2523.96706
2008	2525.23931	2525.23931	2524.64205
2009	2526.28713	2526.28713	2525.68799
2010	2527.33488	2527.33488	2526.86881
2011	2528.38256	2528.38256	2527.41879
2012	2529.43016	2529.43016	2529.37804
2013	2530.47770	2530.47770	2530.27367
2014	2531.52516	2531.52516	2531.54396
2015	2532.57254	2532.57254	2531.96938
2016	2533.61986	2533.61986	2533.06732
2017	2534.66710	2534.66710	2534.40861
2018	2535.71428	2535.71428	2535.50589
2019	2536.76138	2536.76138	2536.09188
2020	2537.80840	2537.80840	2536.86749
2021	2538.85536	2538.85536	2538.55983
2022	2539.90224	2539.90224	2539.99841
2023	2540.94906	2540.94906	2540.53166
2024	2541.99580	2541.99580	2541.99873