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RIEMANN HYPOTHESIS

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CMI MILLENNIUM PRIZE LECTURE

HARVARD APRIL 15 2026.

STATEMENT (RIEMANN 1859)

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\substack{p \\ \text{PRIMES}}} (1 - p^{-s})^{-1}$$

(UNIQUE FACTORIZATION)

EXTENDS TO A HOLOMORPHIC (EXCEPT FOR A
SIMPLE POLE AT $s=1$) FUNCTION OF THE COMPLEX
VARIABLE s AND SATISFIES
FUNCTIONAL EQUATION:

$$\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) = \xi(1-s)$$

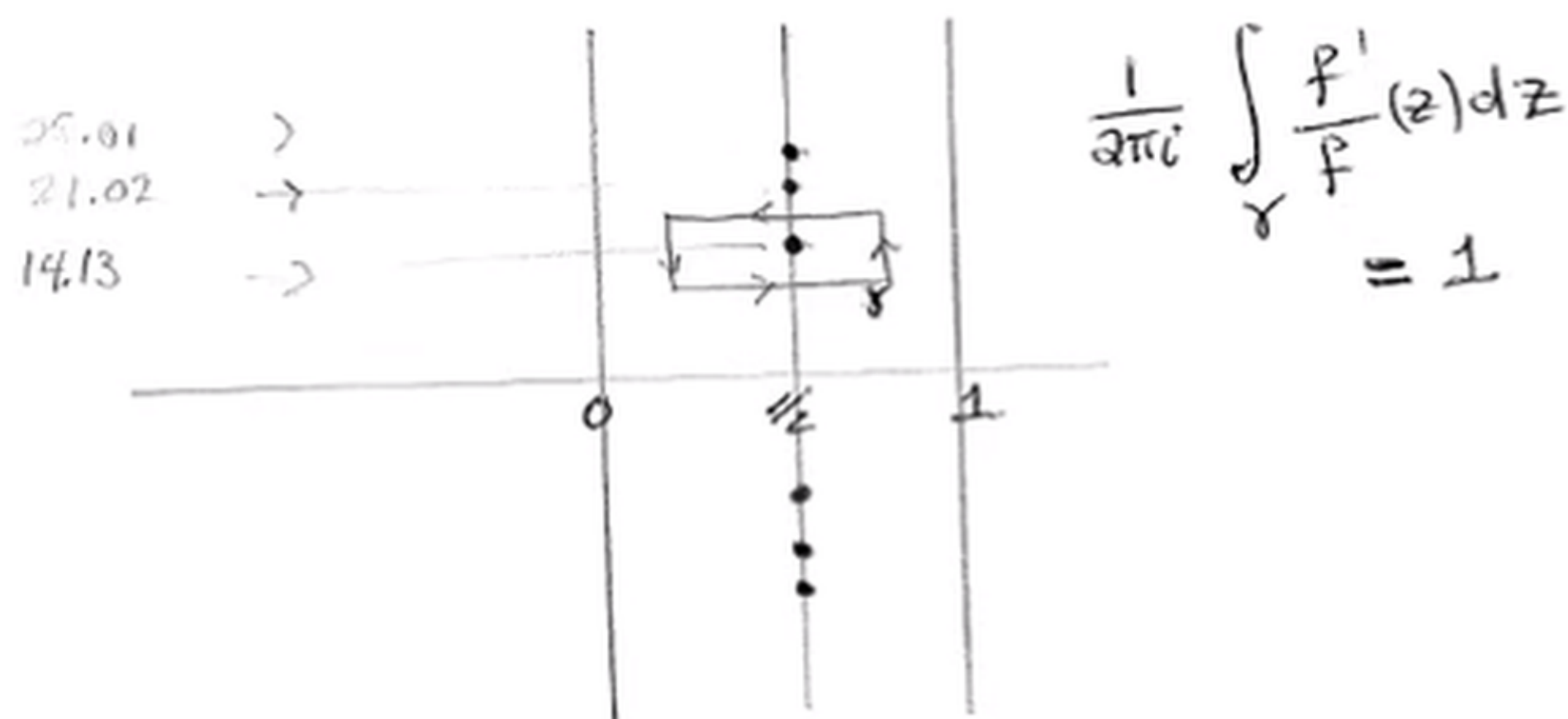
RIEMANN "HYPOTHESIS" (RH)

ALL THE ZEROS ρ OF $\xi(s)$
HAVE $\operatorname{Re}(\rho) = \frac{1}{2}$

- (2)
- AS WAS UNCOVERED BY SIEGEL (1932), RIEMANN COMPUTED THE FIRST FEW ZEROS AND CHECKED THAT THEY SATISFIED HIS HYPOTHESIS.

HOW? HE NEEDED A GOOD APPROXIMATION TO $\frac{\zeta'}{\zeta}(s)$; IN $0 < \text{Re}(s) < 1$.

NOW KNOWN AS THE RIEMANN-SIEGEL FORMULA AND WHICH REMAINS THE GOLD STANDARD IN SUCH CALCULATIONS "CONDUCTOR TO ∞ ".



- THE LARGEST ZERO THAT APPEARS TO HAVE BEEN COMPUTED IS AROUND THE 10^{36} -TH ZERO (BOBER-HIARY).
- ALL ZEROS THAT HAVE BEEN COMPUTED SATISFY RH AND ARE SIMPLE (IF NOT WE WOULD NOT BE ABLE TO VERIFY IT NUMERICALLY).

(3)

FOR RIEMANN IT WAS A WORKING 'HYPOTHESIS'
SINCE WITH IT HE COULD COMPLETE HIS AIM

FROM HIS EXPLICIT FORMULA:

$$\psi(x) := \sum_{p^k \leq x} \log p = x - \sum_{\substack{\rho \\ \text{ALL ZEROS}}} \frac{x^\rho}{\rho}$$

THAT $\psi(x) \sim x$, AS $x \rightarrow \infty$.

THAT IS THE PRIME NUMBER THEOREM (PNT).

• WHILE RIEMANN DID NOT REVISIT HIS
HYPOTHESIS AS FAR AS WE KNOW HIS APPROACH
TO THE PNT VIA $\zeta(s)$ LED TO
A PROOF OF THE LATTER BY
HADAMARD AND DE LA VALLEE POUSSIN (1896)

THEY SHOWED THAT

$$\operatorname{Re}(\rho) < 1 \quad \text{FOR ALL ZEROS } \rho.$$

(4)

RH HAS CHARACTERISTICS OF A GOOD PROBLEM:

- (a) ELEGANT; EASY TO STATE; APPEALING
- (b) FALSIFIABLE

BUT WHY ALL THE FUSS?

- PROVING RH WOULD BE A MONUMENTAL ACHIEVEMENT, EARN THE CLAY PRIZE MONEY AND MUCH MORE. WE EXPECT THAT IT WOULD ENTAIL NEW UNDERSTANDING ABOUT THE INTEGERS AND THAT IT WOULD APPLY TO MORE GENERAL ZETA FUNCTIONS BELOW. IF NOT IT WOULD BE DISAPPOINTING AS $\zeta(s)$ ITSELF CONCERNS THE ARCHIMEDEAN DISTRIBUTION OF PRIME NUMBERS WHICH IS LIMITED.

A TYPICAL SHARP CONSEQUENCE IS

ALMOST ALL INTERVALS $[x, x + (\log x)^{2+\epsilon}]$

~~CONTAIN A PRIME NUMBER (SELBERG RH3)~~.

(5)

- DISPROVING RH BY SAY FINDING A ρ WITH $\text{Re}(\rho) > \frac{1}{2}$ AND AT HEIGHT SAY 10^{15} SHOULD EARN THE PRIZE MONEY (HOWEVER READ THE CLAY LEGAL FINE-PRINT CAREFULLY!)

WHAT WOULD THE IMPACT BE? (NICK TREFFETHEN)

CURIOUSLY BECAUSE OF A FEATURE OF "ZERO REPULSION" AND LANDAU-SIEGEL ZEROS SUCH A ρ COULD PROBABLY BE USED TO RESOLVE SOME EFFECTIVITY QUESTIONS.

BUT THE REAL IMPACT WOULD BE FOR THOSE LIKE MYSELF WHO BELIEVE THAT RH AND MUCH MORE IS TRUE, ONE SUCH ZERO WOULD BURST OUR BUBBLE.

- WHILE THE NUMERICAL EVIDENCE FOR RH APPEARS TO BE VERY CONVINCING ONE SHOULD TEMPER THIS WITH THE "COMPLEXITY" THAT IS THE LOCAL DENSITY OF ZEROS AT HEIGHT T IS ONLY $\frac{1}{2\pi} \log T$ WHICH IS NOT SO LARGE.

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GENERALIZED RIEMANN HYPOTHESIS GRH

THE RIEMANN ZETA FUNCTION HAS NATURAL GENERALIZATIONS TO DEDEKIND ZETA FUNCTIONS AND L-FUNCTIONS OF AUTOMORPHIC FORMS ON GL_n . THE ANALOGUE OF THE RIEMANN HYPOTHESIS FOR THESE IS A POWERFUL WORKING HYPOTHESIS AND IF TRUE A PROFOUND FEATURE OF WHOLE NUMBERS.

- FOR $K/\mathbb{Q}$ AN EXTENSION OF DEGREE n AND DISCRIMINANT Δ_K , ITS ZETA FUNCTION

$$\zeta_K(s) = \sum_a N(a)^{-s} = \prod_P (1 - N(P)^{-s})^{-1}$$

SATISFIES (SAY K HAS n REAL EMBEDDINGS)

P.E.: $\xi_K(s) := |\Delta_K|^{s/2} (\pi^{-s/2} \Gamma(s/2))^n \zeta_K(s) = \xi_K(1-s)$

$D = \Delta_K$ IS THE "CONDUCTOR" MEASURING RAMIFICATION COMPLEXITY AND

$$\frac{\log D}{2\pi}$$

IS THE COMPLEXITY IN TERMS OF THE LOCAL DENSITY OF ZEROS.

(7)

GRH: ASSERTS THAT THE ZEROS
OF $\zeta_k(s)$ (AND OF ALL AUTOMORPHIC L-
FUNCTIONS) HAVE THEIR REAL PART EQUAL TO $\frac{1}{2}$.

THE L-FUNCTION AND MODULAR FORMS
DATA BASE

WWW.LMFDB.ORG

CONTAINS A WEALTH OF INFORMATION ABOUT
SUCH ZETA FUNCTIONS OF SMALL CONDUCTOR
AND SMALL n ; INCLUDING CHECKS OF GRH.

• THE MODERN ANALYTIC THEORY OF
L-FUNCTIONS CONSISTS OF APPROXIMATIONS
OF GRH

- NONVANISHING THEOREMS
- DENSITY THEOREMS
- SUBCONVEXITY

WHICH OFTEN SERVE AS SUBSTITUTES
FOR GRH IN APPLICATIONS.

A SAMPLE:

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PNT ASSUMING RH
(RIEMANN)

PROVED UNCONDITIONALLY
1896.

"PRIMES ARE $\mathbb{P}$ "

THERE IS A POLYNOMIAL
IN $\log n$ TIME ALGORITHM
TO DETERMINE WHETHER A
LARGE n IS PRIME OR NOT
ASSUMING GRH (G. MILLER 1975)

UNCONDITIONALLY
AGRAWAL - KAYAL
- SAXENA
2002.

THE ANDRE-OORT CONJECTURE
CONCERNS THE ZARISKI
CLOSURE OF A SEQUENCE
OF SPECIAL (CM-POINTS)
POINTS IN SHIMURA VARIETIES
WAS RESOLVED BY
EDIXHOVEN
KLINGLER - YAFAEV
ASSUMING GRH (20/4)

UNCONDITIONALLY
PILA - SHANKAR - TSIMERMAN
(ESNAULT - GROECHENIG)
2022

THOUGH TO MAKE
THE UNCONDITIONAL
PROOF EFFECTIVE
ONE NEEDS GRH.

(9)

• IRREDUCIBLE POLYNOMIALS OVER $\mathbb{Q}$

$$P(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1x + 1$$

WITH $a_j \in \{0, 1\}$ CHOSEN WITH PROB $1/2$ (IID).

THEN ASSUMING GRH

PROB(P) IS IRREDUCIBLE TENDS TO 1 AS $d \rightarrow \infty$
(PROVED BY BREUILLARD-VARJU; CONJECTURED BY
ODLYZKO-POONEN.

• CUBIC DIOPHANTINE EQUATIONS:

A CENTRAL CRITICAL PROBLEM IS TO $N(T)$
GIVE SHARP BOUNDS FOR THE NUMBER $N(T)$ OF
INTEGERS x_j IN $\mathbb{Z}^6$, $|x_j| \leq T$ SATISFYING

$$x_1^3 + x_2^3 + x_3^3 = x_4^3 + x_5^3 + x_6^3$$

HUA SHOWED $N(T) \ll T^{7/2}$

HOOLEY ASSUMING GRH FOR HASSE-WEIL
ZETA FUNCTIONS OF ASSOCIATED 3-FOLDS

$$N(T) \ll_{\varepsilon} T^{3+\varepsilon}, \varepsilon > 0 \text{ WHICH IS SHARP IN THE EXPONENT.}$$

(10)

ANALOGUES

THE SETTING IN WHICH GREAT PROGRESS HAS BEEN IS THAT OF FUNCTION FIELDS

RATIONALS $\mathbb{Q}$ REPLACED BY $k = \mathbb{F}_q(t)$, $\mathbb{F}_q$ FINITE FIELD

INTEGERS BY $\mathbb{F}_q[t]$ POLYNOMIALS

PRIMES IRREDUCIBLE POLYNOMIALS

$\text{NORM}(f) = q^{\deg f}$; $T = q^{-s}$ IS THE PREFERRED VARIABLE

$$\zeta_k(s) = \sum_{(f)} N(f)^{-s} = \sum_{(f)} T^{\deg f} = \frac{1}{(1-T)(1-qT)}$$

IF K IS A FINITE EXTENSION OF k (OR A CURVE OVER $\mathbb{F}_q$) THEN ITS ZETA FUNCTION TAKES THE SHAPE

$$Z_K(T) = \frac{P_K(T)}{(1-T)(1-qT)}$$

P_K HAS DEGREE $2g$ WHERE g IS THE GENUS OF K ; SO # OF ZEROS "log cond" IS g .

(10')

$Z_K(T)$ HAS A FUNCTIONAL EQUATION
INVARIANCE UNDER $T \rightarrow 1/q \cdot T$

• ARTIN CONJECTURED THE ANALOGUE OF THE GRH; ALL THE ZEROS HAVE $|p| = \sqrt{q}$.

• WEIL FORMULATED AND CONJECTURED GENERALIZATIONS TO VARIETIES OF $\mathbb{F}_q$.

THE PROOFS OF THESE IS ONE OF THE HIGHLIGHTS OF 20TH CENTURY MATHEMATICS

• WEIL FOR CURVES

• GROTHENDIECK, DELIGNE FOR VARIETIES.

CRUCIAL TO THESE DEVELOPMENTS IS THE LINEARIZATION WHICH REALIZES THE ZEROS AS EIGENVALUES OF THE FROBENIUS MORPHISM $x \rightarrow x^q$ ACTING ON COHOMOLOGY GROUPS.

(11)

IT IS WORTH NOTING THAT THE PASSAGE FROM LINEARIZATION TO SHOWING THAT THE ZEROS LIE ON A LINE (OR CIRCLE) IS A MAJOR STEP (NO REMARKABLE "HILBERT-POLYA" TYPE SELFADJOINTNESS MIRACLE).

IN GENERAL (DELIGNE) IT INVOLVES DEFORMING THE VARIETY IN A FAMILY AND USING THE GLUE OF ^{THE} MONODROMY SYMMETRY TO GO FROM ONE TO ALL.

CHECKING THE CRITERIA FOR A GOOD PROBLEM

GRH

(1) ELEGANT AND FALSIFIABLE

(2) IF TRUE IT IS A POWERFUL TRUTH ABOUT NUMBERS

(3) IT HAS PROFOUND ANALOGUES AND GENERALIZATIONS IN RELATED FIELDS.

RH AND GRH WHERE DO WE STAND TODAY?

I REVIEW SOME AND ADD TO MY 2018 BRISTOL LECTURE

"COMMENTARY AND COMPARISONS OF SOME APPROACHES TO GRH"

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$$

(i) THE LHS ALLOWS ONE TO STUDY $\zeta(s)$ USING THE ADDITIVE STRUCTURE OF $\mathbb{R}$ AND OF $\mathbb{Z}$ AS A DISCRETE SUBGROUP WITH COMPACT QUOTIENT. IN PARTICULAR THE POISSON SUMMATION FORMULA WHICH LEADS TO THE ANALYTIC PROPERTIES

(ii) THE RHS CAPTURES THE MULTIPLICATIVE STRUCTURE OF $\mathbb{Z}$.

(13)

THAT ONE HAS TO USE BOTH (i) AND (ii)
IS MADE CLEAR BY

DAVENPORT-HEILBRONN (1936) (NO EULER PRODUCT)

LET $Q(x,y)$ BE AN INTEGRAL BINARY QUADRATIC FORM OF
DISCRIMINANT $d < 0$ FOR WHICH $k(d)$ IS ODD AND NOT 1.

THEN

$$J_Q(s) = \sum_{(x,y) \neq (0,0)} Q(x,y)^{-s} \quad \text{HAS INFINITELY MANY ZEROS IN } \operatorname{Re}(s) > 1.$$

SELBERG (1980) SHOWED THAT $J_Q(s)$ HAS
A POSITIVE PROPORTION OF ITS ZEROS ON
 $\operatorname{Re}(s) = \frac{1}{2}$ (MATCHING HIS RESULT ON $\zeta(s)$) SHOWING
THAT SUCH RESULTS ARE STATISTICAL IN NATURE.

- MIXING THE ADDITIVE AND
MULTIPLICATIVE STRUCTURES EFFECTIVELY
REMAINS A CHALLENGE.

(14)

- IN THE FIRST 30 OR SO YEARS AFTER RIEMANN'S PAPER IT WAS NOT CLEAR IF IT WAS A DIFFICULT PROBLEM OR NOT (AS RIEMANN HIMSELF EXPRESSED). HOWEVER, BY NOW IT IS CLEAR THAT IF THERE IS SOME ELEMENTARY SOLUTION IT IS WELL HIDDEN.

- THERE ARE PLENTY EQUIVALENCES OF RH IN THE LITERATURE. IT IS DIFFICULT TO PREDICT WHETHER ANY OF THESE OFFER INSIGHTS OR NOT.

AS LONG AS THEY YIELD NO NEW RESULTS THAT HAVE APPLICATIONS ALONG THE LINES THAT RH IS A WORKING HYPOTHESIS, THEIR RELEVANCE ~~REMAINS UNCLEAR.~~

(15)

FOR EXAMPLE, IF $\lambda(n) = (-1)^{\Omega(n)}$ IS THE PARITY OF THE TOTAL NUMBER OF PRIME FACTORS ~~NUMBER~~ OF n THEN

(i) $\sum_{n \leq x} \lambda(n) = o(x)$ AS $x \rightarrow \infty$ IS EQUIVALENT TO THE PNT.

(ii) $\sum_{n \leq x} \lambda(n) = O_\epsilon(x^{\frac{1}{2}+\epsilon})$, $\epsilon > 0$ IS EQUIV. TO RH.

WHILE ELEMENTARY PROOFS OF (i) HAVE BEEN FOUND*, MOST RECENTLY A LOVELY ERGODIC THEORETIC PROOF BY RICHTER AND MCNAMARA (2021)*, THESE OFFER LITTLE TOWARDS THE QUANTIFIED (ii)

(ii) SAYS THAT THE $\lambda(n)$ 'S HAVE ± 1 VALUES MIMICKING RANDOM ONES, BUT SHOWING RANDOMNESS FEATURES IN SOMETHING EXPLICIT IS DIFFICULT.

(16)

• WITH THE SUCCESS OF THE LINEARIZATION THROUGH COHOMOLOGICAL METHODS IN THE FUNCTION FIELD, ONE MIGHT HOPE FOR SOMETHING SIMILAR OVER $\mathbb{Q}$.

• A PROMISING INDICATION OF THE SPECTRAL NATURE OF THE ZEROS OF ZETA FUNCTIONS IS

DYSON-MONTGOMERY (1973)

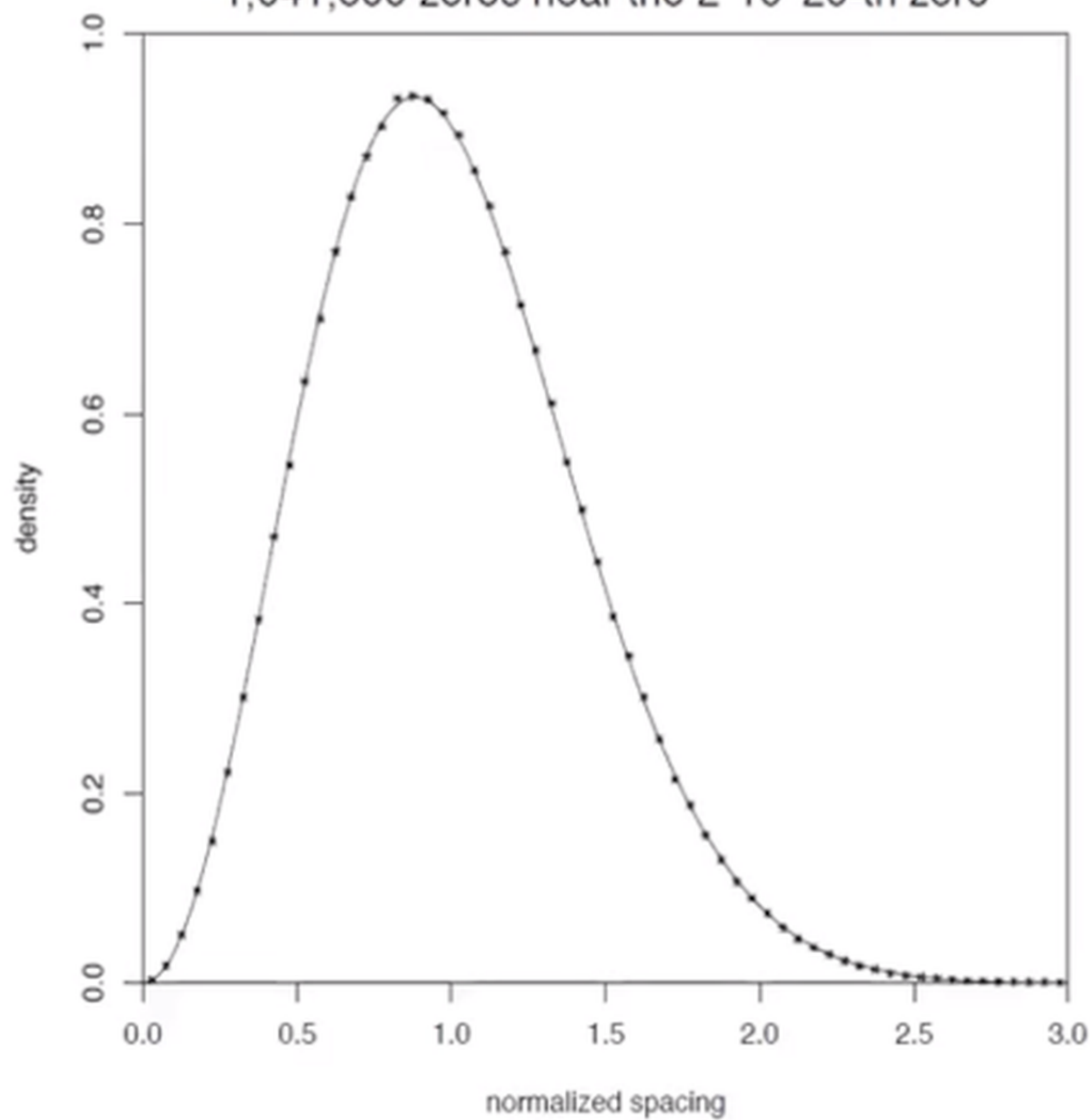
THE LOCAL SPACING STATISTICS

(I.E. IF $\rho = \frac{1}{2} + i\gamma$, $\frac{\log T}{2\pi}(\gamma - \gamma')$
CORRESPONDING TO ρ, ρ' WITH $\gamma \sim T$)

OF THE HIGH ZEROS OF (ALL!) ZETA
FUNCTIONS FOLLOW THE STATISTICAL LAWS
OF RANDOM HERMITEAN MATRICES

"GUE"

Nearest neighbor spacings among
1,041,600 zeros near the $2 \cdot 10^{20}$ -th zero



(17)

• THIS HAS BEEN VERIFIED NUMERICALLY AND THEORETICALLY IN RESTRICTED RANGES (ASSUMING GRH). IT IS VERY CONVINCING AND HAS LED TO AN ACTIVE STUDY OF SUCH STATISTICS OF ZETA FUNCTIONS AS WELL AS FAMILIES OF SUCH.

• THE SOURCE IN THE FUNCTION FIELD OF SUCH A CONNECTION IS UNDERSTOOD (KATZ-S 1998) AND ARISES THROUGH SCALING LIMITS OF MONODROMY GROUPS OF FAMILIES OF ZETA FUNCTIONS.

ONE LINEARIZATION THAT WAS SUGGESTED BY PAUL COHEN IN 1977 AND WHICH HAS BEEN ADVANCED BY CONNES (1999) AND MEYER (2005)

IS AN ACTION OF MULTIPLICATION OVER ADDITION ON CERTAIN ADEL/IDELE SPACES.

(18)

$\mathbb{A}$ RING OF ADELES OF $\mathbb{Q}$ $(a_0, a_2, a_3, \dots)$
(ADDITIVE) $a_p \in \mathbb{Z}_p$ FOR a.e. p .

$\mathbb{A}^\times$ RING OF IDELES $(a_0, a_2, a_3, \dots)$
 $a_0 \neq 0, a_p \in \mathbb{Z}_p^\times$ FOR a.e. p .

THE STANDARD SPACES ARE (TATE "VALUATION VECTORS")

$\mathbb{A}/\mathbb{Q}$, $\mathbb{A}^\times/\mathbb{Q}^\times$

CONSIDER $\mathbb{A}/\mathbb{Q}^\times$ VERY SINGULAR QUOTIENT

THE ACTION $x \rightarrow xy$, $y \in \mathbb{A}^\times/\mathbb{Q}^\times$, $x \in \mathbb{A}/\mathbb{Q}^\times$

MIXES THE STRUCTURES.

- ONE CAN INTRODUCE SUITABLE TOPOLOGICAL VECTOR SPACES OF FUNCTIONS ON $\mathbb{A}/\mathbb{Q}^\times$ SUCH THAT THE TRACE OF THE ABOVE ACTION YIELDS THE RIEMANN-GUINAND-WEIL EXPLICIT FORMULA (MEYER), IN PARTICULAR ONE HAS THE ZEROS OF L-FUNCTIONS INTERPRETED AS EIGENVALUES OF THIS ACTION.

CONNES ARXIV 2602.04022 GIVES AN UPDATE ON HOW ONE MAY USE SUCH A SPECTRAL INTERPRETATION. ALSO INCLUDED IS AN ENTERTAINING LETTER TO PROFESSOR BERNHARD RIEMANN.

(19)

GIVEN ITS VERY STRONG IMPLICATIONS GRH
SHOULD BE PROBED IN EXTREME LIMITS AND
FOR THE DIFFERENT SPECIES OF ZETA FUNCTIONS.

RAMIFICATION IN THE $n \rightarrow \infty$ LIMIT:

THE LOCAL COUNT OF THE NUMBER OF
ZEROS OF $\zeta_K(s)$ IS CONTROLLED BY

$\log N_K$ "LOG OF THE CONDUCTOR"

AS SUCH TOGETHER WITH GRH THIS YIELDS
VERY STRONG LOWER BOUNDS FOR RAMIFICATION =
CONDUCTOR, IN TERMS OF THE DEGREE n OF K
(STARK, ODLYZKO, ...)

THESE BOUNDS ARE MUCH SHARPER
THAN THOSE COMING FROM THE GEOMETRY OF
NUMBERS (MINKOWSKI), SO MUCH SO THAT THE
COME CLOSE TO VIOLATING EXPLICIT CONSTRUCT
OF TOWERS WITH LITTLE RAMIFICATION (GOLOD-SHAFAREVIC)

(20)

NO CONTRADICTIONS OTHER THAN MISCALCULATIONS
HAVE EMERGED FROM THESE.

THERE IS ANOTHER SPECIES OF ZETA
(OR "L") FUNCTIONS THAT COME FROM
TRANSCENDENTAL AUTOMORPHIC FORMS (CUSPIDAL)
ON $GL_n / \mathbb{Q}$ WHICH ARE
EVERYWHERE UNRAMIFIED (MAASS FORMS)

THESE ELUSIVE FORMS ARE THE BUILDING-
BLOCKS OF THEORY AND ARE THERE ONLY
BECAUSE OF THE "ARCHIMEDEAN PLACE".

THEY ARE EIGENFUNCTIONS OF DIFFERENTIAL
OPERATORS ON THE LOCALLY SYMMETRIC SPACES

$$X_n = SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) / SO_n(\mathbb{R})$$

- THE "ANALYTIC CONDUCTORS" OF THEIR ZETA
FUNCTIONS; THAT IS THE LOCAL DENSITIES
OF THEIR ZEROS, INVOLVE THEIR EIGENVALUES.

(21)

GRH $\Rightarrow$ VERY STRONG BOUNDS FOR
THEIR EIGENVALUES AS WELL AS A
PAUCITY OF SUCH FORMS.

IN PARTICULAR IN THE $n \rightarrow \infty$ LIMIT
IT IMPLIES VERY STRONG CONCENTRATION
PROPERTIES FOR FUNCTIONS ON THE
CORE OF X_n , AS $n \rightarrow \infty$.

LARGE n -
THIS LIMIT HAS NOT BEEN PROBED
AND I BELIEVE ^{WILL BE} ~~IS~~ A REAL STRESS
TEST FOR GRH.

THE VERY GENERAL

(AI ?)