Forbidden symmetries
Sir Roger Penrose

(First 20 minutes of the talk)

Well, thank you very much for that introduction, I hope I can live up to it. First of all let me... In fact, the main thing I want to talk about is something which is not quite finished, namely, well, you see, what you see in front of you up here is actually a drawing or whatever they do these days with computers, I suppose, of the new mathematics building that's the only thing which is in color, which is not quite finished. It's finished enough that I have an office there somewhere, I can't really get into it, there's only one chair and two desks and a whole lot of crates. So it's not very hospitable yet, but that's partly my fault, because I collected too many things in my other office.

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But right here will be an area tiles with a particular arrangement which is based on things that I've been doing. So I want to explain that. That's really the purpose of this talk. I'll come to the detailed explanation of what's going on there towards the end of the talk. I shall say lots of things about other architectural use of these tilings in other parts of the world, but before doing that, I want to explain the tilings themselves.

Greg Jenkins, Education Media Services,
University of Oxford.

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So let me... well I need to go to the next picture I think. What you can see here, the front entrance is right here and you have to walk over my tiles in order to get in the building. And it's not quite finished you might see some goings on back there where it's not finished there's some other than goings on back in front which aren't in the picture very conveniently where it's not finished either. That's not finished clearly. *(Laughs).* Ah but so and that one in the middle isn't finished I think that's going to be the last tile which is placed in its certain sense, its central in the whole design but you may see some kinds of regularities as you look at it. But not altogether obvious what these regularities are.
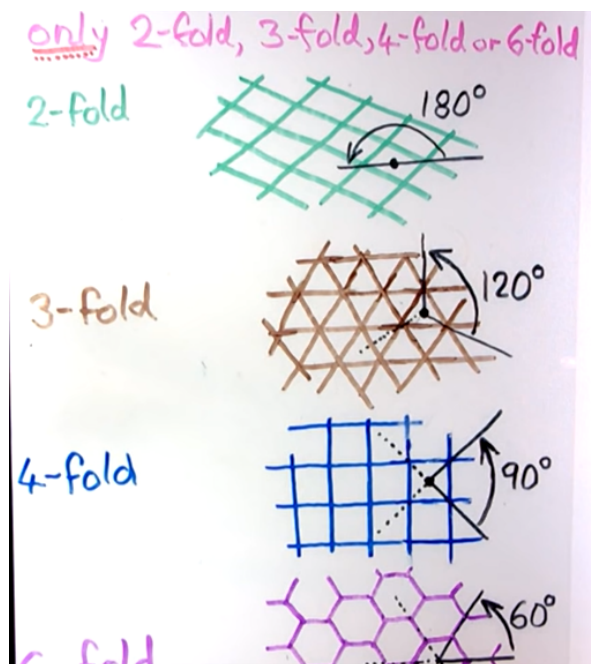

So I shall be explaining what that's all about. But before doing that let me explain the general idea of what crystal symmetries are allowed and what aren't. And so can I move to the visualizer please... Thank

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you. I'm using this kind of old-fashioned technology because that's the only one I understand. These are the crystallographic symmetries. So we have two-fold symmetries a regular pattern of parallelograms will have a two-fold symmetry about the center of each parallelogram. Three-fold if you have equilateral triangles then about the center of any triangle you will have the whole pattern has a three-fold symmetry, you rotate 120 degrees, and the whole pattern goes back into itself. Four-fold core squares that's the most familiar. And six-fold we have the familiar pattern of hexagons. And it's a theorem of mathematical theory that these symmetries are the only ones that you can have. When I say that, I mean in addition to translational symmetry so you have to have a translational symmetry that means you slide the whole picture parallel to itself in some directions and the pattern goes into itself.
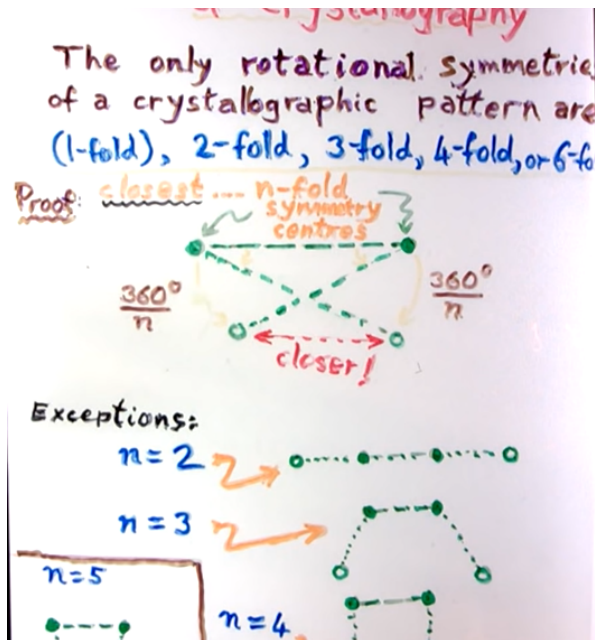
So you want to have a rotational symmetry *and*, together with that, a translational symmetry. And I want to give you a quick argument to show that that is the case : these are really the only symmetries you can have ; and this is a sort of standard theorem. Suppose you have a pattern of points or something something which isn't like a fractal. So they're discrete and have sort of finite minimum distances between points and things that you're... Just you can imagine some pattern. And that pattern has a translational symmetry so you could slide it along and it goes into itself but also rotational symmetry so you rotate through 360 degrees over n where n is your degree of symmetry and the pattern goes into itself. So what I'm going to propose then is that you have such points of symmetry. So here's a point of n-fold symmetry and there's another point of interest : if there's *one*, there must be *another* because if the whole pattern slides along, then this will go into another. It has to be another n-fold symmetry point. So it's gotta have more than one. If it's got to have more than one, there will be somewhere in the pattern, two of them which are as close as possible. So I'm going to choose those, as long as it's not a fractal or something silly like that, choose a pair of points which are as close as they can be, and that's those ones up there. Now you see I'm going to rotate this one by 360 degrees over n into that point, and this one in the other direction through 360 degrees over n into that point, and these two will then be closer, which contradicts this being the closest. So that's a contradiction, unless n equals two, when they're not closer, they are lot further off. When n equals three they're not closer, they're further off. When n equals six, that's above there. When $n = 5$, they coincide so you get away with it there. In the case of n equals four of course they are at same distance as before, so that's okay. But n equals five for example, they're closer and anything larger than six they'll cross over like here and they'll certainly be closer;
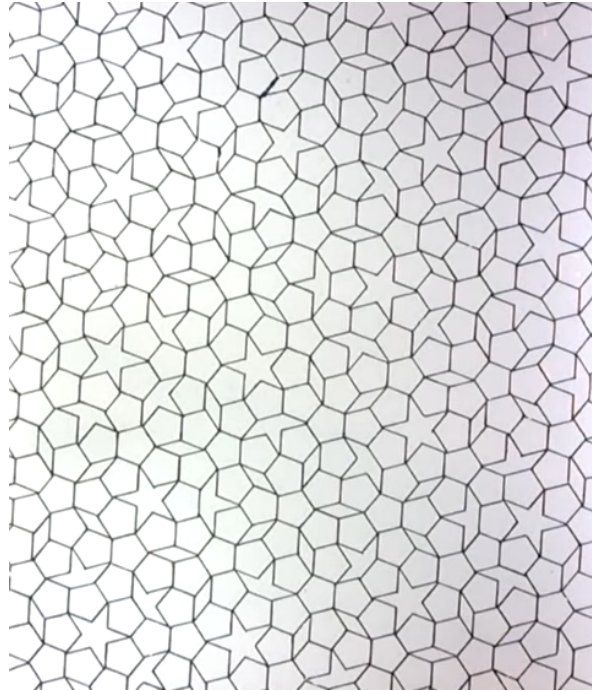
So that tells you that the only crystal symmetries you can have are those ones given : two, three, four, and six. Okay, straightforward, okay.

Well, what about this pattern well it has a lot of regularity to it and it has a sort of fivefold Ness to it also in fact you can see various regions which are five-fold symmetry up to a point and which have translational symmetry up to a point in fact let me tell you that if you give me any percentage less than 100% so in 99.9 percent then I could slide this picture over itself so the pattern agrees to with what you had before to that percentage 99.9 percent of whatever it was and also has the rotational symmetry fivefold symmetry two ninety-nine point whatever it is and you might say well why doesn't the theorem work and the argument is well you go back to the proof here[1] and you see if this point up here wasn't a perfect one you see suppose it was only ninety-nine point nine percent point and that's a ninety-nine point nine percent point then that's likely to lose just a little bit of accuracy, it'll be nine nine point eight likely, and that one's probably ninety-nine point eight so that although they're closer they're not quite so good. So you *lose* a bit of the symmetry. But each time you perform this argument here, but the points may be closer. So there's a trade-off between those two things. And that's exactly what happens with this pattern.

So anyway, I'll just show you that. I think it's worth pointing out a number of features of this pattern... Mmmm, for example, wel,l actually have-you these rings here. I'll talk about those later.

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[1]See picture 4.
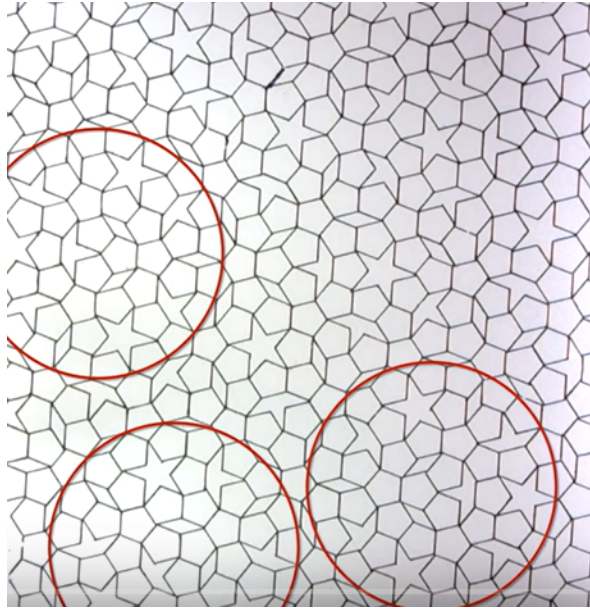
That's a 10-folded regular decagon. And every time you find one of those regular decagons, there's always ten pentagons surrounding it. That's always the case, wherever you find one, there is another one here. Sometimes you find them overlapping, like this one and this one,
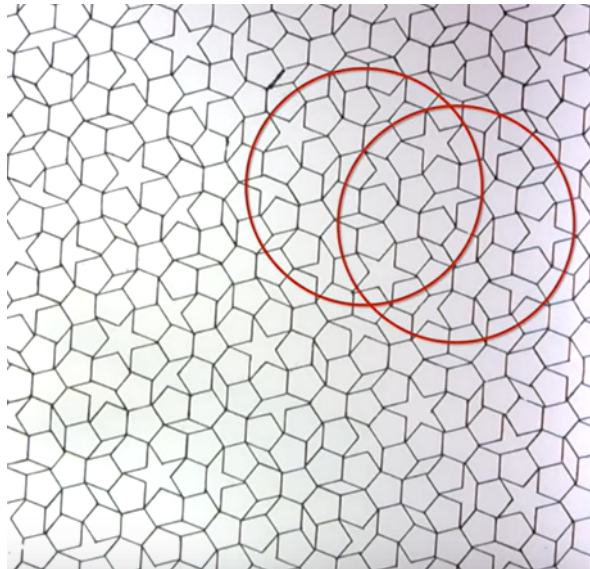

and you still get rings of ten pentagons, they just happen to go through each other that's all, so that's a general feature.

Another feature for example, which is perhaps a little more obvious from where I am, well, it depends where you're sitting, if you're sitting up the edge, it's probably fairly obvious, that you see these lines line up.
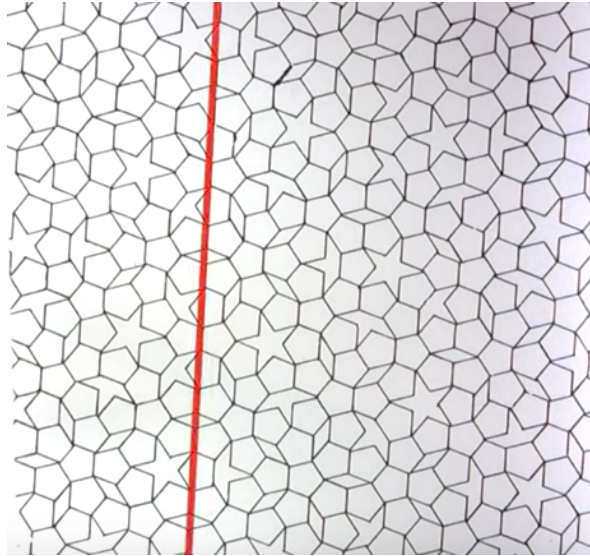
Here I have wherever you find a line in there in the picture and you put your ruler along that you just find other lines lying in that and the density of them just doesn't fall off, it doesn't matter where you do it.
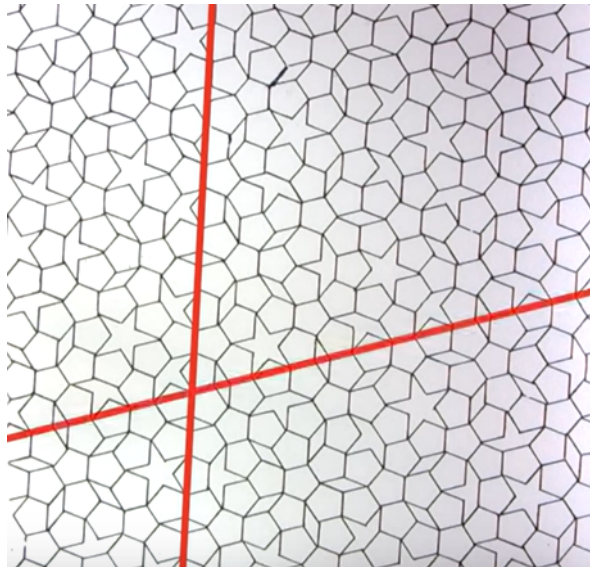

So the pattern has a lot of regularity about it and this regularity, well, it's not completely obvious from the way... Well, I'm going to tell you how it was constructed first of all. It's constructed from something very simple : here we have a regular pentagon, subdivided into six smaller ones I think it's easy if I just do it down here regular pentagon six smaller ones with a few little gaps here.
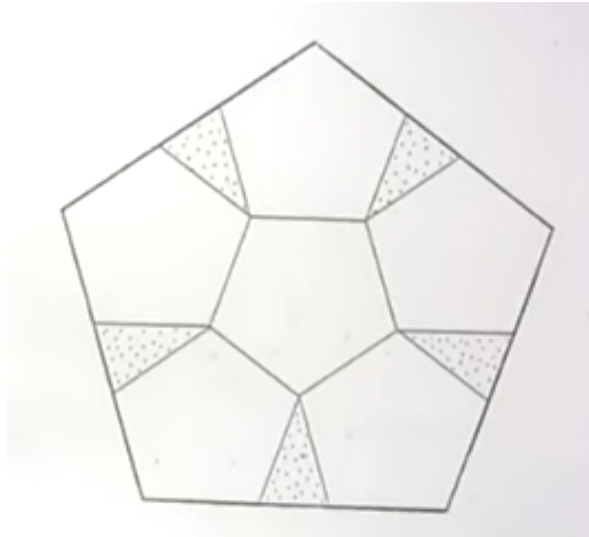
Now what I'm going to do is blow this up so that the smaller pentagons are the same size as the original one, and then subdivide each of those blow it up, subdivide, blown up, subdivide... Now if I do that it doesn't quite work : et me just imagine I now have blown it up so this was a bigger pentagon, sort of out here and this was subdivided, then I've start dividing again, and you notice there was a little gap here and that little gap joins up another little gap so we have a little rhombus shape, hole in the pattern.
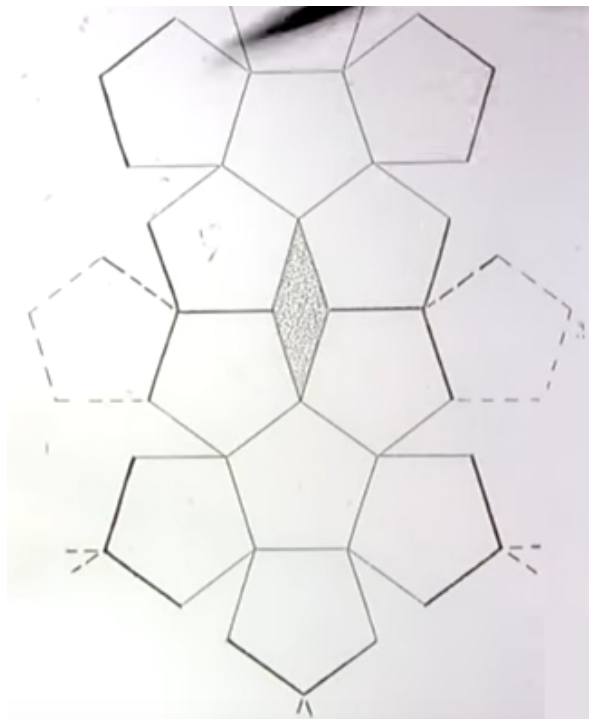

Okay. Now I'm going to do it again. And when I do it again, if I subdivide this, there's going to be a little gap there, so this rhombus will grow spikes. And it will look like that.
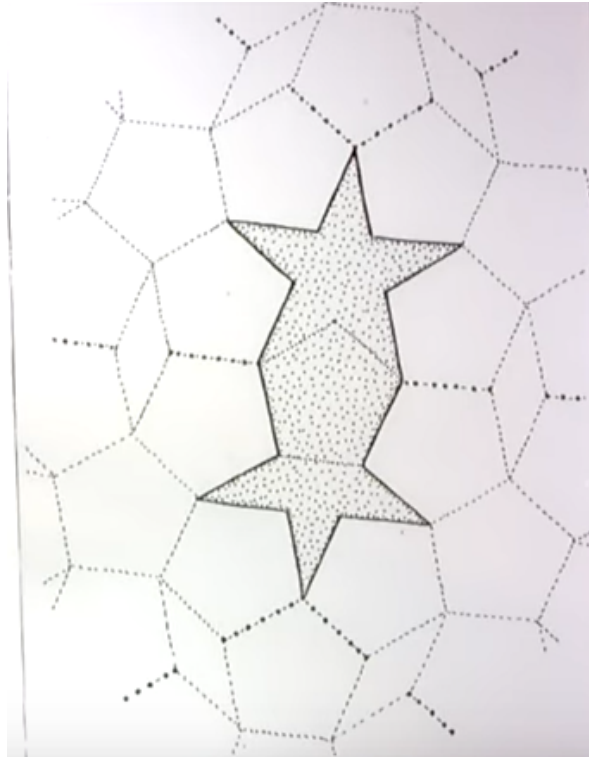
Now when it does that, I found there's just room to put another pentagon inside there. *(Laughs)*. Okay and then the next step, this will grow spikes and you get something which looks like... Well, you see, I should point out that the shapes I have, the original pentagon, this star now which I'm calling a pentacore that's a pentacore, and this thing I'm calling the justice cap, well it's more like a justice cap, this way up. So they're justice caps, pentacores and rhombuses, we had the little rhombuses before now when you grow spikes for each of these shapes which you go at the next stage of hierarchy you can always find that you can fill out the gaps with shapes that you had before, so you can blow up, fill it up with these shapes, blow it out, fill it up. And that will end up with something like this, well, it will, if you've been just slightly careful about this little subtlety which I have to mention.
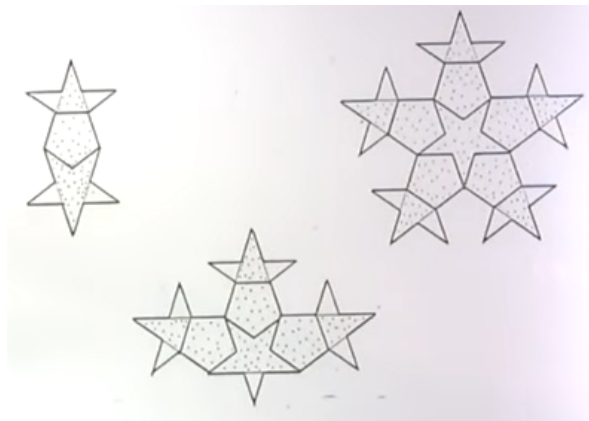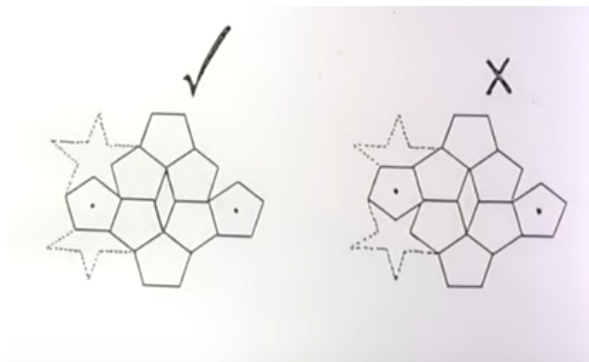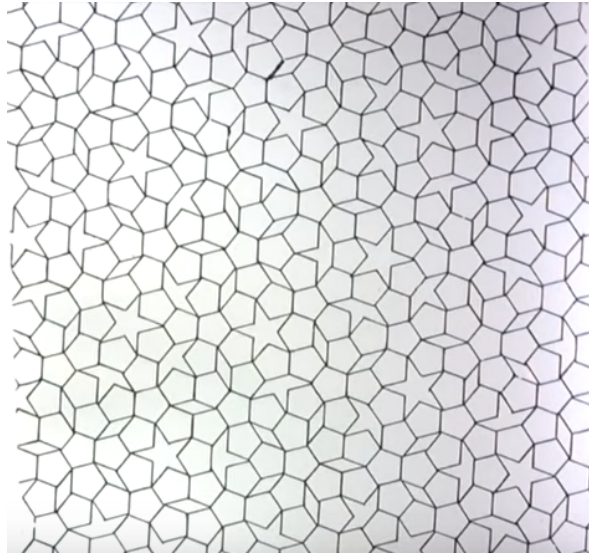
It's interesting because I discovered after having done this because a Japanese man had done exactly the same thing except that he made the wrong choice here, and that doesn't lead to what you get.

The other thing I wanted to say : now I put a tick on this, and a cross on this, so here is the rhombus with its spikes and I put a pentagon there like that, remember, you see this, that's the rhombus, with its spikes, and I can put a pentagon either up here or down there, there's a choice. Now what I want to do is to force one of those two choices. And if you look at this spiky rhombus here, does that go down or top ? Well, what you do is you look one side or the other and I'm just telling you this, but it does happen, that you always find a pattern of pentagons like this, except this one maybe *(pointing the pentagon at the right of the picture, in the "good" picture on the left)* at bottom or at top, you then reflect in the middle rhombus so that one goes to there if that's on the bottom, if this is on the bottom, that has to be on the bottom ; if this one is on the top, that one will be on the top. If you make the choice the Japanese man took, this one then you find that the next stage, it goes wrong, whereas this one keeps going forever and so the pattern with the subdividing blowing up, subdividing, blowing up, will just get bigger and bigger and bigger, and cover as much of the plane as you choose, in fact will cover the entire plane.

Okay now I want to show that of the backwards now. Suppose we have a big pentagon here.

I hope most of that Pentagon is on the screen, there's another one sitting up here, and another one sitting up here, and one sitting there, and sitting here, and so on. But there's a big pentagon and what I want to do to that big Pentagon is subdivide in the way I've been describing, here we go.
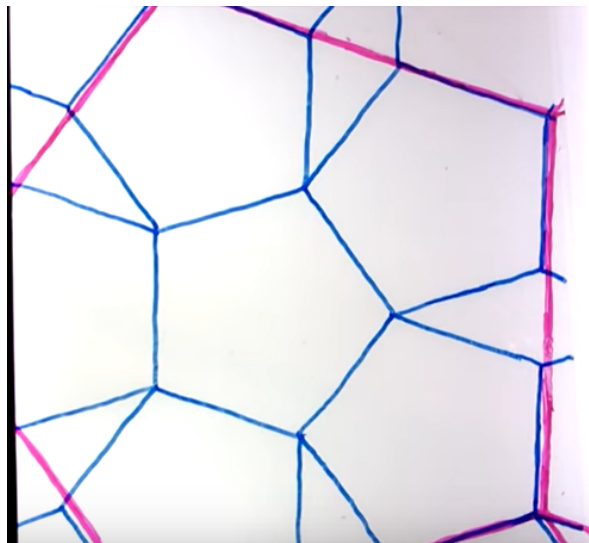
And then I subdivide that again there yeah and then I subdivide that again and you get the pattern I just showed you. Okay.
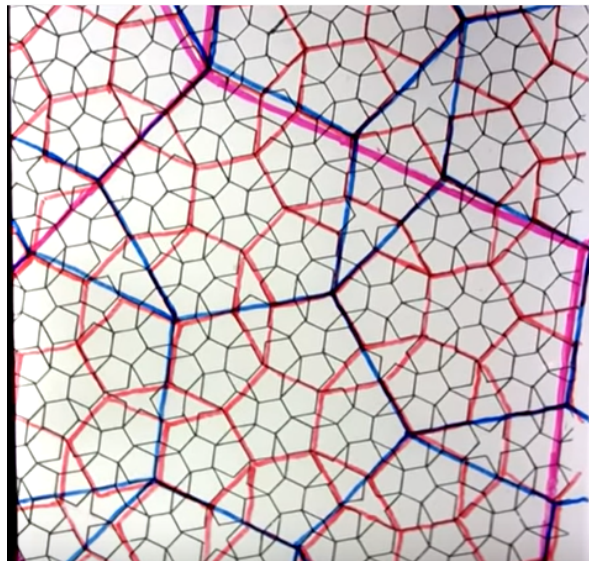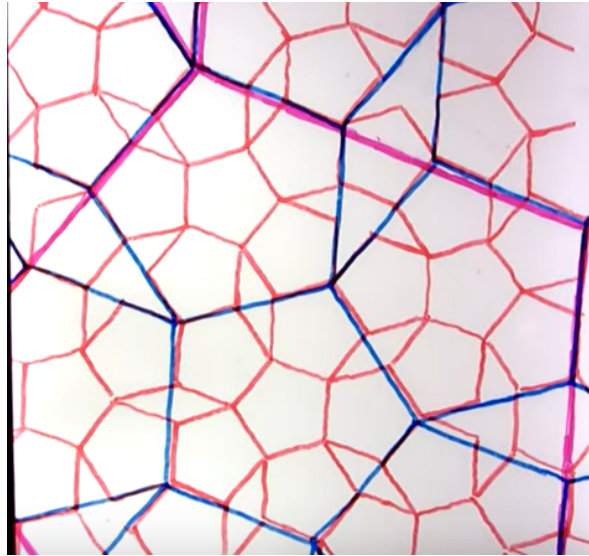
Now, where wis that big Pentagon ?

(Laughs)

Now, you see, there's something interesting about this, that the pattern has a greater uniformity, than you might have thought from the hierarchical construction. In fact, you know I have to find it, I can probably find it, but I won't even try. It has a greater uniformity than the hierarchical organization seems to suggest. But yet their hierarchy is hidden in that all the time. Now sometimes people point out that this kind of thing was in ancient Islamic art And there's an article here I'll show you.
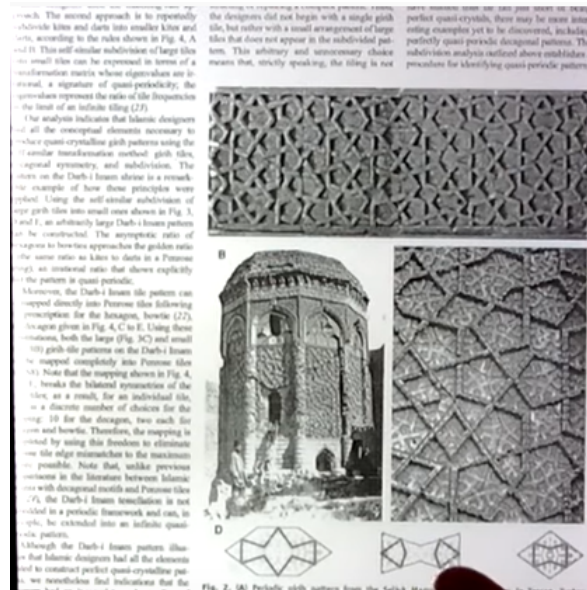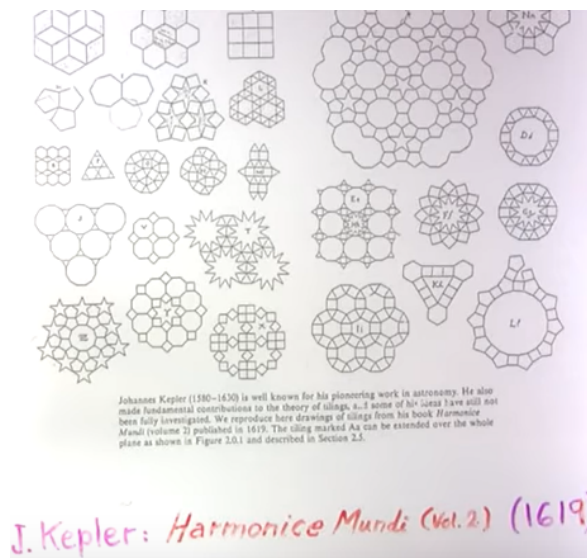
Fig. 2. (A) Periodic girih pattern from the Selûk Mosque in Sivas, Turkey.

And there's a lot of very interesting ancient Islamic art, where you see regions of five-fold and ten-fold symmetry, fascinating stuff, but I haven't seen any indication of the very kind of strict structure that scones the hierarchical arrangement in that strict sense, or anything like that. Nevertheless they're fascinating but it's not so clear what the deep connection is with the things I've just been showing you.

However if you go a little bit more recently than these ancient Islamic things, namely to 1619, then we find in the works of Johannes Kepler, the famous astronomer, in a book he wrote called *Harmonice Mundi*. And in that book, you will find these fascinating pictures.

Johannes Kepler (1580–1630) is well known for his pioneering work in astronomy. He also made fundamental contributions to the theory of tilings, a...t some of his ideas have still not been fully investigated. We reproduce here drawings of tilings from his book *Harmonice Mundi* (volume 2) published in 1619. The tiling marked Aa can be extended over the whole plane as shown in Figure 2.0.1 and described in Section 2.5.

J. Kepler: *Harmonice Mundi* (Vol. 2) (1619)

Now I should say that my father owned a copy of this book, and I had seen this picture. I'd seen it although it wasn't in my mind when I started doing this, except somehow, I was with no doubt somehow influenced to think that pentagons unloss a dead loss (?) you see what he's done, fascinating things with pentagons and other things with pentagons. But this design in particular I want to point out. And as I later discovered to my surprise : here we have that picture drawn bigger, that's the very same picture in Kepler
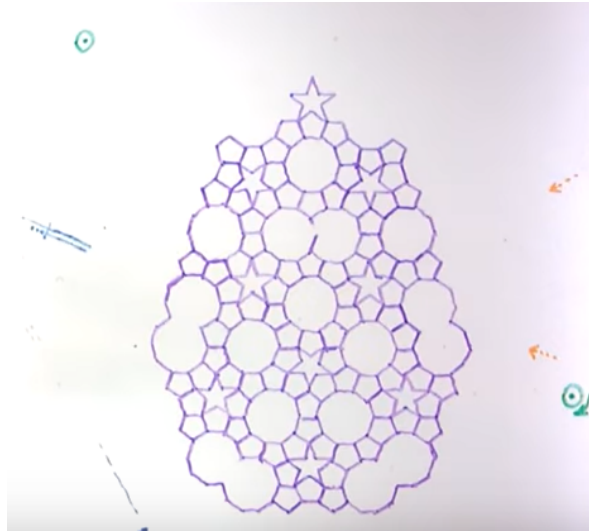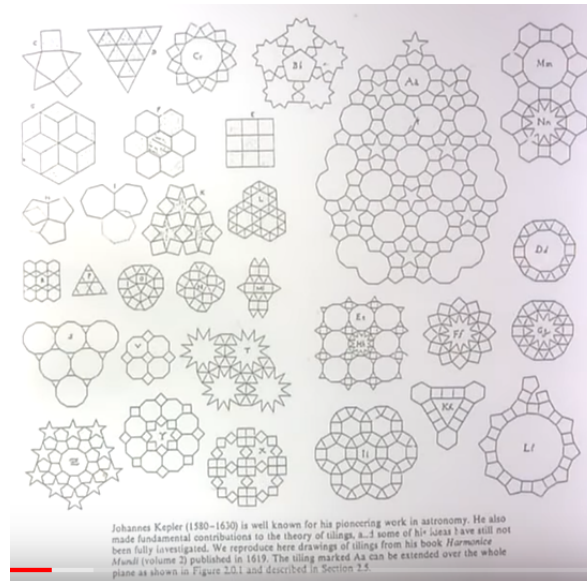
including a little line there[2], which I don't quite know why he put it there but he did.
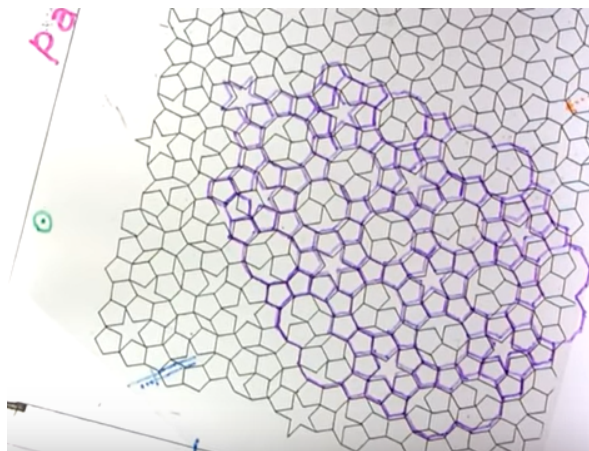

Now here is my Pentagon pattern. Now I've put some marks here if I can find exactly where. You will see that the Kepler pattern exactly fits this, including this little line there. So what was he doing ? I don't know ; I suspect he was probably thinking, I mean, they didn't even know about atoms and so on, and crystals, I suspect that he was interested in maybe some kind of atomic arrangement and perhaps, he knew about crystals and he perhaps wondered where the biological things. Because you often see five-fold symmetry and so on in biology that perhaps that was something underlying that. I have no idea, and I have no idea how he intended to continue this pattern. There is people who have tried other continuations which I don't believe. I think his continuation was probably much more like what I was doing here, I wouldn't be at all surprised. But it is fascinating that he did indeed do, that he has had this great interest in these things. He was interested in this thing called the Kepler problem which is packing spheres as closely as you can in three-dimensional space, and this problem was unsolved for a long time and only fairly recently was it sort with the aid of a computer, I may say, and it turned out Kepler was right and what he suggested.

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[2]Showing a little line in two big decagons, in the top middle of the figure.

Johannes Kepler (1580–1630) is well known for his pioneering work in astronomy. He also made fundamental contributions to the theory of tilings, and some of his ideas have still not been fully investigated. We reproduce here drawings of tilings from his book *Harmonice Mundi* (volume 2) published in 1619. The tiling marked Aa can be extended over the whole plane as shown in Figure 2.0.1 and described in Section 2.5.

You also see are other symmetries displayed like these[3] and in fact you find these things in quasi-crystals also ; I wasn't going to show you too many of these other ones, it just takes too long,
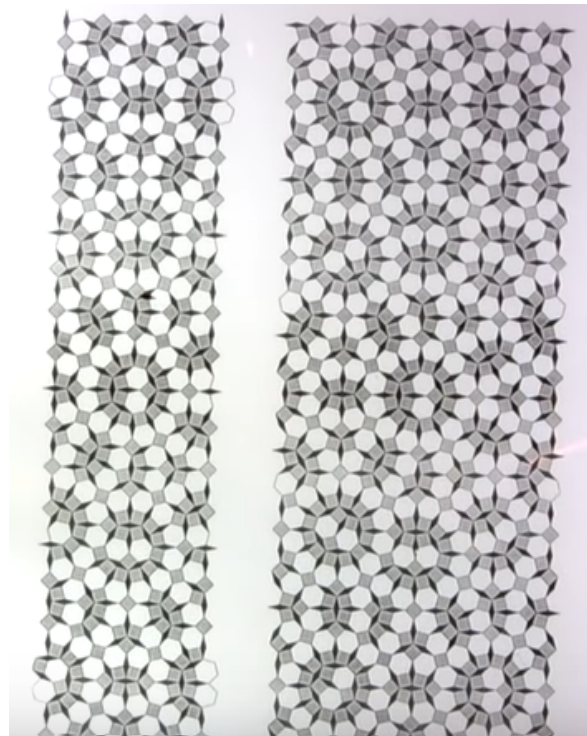
I've just shown you here 12-4-1. This is by some people (Galen and H.-U.Nissen[4]) and it's a design (Nisssen is an experimental person and he'd actually spotted twelve-fold symmetries in certain apparently crystalline materials and he showed me the diffraction pattern it's got a curious story showing the diffraction pattern and I saw this pattern of little spots where you get diffraction electrons really bounce off in certain directions. And where have I seen that pattern before ? I couldn't think, I've seen it before, from

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[3](pointing F)
[4]?

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where, I've seen it before and I realize it's right here. It is this little [5] little spikes are on the corners of this little pattern in Kepler. What was he doing ? I haven't the foggiest, but it's fascinating to see how much there was in that little set of pictures of Kepler.

Here's another one with that 12-fold symmetry. 12-fold ones are rather nice. There are also eight-fold ones which Robert Ammann and someone else[6] produce. They're never found them quite so attractive I think that the five-fold ones and the twelve-fold ones are also particularly nice. So but I'm not showing any more version for in so I don't think they've been used in architectures far enough. Okay, so, that's the basic idea.

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[5] (showing *F* figure in Kepler *Harmonice Mundi*)
[6] Beenker ?