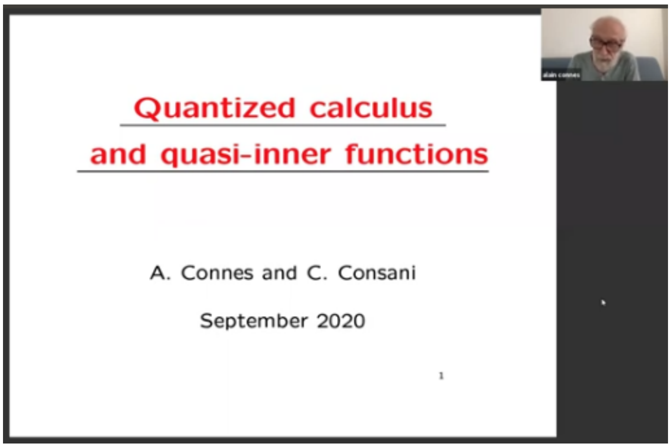
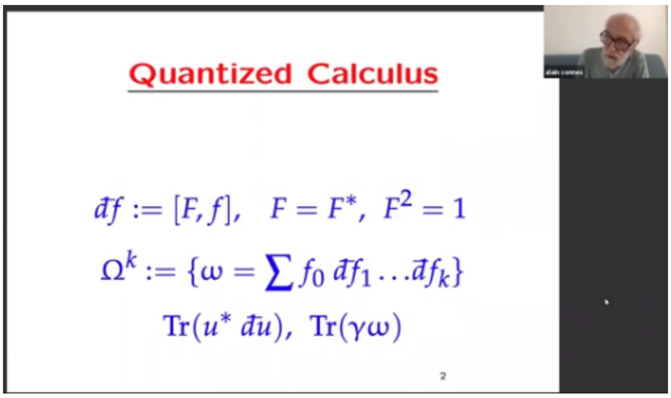
Quantized calculus and quasi-inner functions
Alain CONNES

Introduction by Walter van Suijlekom to the NCG seminar.

Thanks a lot, Walter, and indeed, for me, it's a great pleasure, really, to give this inaugural talk for the seminar, and what I will explain is something which has to do with the quantized calculus and which is a joint work with Katia Consani. The content is a paper that we put on the arxiv in august of this year, but it also deals a lot, as you will see, with a previous paper that we have put earlier, like in june of the same year.

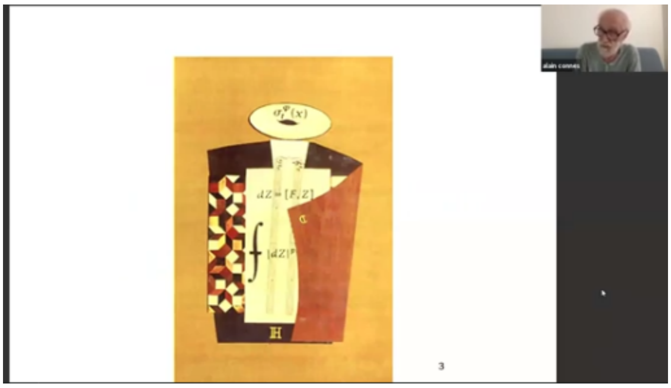


What I will discuss will be the link precisely between the quantized calculus and number theory. And I mean the quantized calculus... I want to give you like a, you know, a very short introduction to it, and the idea is very very simple. The idea is the following... It's that after all, there is a framework where the calculus, infinitesimal, and many things... have their place, and which really comes from quantum mechanics. So the scenari, if you want, the stage, is quantum, in the sense that you have a Hilbert space around, and that, whatever you are considering are sort of realized as operators in Hilbert space.

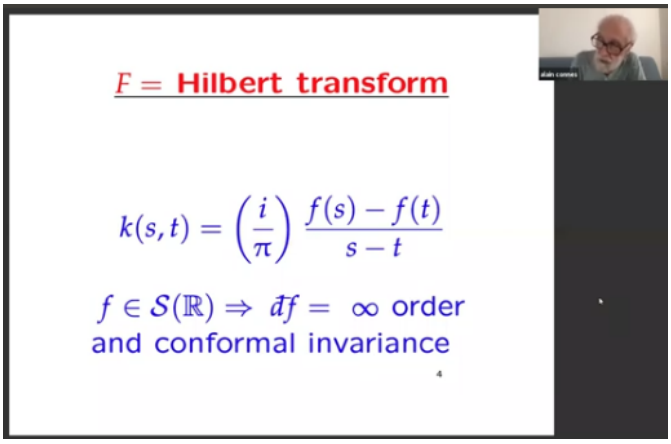


This video can be followed here https://www.youtube.com/watch?v=mmbOQ6h1Qk8.
Transcription Denise Vella-Chemla, 10.10.2020.

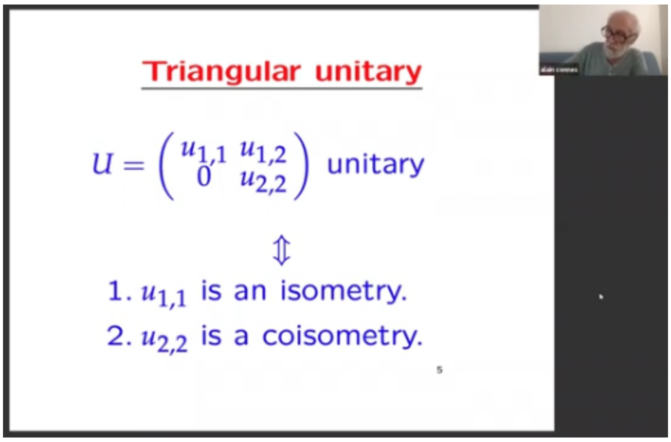
Now the key equation is the equation that defines the differential. So if you have a function, let's think of it as a function but it's an operator, and you define its differential by the commutator that is an operator F. And this operator F is remarkably simple in the sense that it's a self-adjoint, unitary operator of square 1. So it's an operator which is given as twice a projection minus the identity. And the fact that it is of square 1 tells you that when you take twice the differential function, you get 0. So you can argue with me, you can say "well, now, what you say is wrong" except that when you take the differential one-form which is already a differential, you take the graded commutator. So instead of taking F commutator F, you take $FT + TF$. And if you do that, you'll find out that because F^2 is 1, the iterated commutator is 0. So you can define differential forms. So differential forms are just those operators which can be written as combinations of products of one-forms and (of course, you can put in front a function), but you could also put that function on the right hand side because it's a bimodule by construction. Now the beauty of that is that extremely simply and from the very start, you get index formulas. And the simplest index formula is the index formula that gives you a winding number of a function on S^1 as the trace of the quantized differential of u^*, u^* is the complex conjugate of this function of modulus 1, times the differential of u. And more generally, when you deal with a space whose dimension is even and not odd, you have to use, as in the Kasparov theory, you have to use a Z_2-graded, called little gamma (γ) and the formula for the index is again very very simple because you integrate the differential form by taking the grading trace for even differential forms. So all of that is fairly standard and if you want just to... first of all, let me show you



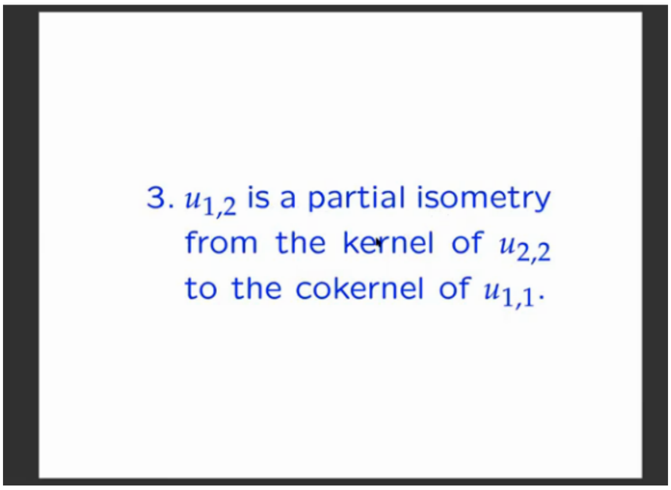
what was the cover of my book because that picture is telling in the following sense that if you look at the differential dZ equals F commutator Z (d$Z = [F, Z]$), it's written just at the level of the neck of the picture, but if you look a little bit more, downstairs, what you find is the following : you have to understand it like this : you see, normally, when you have a function that for instance parametrizes a Julia set or something like that, by the Riemann mapping theorem because the Julia set will separate the convex plane in two domains, one simply connex, so you can map it to the disk, but the function that maps it to the disk, that maps rather the circle to the Julia set, is a very discontinuous function, I mean it's a function which is not at all differentiable, it's continuous because it's join kernel which doesn't care, but it's extremely irregular so if you would take its diffential at the distribution, this would make sense, but you would have no way to take its absolute value and raise it to a power which is not an integer, and even an integer wouldn't do. But here, what you can do, you can take the quantized differential, raise it to a power which is a real number, which is the outer-dimension of the Julia set, and with my australian collaborators, Sukochev, Zanin, and their collaborators, what we have done, we have been able, in the recent years, to put on a completely rigorous ground what in my book I had described a little bit at a heuristic level, namely the fact that this type of formula does give you the conformal measure on the Julia set or more generally boundary of the quasi-fuschian groups. So I mean this calculus has a lot of power. In fact, one should keep in mind constantly to compare this calculus with the ordinary calculus and the calculus of distributions and what I will explain today is the role of this calculus with respect to the Riemann zeta function.



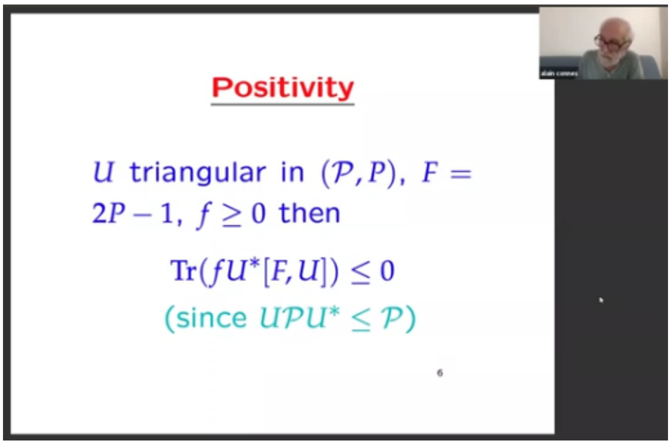
I mean, when you take the simplest example of this calculus, you look at functions on the line, so they are just ordinary functions of a real parameter, you view these functions as operators, in L^2 of the real line ($L^2(\mathbb{R})$), and when you take their commutator with the Hilbert transform, what you find out is that the quantized differential now is no longer just the usual differential, the usual derivative, the derivative is the diagonal values of course because it's f of s minus f of t over s minus t $\left(i.e.\ \frac{f(s)-f(t)}{s-t}\right)$ up to a normalization factor, but what is really important is that the regularity of the function is actually governing the order of the infinitesimal that you have on the right hand side because after all, the differential should be an infinitesimal. And if the function is Schwartz space, it's an exercize which you can prove in several ways, to show that the quantized differential of the function is an infinitesimal of infinite order. Infinite order means that, if you want, what's is quite nice with the infinitesimals is that they have an order. So for instance if you have eigenvalues of the order $\frac{1}{n}$ (1 over n), when n goes along, then the infinitesimal has order 1. If you have eigenvalues that decay very very very fast, then the infinitesimal has infinite order and that is the case here. Another very crucial property of this calculus, even in one dimension, is that it is conformally invariant. By conformally invariant, what I mean is that if I replace the line by the circle, then I will be able to get an isomorphism of calculus, from the calculus on the line with the calculus on the circle. Now you could argue with me "Okay, but how can you relate the line with the circle ?". Well, the line, if you compactified by a single point, it's $\pi^1(\mathbb{R})$. And how do you relate it to the circle ? Well you relate it as domains in the complex plane, which are conformally equivalent, so you see the line as the boundary of the upper-half plane and you see the circle as the boundary of the unit disc. Now the two things are related by a conformal transformation. You have a little bit to be careful in defining the transformation on the vectors, not on the functions, because on vectors you want that the inner product of $L^2(\mathbb{R})$ transforms into the inner product of $L^2(S^1)$. So you have to introduce, you know, something which is like the squareroot of the denominator, when you take the conformal transformation. The denominator is always squared by construction. So you do that and everything works fine. And then, one proofs that if you are in $S(\mathbb{R})$, the differential is of infinite order, is that if you are in $S(\mathbb{R})$, and you look at a function in $\pi^1(\mathbb{R})$, then it turns out that the function is smooth because at infinity, it behaves very well. So when you look at it as a function on S^1, it's a smooth function and when you look at the calculus on S^1, because it's related to the complex structure, this calculus which is given by this f, f is twice the projection minus 1 where there, the projection will be the projection on the Hardy-space, H^2 of the disc (i.e. $H^2(D)$). So in fact it's very easy then, very easy, it's a very easy exercize to prove that if you have a $\mathbb{C}^\infty$ function on S^1, its quantized differential is of infinite order. You need just to look at the Fourier coefficient which have to decay very fast. So all of that is very simple and however okay, there is one typical feature which we shall see immediately of the calculus which, if you would look at ordinary calculus, ordinary differentials, ordinary functions, and so on, would not be transparent but which becomes completely transparent with this calculus. And what I want to explain is the fact that we shall have handle on positivity. So we shall get an immediate handle on positivity, of the index, if you want, on the thing which I was explaining before, the trace of U^*VU so we shall get this handle because of the fact that when we look at unitary operators,



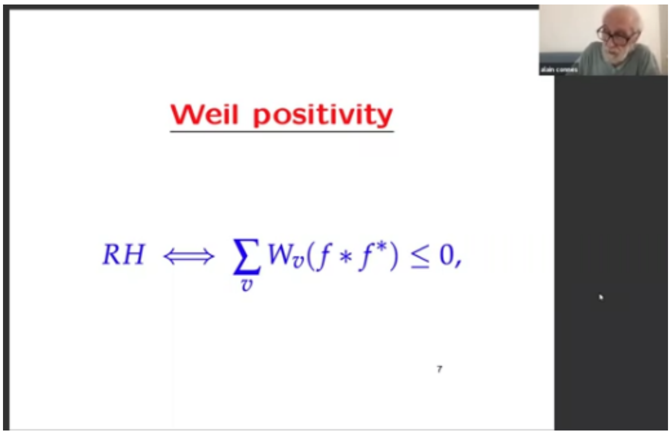
we can ask "when is it that in a two by two (2×2) matrix decomposition of an operator, which is typical of what happened in a quantized calculus, what happens if the matrix representing the unitary is triangular ?" Okay, so I will just state a simple exercize that you can do in your head, when I'm talking, which is to characterize those 2×2 matrices which form a unitary operator but which are triangular, there's a zero in the 2,1 place. It's a very simple exercize to show that this is equivalent to three things, the third one will be on the next page. The first one is that $u_{1,1}$ is an isometry. So $u_{1,1}^* u_{1,1}$ is equal to one. The second is that $u_{2,2}$ is a co-isometry. So $u_{2,2}u_{2,2}^*$ is equal to one.



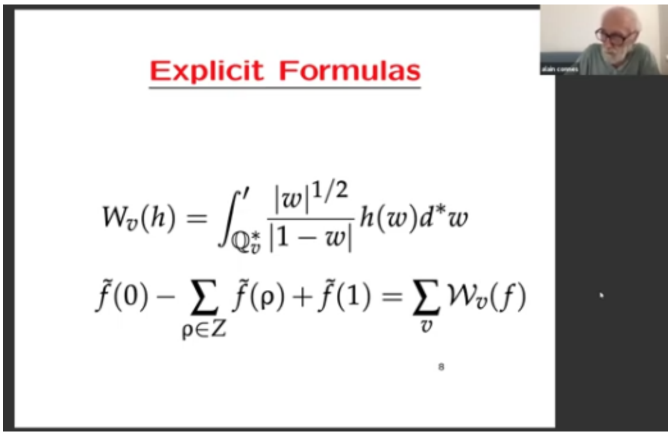
And the third one is that $u_{1,2}$, the one which is off the diagonal, is a partial isometry which goes from the kernel of $u_{2,2}$, remember, $u_{2,2}$ is a co-isometry so it might have a kernel, to the cokernel of $u_{1,1}$, again, $u_{1,1}$ is only an isometry so it might not be surjective so it must have a cokernel. So the conditions are like this, they are very simple conditions, it is very easy to check this exercize, but they will play a key role in what we are doing. And the reason why they will play a key role is that when you have such a triangular unitary, with respect to the F of the quantized calculus, you take the F, the F decomposes the Hilbert space in a sum of two Hilbert spaces, any other operator has a 2×2 matrix decomposition, and now you can ask the condition that it is triangular. Well if it is so,



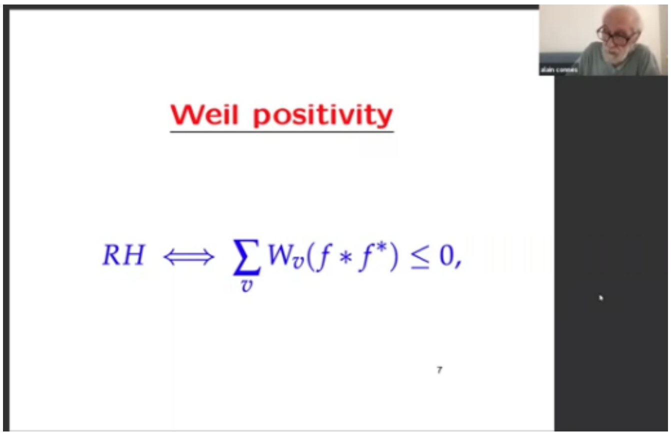
what you'll find, within two minutes, you'll find that you have a positivity for the index, and even a strengthen form of positivity. So the positivity for the index will just be the fact that the trace of $fU^*[F,U]$ is positive, I prefer to put it in the negative because as you will see later. So I take my F to be of the form $2P-1$, it has a complement which I call curly $\mathcal{P}$ ($\mathcal{P}$ is juste $1-P$) and now what I state is that there is a negativity fact which is again very simple to check which is that the trace of the product of f by what gives the index, namely the logarithmic derivative of U, the quantum logarithmic derivative, $U^*[F,U]$ will be negative. Then you prove this, well, it's not such a problem to prove that $U^*[F,U]$ is negative. First of all, you can see that $U^*[F,U]$ is self-adjoint, because F is self-adjoint, and you conjugate F and you substract, so it's self-adjoint, but then what you find out is that for this projection $\mathcal{P}$ what you find is that $U\mathcal{P}U^*$ is lesser than $\mathcal{P}$. So by playing around, you find that this operator in U^* F commutator U (i.e. $U^*[F,U]$) is negative. But the product of a positive operator by a negative operator has a negative trace, we know that. I mean the trace of the product of two positive operators is positive. I mean, one should remember that because it's not only true that the trace of a *positive* operator is positive, no, the trace of the *product* of two positive operators is positive. How do you prove it ? Well, I want to prove that trace of the AB is positive, A and B are positive. So what do I do ? I take the squareroot of A. So I write the trace of AB as a trace of squareroot of A times B times squareroot of A. And that's of course positive. So that's done. Okay ? So that's what we have, we have a fairly general kind of index sign, which is a general fact about triangular unitaries. And I prefer as I said to write it as a negative fact. Okay. Now, here comes the first crucial point.



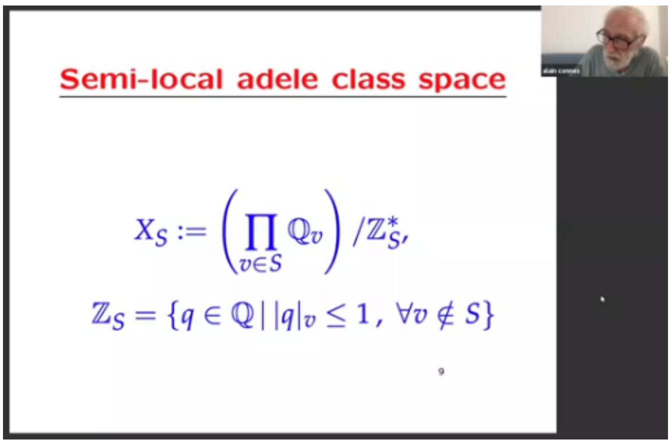
The first crucial point is that the Riemann Conjecture RH is also equivalent to a negativity statement. Oh, you can say positivity if you like to play around, but... and what is this statement ? This statement, it took us a lot of time to really *not* understand, because understand, you can understand a statement, it's a very different thing from having absorbed it. So I mean it's a fact that the Riemann Hypothesis, even though it's an hypothesis which involves infinitely many primes, and which is concerned with the distribution of primes, it turns out to be equivalent to this statement, and in this statement, if you take f to be of compact support, and that ??..., so f is a test function, I will explain well it is, and so on, but the *key fact* is that if f has compact support, the number of primes which are involved in the right hand side is finite, which is amazing. It's amazing because this Weil's equivalence is telling you that even though the Riemann hypothesis is something which involves infinitely many primes, in fact, in order to prove it, you only have to consider finitely many primes at a time. And it's an if and only if. That says that it is not that you are making a bounds, you know, that you are going to prove it for finitely many primes and then... fine, okay, no, it's equivalent.



And what are these W_v, well, they are given in what are called the explicit formulas which were certainly known to Riemann. They are called Riemann-Weil formulas. Now, in modern terms, of course, okay, you have to explain they are integral over the multiplicative group of the p-adic fields $\mathbb{Q}_p$ which for the archimedean place is just the real line and then you do an integral. This integral is tricky, and it's tricky for the following reason : that the test function doesn't need to vanish at the point 1. So the integral will a sort of diverge at $w = 1$ and you have to take in a very careful manner a principal value. So a principal value, what does it mean, it means that what you do is that you substract from h a certain choice function that actually has the same value as h at 1 and then you do a little bit of moving around. So what I wrote there is the integral prime ($\int'$), so I mean in my paper in 98 I evolved a lot on that but I mean okay, it's well defined, but you have to be extremely careful when you work with it, because if you would prove positivity for a wrong normalization, you would have done nothing. This equivalence of Weil is based on what are called the explicit formulas. And these explicit formulas tell you that if you take the sum of the Fourier transform (here it's a Mellin transform but the two are related) on the zeros of the Riemann zeta function, you put a minus sign in front, you add two boundary values which are the evaluation at the poles of the complete Riemann zeta function and then what do you get, you get this sum of the local contributions. And from that, you can infer what I was saying before, namely that RH is equivalent with this negativity statement.

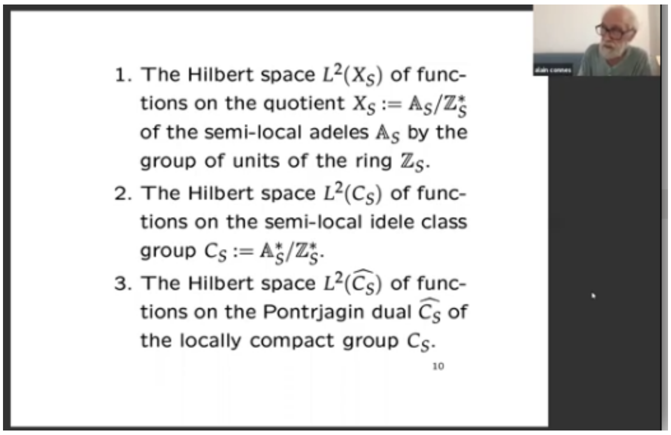


Okay ? But now what is the link ? As I said, there is this remarkably striking feature, by the Weil equivalence, that you only need to look at finitely many primes. And I said also, you know, it took years before this statement could sink in. And when it has sank in, I realized that what I had developed in my paper in 1998

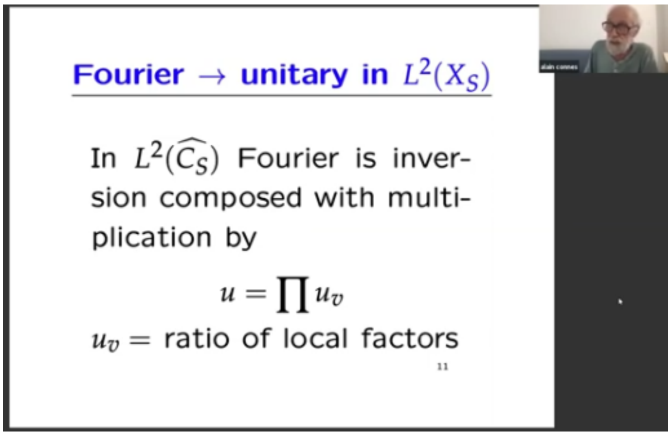


should play a role. So what did I devise in that paper in Selecta in 1998 ? Well I had devised a global space that is the adele class space and where, it was okay, one could find the spectral realization and so on and so forth. However, this global space had something which was extremely natural and delicate. And that was due to the fact that when you look at adeles, the restricted product of local field over all places, and when you look at ideles, same thing, but you only look at the invertible elements, then it turns out that when you look at both of them as a group, you look at adeles as an additive group and at ideles as a multiplicative group, then it turns out that the Haar measures for these two groups are singular with respect to each other and that makes thing extremely delicate and complicated. But this pathology if you want, this difficulty doesn't arise when you take a finite product. And this finite product is not the full adele, but it's only taking a finite number of places I call S (finite set of places) which contains the archimedean place, and I now take the product of the local fields over this set. And I will divide it by what should divide it if it were infinite and I would divide by the rational numbers, the multiplicative rational numbers, but I divide it by the invertible elements in a certain ring which is associated to set S and to make thing very concrete, let me take S to be given by the archimedean place and the prime 2. So what would be the ring $\mathbb{Z}_S$? Well, it would be the ring of all rational numbers whose denominator is a power of 2. You see, I mean, when you take rational numbers whose denominator is a power of 2, you can add them, you still have a denominator which is a power of 2 and you can multiply them. So you get a ring. Now in that ring, not every element is invertible, the elements which are invertible are only the powers of 2, plus or minus. So this fact that I mention generalizes, and now we are going to reach the heart of non-commutative geometry because when you divide the product of these local fields $\mathbb{Q}_v$ by the multiplicative group of the invertible elements of this ring $\mathbb{Z}_S$ (i.e. $\mathbb{Z}_S^*$), then as soon as you have more than three places let's say, this is a non-commutative space. You can understand why it's a non-commutative space, because, for instance, if I would divide $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$ by powers of 2 and 3, you will see that the way they act on the real line is dividing the real line by powers of 2 and 3, and of course, this is a non-commutative space. But the amazing fact is that, at the level of measure theory, it's not a non-commutative space. In other words, this space X_S, as you know, in non-commutative geometry, we always look at a space with several pairs of glasses. And the pair of glasses which is like the coarsest is measure theory. In other words, what we do is that we take our space and we look at it from the point of view of measure theory that means that we can neglect sets of measure 0. When we do that, it turns out that the semi-local adele class space is a standard space. Why ? Because the measure is carried by the idele. Namely, the measure is carried by those elements in the product of the local fields which correspond to invertible elements, because the rest is of[1] measure 0. You see if one of the elements is zero, so it is not invertible in the local field, then this will be of measure 0 for the additive Haar measure. So in fact what we find, which is totally different from the global aspect, is that in the semi-local aspect, for measure theory, it's perfect. So the fact that it's perfect from the point of view of measure theory allows us to have a perfect Hilbert space.

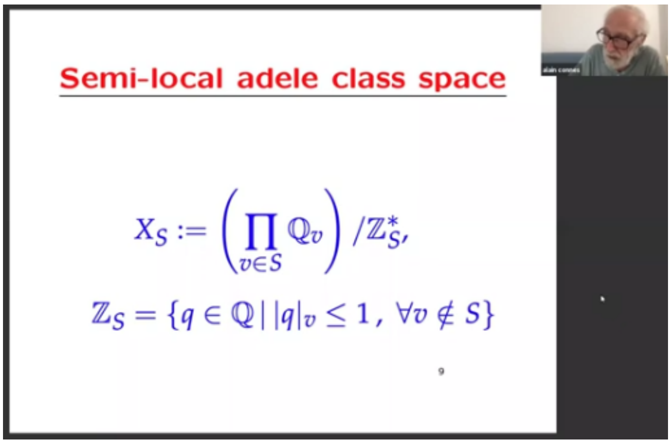
[1] if ?



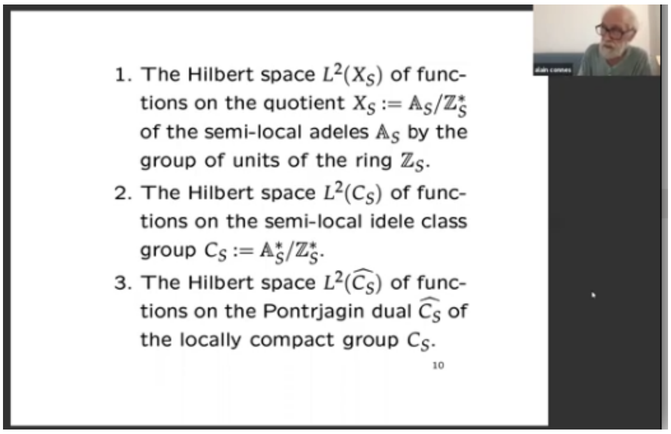
So the Hilbert space L^2 of this quotient space X_S (i.e. $L^2(X_S)$), because the measures are the same, namely they are equivalent, it makes perfect sense, and it can be defined, okay, you can define the inner product, as you would do in the global case but okay, everything is fine, and it turns out to be canonically isomorphic, but the isomorphism is non-trivial, with the Hilbert space of functions on this semi-local idele class group, which is a multiplicative group, and which is the quotient of the invertible elements in adeles, which are the ideles, divided by $\mathbb{Z}_S^*$. And finally, it's also isomorphic to what you obtain when you take the group $\widehat{C_S}$ and you take its Pontrjagin dual. Of course, you know, it's always true that when you take L^2 of a locally compact group, it's isomorphic to L^2 of the Pontrjagin dual. So you can do that here. You should keep in mind that you have three Hilbert spaces. They are isomorphic to each other, but the isomorphisms are very significant and non trivial. And each time you think about one of them, you have to think in terms of its meaning. And then, you will be surprised by the translation. So the thing that plays the absolutely fundamental role is taking place in fact in the first one. Because you could say "okay why do they consider the three, euh, how to say, incarnations, these three avatars of the same Hilbert space ?", well... When you look at the first one, which is the quotient of adeles by $\mathbb{Z}_S^*$, it turns out that the Fourier transform



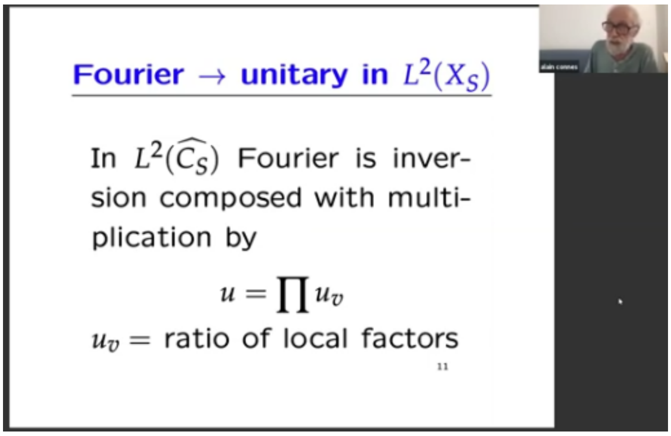
defines the unitary in this space. What is the Fourier transform,



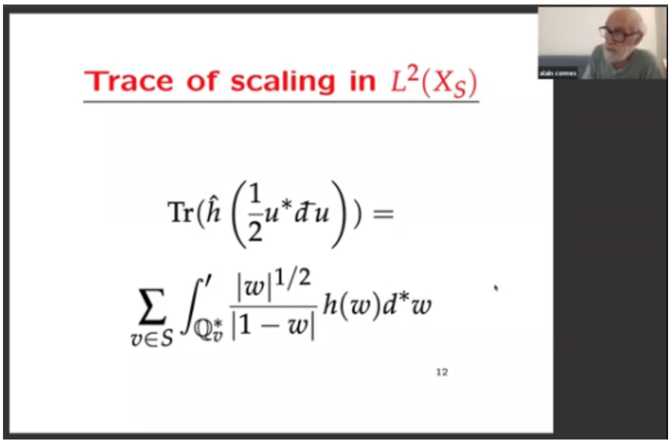
Well the Fourier transform, what you do is that you take the Fourier transform on each local field $\mathbb{Q}_v$ and then, because the inner product is sort of divided by $\mathbb{Z}_S^*$, you have to prove, this requires a little bit of doing, that the Fourier transform passes to the quotient. Okay, you can prove that. But the Fourier transform is really like you want the tensor product of the Fourier transform at each place, and it's remarkable that it passes to the quotient. So it passes to the quotient, and when you are in the quotient,



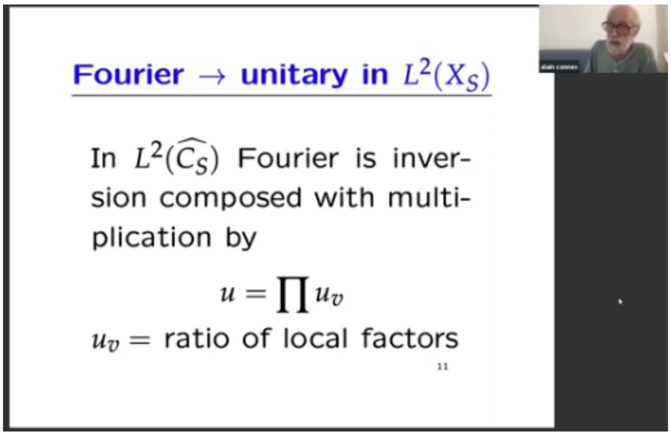
then what you can do is that now you can play the game of passing from one Hilbert space to the other and understand how it becomes. Now I suppose that I don't have to recall the work of Tate on the local functional equation. So what Tate did, he showed that you know, if you consider a *local* field, so any of the $\mathbb{Q}_v$, it could be $\mathbb{R}$, and if you consider Fourier transform in $\mathbb{R}$, let's think about Fourier transform and I hope I can teach you something that you don't know but it's very well known, that if you take Fourier transform on the real line, then when you Fourier transform a function that is scaled, it's scaled for instance by φ, okay, then you will get the same function, as the Fourier transform of the original one, except that you rescale by $\frac{1}{\varphi}$. Okay, there is a scaling factor but let me ignore that. So if you think about that, you will reflect and you will say, okay, what does it mean ? It means that if I compose my Fourier transform with the inversion of the variable, so I replace the variable x by $\frac{1}{x}$, then what happens is that after all, if I rescale my variable starting from, then the composition of the inversion with the Fourier transform, I rescale by the same amount. What does this say, for fundamental algebraists ? It says that the scaling group commutes with the inversion composed with Fourier transform. But the scaling group, when we consider von Neumann algebra it generates, it generates a maximal abelian von Neumann algebra in L^2. So if something commutes with the scaling, then it has to be a scaling. Well, not to be the scaling by a single element but a scaling by a scaling convolution. So when one applies this, one finds that locally, for each place v, the Fourier transform composed with the inversion is in fact given by a scaling by a certain function, and this function was investigated by Tate, and this function is the ratio of the local factors. So now, the miracle if you want of the semi-local adele class space is that the same thing holds when you take now the finite product of these local fields, but you divide by this group, then what happens, again,



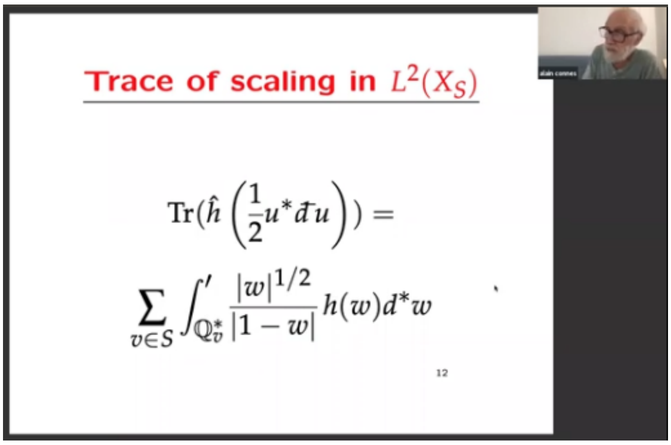
is that the Fourier transform composed with inversion is in fact given, when you pass to the dual, given by scaling but in the dual group, scaling becomes multiplication. So it's a multiplication by a product of terms. Okay. So that what we have. And without the semi-local adele class space, what you would get is a single u_v but you would never get their product. So how do you use this product ?



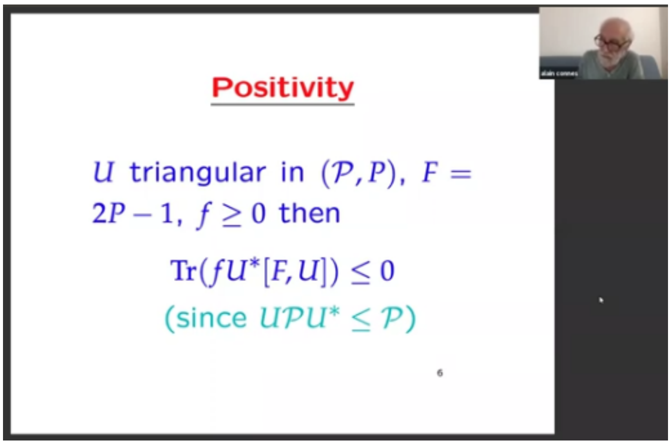
You use this product because now you can combine it with the quantized calculus, and you get a formula for the Weil explicit formula, which is now expressed in the terms of the quantized calculus. So what is this formula ? On the second line, you have the terms which intervene with these very precise principal values in the Weil formula and what do you have on the other side. Well, on the other side, you have the same type of expression that I was claiming we had a control about the positivity or about the negativity before, namely it's a trace of the Fourier transform of the function h times the quantized differential, ok there's a factor of $\frac{1}{2}$, ok, fine, and what is the $u^* đu$, it's exactly the expression that I was using for the index. And what is the u, well the u is the 1 that we had before,



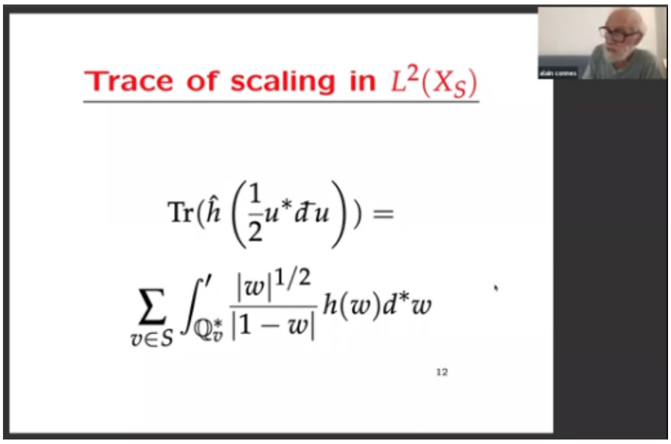
it's the product of the u_v.



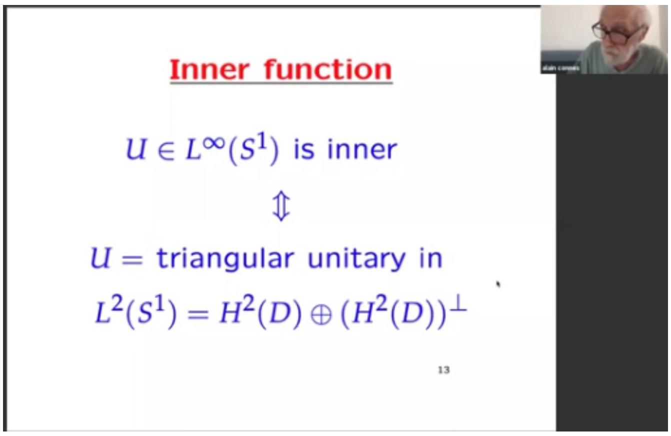
Now you know, this is of course the semi-local trace formula which I had proved in my paper in 1998. But I was always puzzled by the fact that this semi-local trace formula was giving you a sum over places. Why was it puzzling ? It was puzzling because after all, you take the product of the $\mathbb{Q}_v$, the product of the local fields, so when you take a product of local fields, you would expect that the trace will give you a product, not a sum. So why does it deliver a sum ? It delivers a sum because this $u^*đu$ is like a logarithmic derivative, you see, u is a unitary and when you take $u^*đu$, you are really taking like a logarithmic derivative. Now a logarithmic derivative of a product, of course you have to be careful because things don't commute, but it's fine, I mean, you can play around, what you find is that this trace of this product delivers the right formula for Weil but it has exactly the same aspect as what we had encountered when we were talking about these unitaries, namely it has exactly the same aspect as this trace on



f times $U^*V/U]$[2] which was negative. So we are very interested in this in the sense that

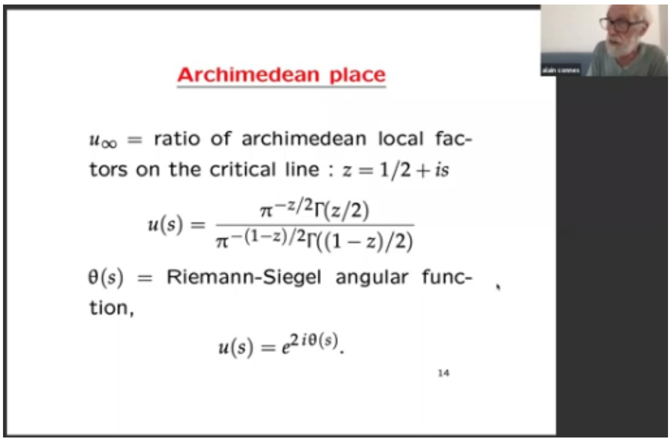


this clearly shows that there is a link between the Weil negativity that we want to prove and type of index formulas that we are very familiar with, in non-commutative geometry, and that, you know, enter directly when you look at the quantized calculus. Then you know, one would say, "okay, well", and this is in fact one should have tried since the very beginning. After all, it could have been that the unitary given by Tate, because after all we are looking at the product of unitaries I mean, would be triangular because we have to look at them at one at a time, because it's true that product of triangular matrices is triangular. So ok, you can look at what happens



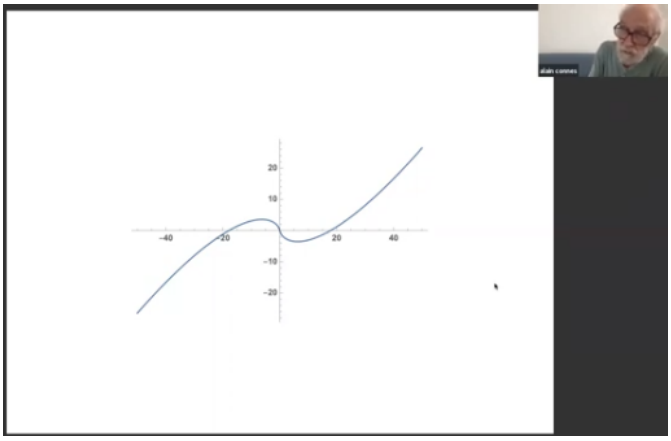
[2] following the line $U^*[f,U]$.

what does it mean, that the unitary matrix is triangular for the quantized calculus ? And in fact, what you find out is that there is an existing theory, which is due to Beurling and Hardy and several other people which tells you that if you take a function that is in $L^\infty(S^1)$ which is a unitary, you find, this is what is called an inner function if and only if the corresponding unitary is triangular in the decomposition of $L^2(S^1)$ according to the natural quantized calculus, namely which has to do with the Hardy-space for the disk ($H^2(D)$) and its orthogonal complement ($H^2(D)^\perp$). Okay, so this is a simple exercize and I recommend for you to look in the Rudin's book[3] and understand the link with positivity and all that. It's all transparent when you think in these terms. So okay, so of course, then you can say Okay, I will try to see,

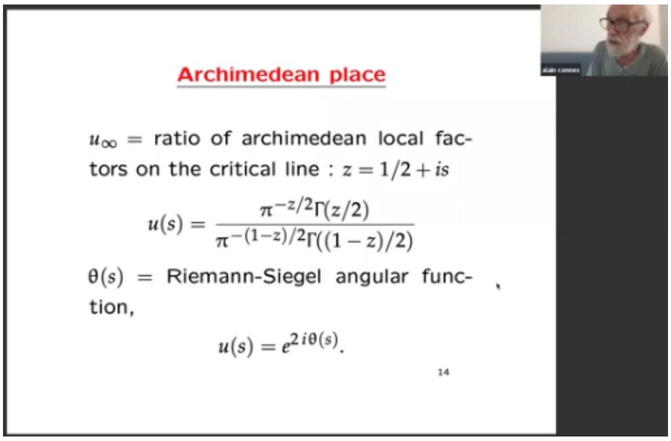


if, at the archimedean place which is the first thing that you need, it is true that this is an inner function. So what is the function ? The function is the ratio of local factors according to Tate, so this ratio of local factors is the following. The local factors at ∞ is π power minus z over 2 times gamma of z over 2 and you have to divide by what would be its complex conjugate on the critical line, which amounts to replace z by $1-z$ (i.e. $u(s) = \frac{\pi^{-z/2}\Gamma(z/2)}{\pi^{-(1-z)/2}\Gamma((1-z)/2)}$). So this is a function. This function at first you view it only for the critical line, but you have to view the critical line as a boundary of the half-plane, and it is the boundary of the half-plane which is located on its left. Now you can compute this function. This function is not difficult to compute and it is given by the exponential of 2 $i\theta(s)$ (i.e. $e^{2i\theta(s)}$) where $\theta(s)$ is what is called the Riemann-Siegel angular function. This Riemann-Siegel angular function is very important because when you look at the Riemann zeta function on the critical line, it's not a real function. To make it a real function without changing its modulus, you have to multiply it by the exponential of $i \times \theta(s)$ where $\theta(s)$ is the Riemann-Siegel angular function and here you get the double because of what I said before. Well there might be a sign, okay. Now what happens is that when you look at this Riemann-Siegel function, at this graph,

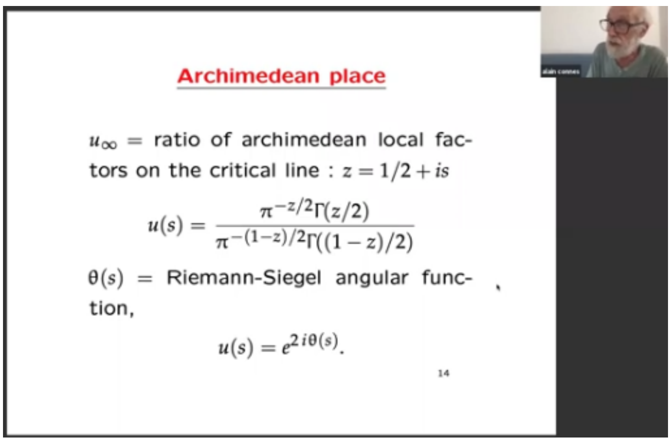
[3] *Complex and real analysis, Chap. 17*



you find out that it couldn't be an inner function, the previous couldn't be an inner function because an inner function would have a derivative which is always of the same sign, always positive. So when you look at this function, you find out that it's not the case, Xian-Jin, you know, a very good Chinese mathematician had proposed some idea which was related to this but I found that it couldn't work because this function is not monotone, okay, Xian-Jin Li. So what happened is that then you say "okay, this is too bad, we can not use these triangular unitary and so on."

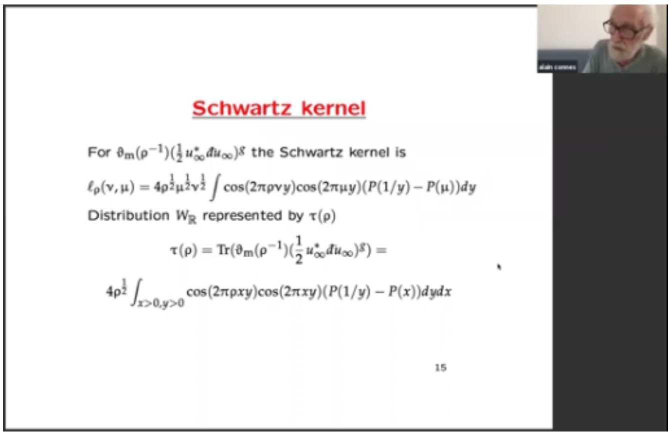


And what we shall see, and this is the topic of my talk today, is that fortunately, there is a beautiful theory of

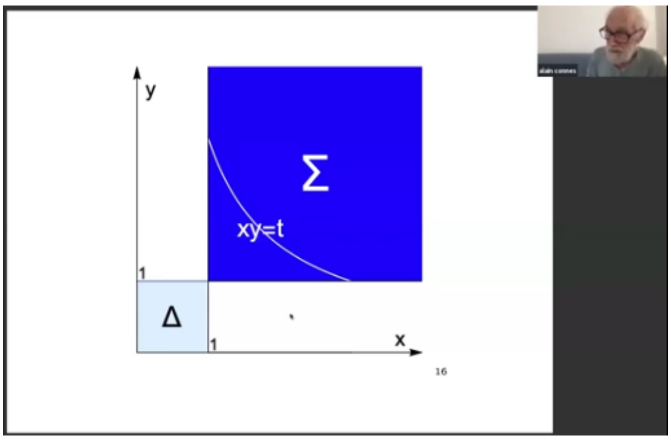


quasi-inner functions, and with this theory of quasi-inner functions, it will repair the fact that this function is not inner. And I will explain that in the two papers with Katia especially in the paper in june, we were able to achieve the goal of proving this Weil positivity for the archimedean place alone, which of course is very far from the global goal, or the goal with finitely many which would suffice, but however, now, we have really the fundamental reason why the Weil positivity should hold.

To give the way how we understood it in the first paper,



what we did was, because Katia always insisted that we should always have both the analytic understanding and the geometric understanding, and if you want to get the geometric meaning of the calculation I was doing in 1998, you have to translate to geometry by looking not at the operator but by looking at the Schwartz kernel of this operator. So when you look at what happens in the semi-local trace formula, the operator that enters is the operator, you don't have to understand too much in details what it is, it is this operator here[4]. Here is the quantized differential[5], and here is the scaling operator [6]. And you look at the Schwartz kernel of this operator, and you can compute it. It's a very tricky operator. And when you compute it, what you find, first of all, you find that you can re-prove the semi-local trace formula using this operator, here we are in the archimedean case, but you find out something that is picturally obvious

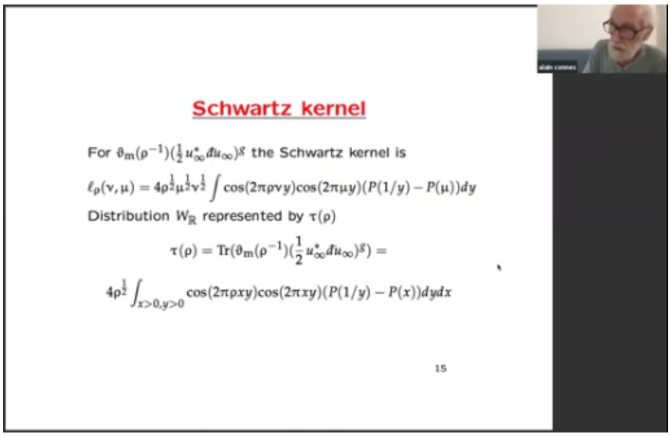


[4]surrounding $\frac{1}{2}u_\infty^* đu_\infty$.

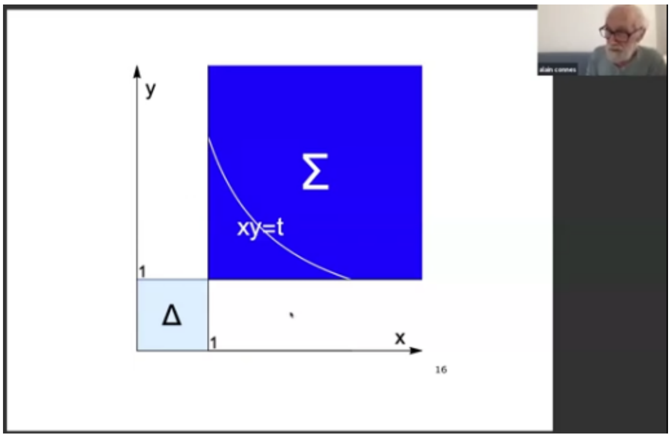
[5]surrounding $(\frac{1}{2}u_\infty^* đu_\infty)^g$.

[6]surrounding $\theta_m(\rho^{-1})$.

because what happens is that the quantized differential, when you do things geometrically, it is given by the commutator with the projection

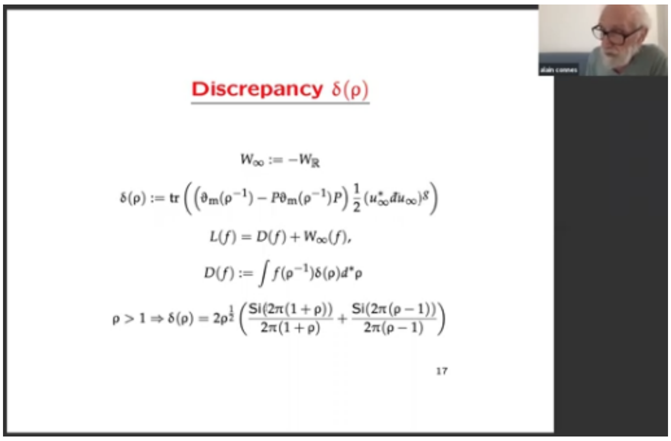


but this projection[7] is no longer the Hilbert transform or anything complicated like that, it's just the projection on the elements which are bigger than 1 *(He laughs)*. So when you consider the commutator and you make a change of variables, you find that

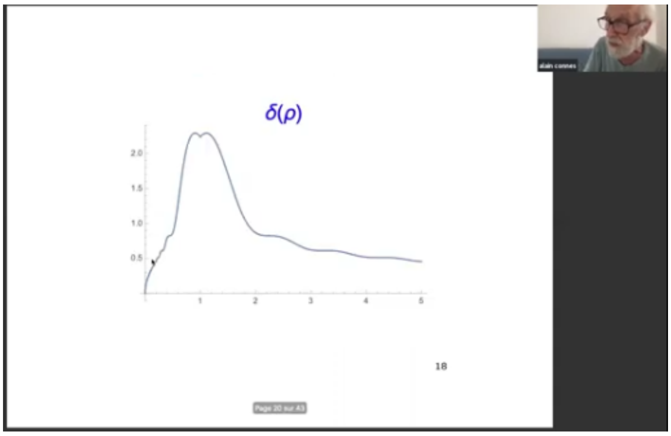


when you write the commutator, it will ignore the regions which are not in the small square and in the big square. And when you compute what happens with the big square, you find that the Weil positivity or if you want the things that you want is obvious, there, because you can write it in terms of projections and so on. And so the new sense is the small square. So when you see that geometrically, you are forced to define what we call the discrepancy.

[7]surrounding $P(1/y)$

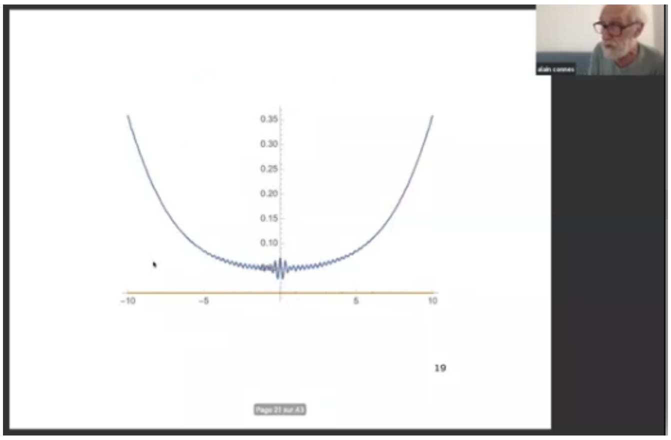


The discrepancy is what is due to the small square Δ and you can write a formula for this discrepancy, okay, you don't have to get lost in the formula, but the discrepancy is something which is a function actually, it's not even a distribution, it's called $\delta(\rho)$ and when you define the functional which is given by this $\delta(\rho)$, we were able to compute it, this function, and for ρ bigger than 1, it is given by an explicit formula. So it's a true function and the symbol that we see here[8] is called the Sine integral. So it is the integral from 0 to x of $\sin x$ over x dx (i.e. $\frac{\sin x}{x}$ dx). It makes sense because it behaves well at 0 and it's a certain function, you can have it on the computer without any problem. And when you plot this function because this equality with the Si and so on is only true for ρ bigger than 1 but the function is symmetric. If you change ρ to ρ inverse (i.e. $\frac{1}{\rho}$), it doesn't change, and its graph is of this form.

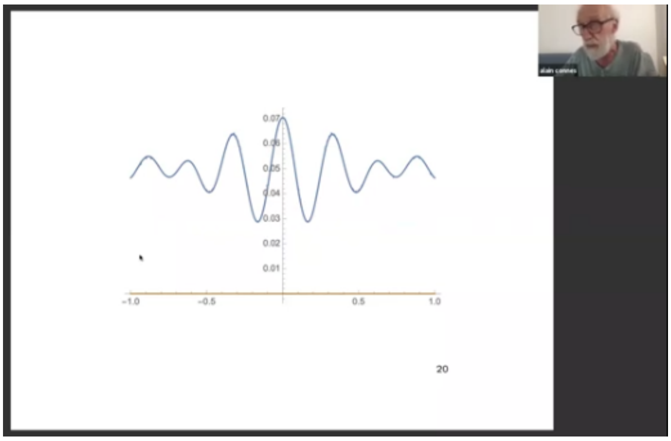


You get a function which has this graph. And it turns out that this peculiar picture of this graph which has this singularity here[9] will plays a role later. But once you have this function then you can say "oh my God !". Now what happens is that theoretically, we can prove that if we add the Weil distribution at the archimedean place, this new distribution, this new $D(f)$, then it should be positive. Alright. What does it mean to be positive ? To be positive means than when you pass in Fourier, the Fourier transform is a positive transform, that all it means. So then what we did... We took the computer ; with the computer, we computed the Fourier transform of this function $\delta(\rho)$, okay. And we tried to see if what we got had into it the derivative of the Riemann-Siegel angular function will be positive. And this is the graph that we got.

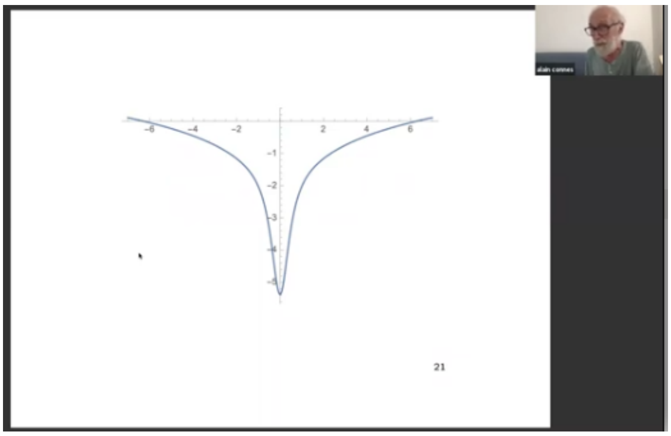
[8] last line, page 17.
[9] showing singularity for $x = 1$.



When you see this graph I called it like the doors of Hell, something like that you know because look at the value which is here[10], it's a very small value, and it has to be bigger than 0. And when you look, when you make a zoom, to see what happens near 0, this is what you see.

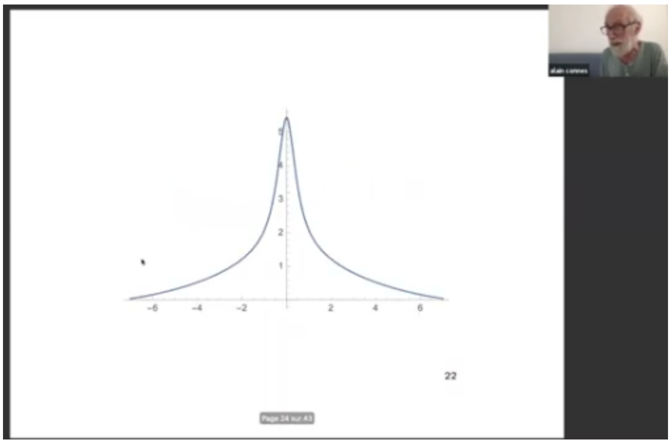


So you see that the sum of these two things, the sum of the derivative of the Riemann-Siegel function (twice the derivative), plus this function is actually positive, but there is better to feel what is going on, you see the negative aspect of the Riemann-Siegel function gives you this graph

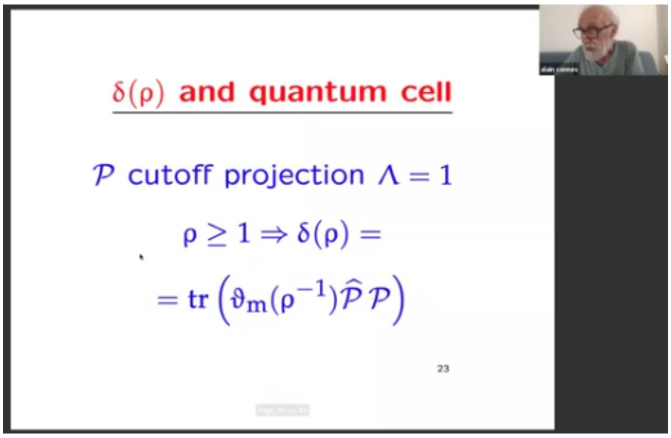


[10]showing the value on the y-axis.

it's a graph which is exactly like this. And now when you look at the Fourier transform of this discrepancy of this function $\delta(\rho)$, this is the graph that you see :

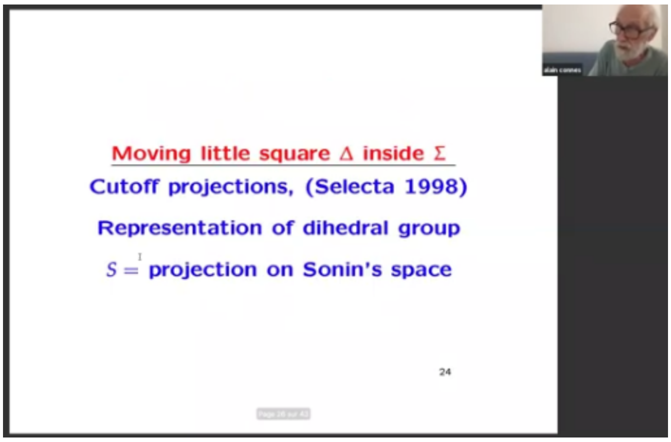


so, *(laughing)* this means that this graph is almost ??? that it compensates exactly for the negativity of the other one. Then of course, you have to prove theoretically that the sum is positive and so on, this is done, no problem, and then you have to control this function $\delta(\rho)$.



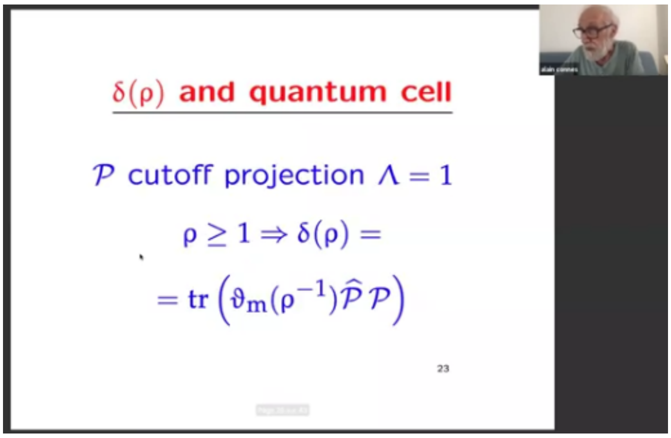
Now what we proved in the paper of june with Katia is that when we looked at this function $\delta(\rho)$, it's in fact related to the quantum cell, namely to something which I had defined in my paper in 98 which is the cutoff projection for the value of the cutoff which is 1. And so for ρ bigger than 1, this $\delta(\rho)$, this discrepancy, is in fact just the trace of scaling multiplied by $\widehat{\mathcal{P}}\mathcal{P}$ where $\mathcal{P}$ is the projection on this cutoff projection, and $\widehat{\mathcal{P}}$ is the Fourier transform. Okay.

So then, if you want, there is quite a work which is done, which we did in that paper, which is the following, which is that when you have a pair of projections, in Hilbert space, well, you know, you feel very well. Because a pair of projections is the same thing as a representation of the dihedral group,



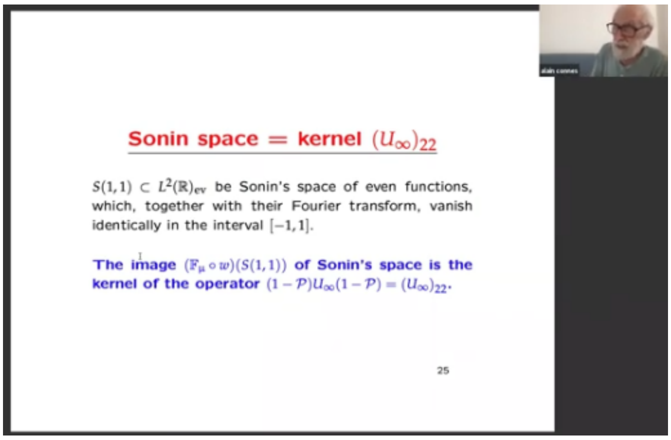
and representations of dihedral groups, they are given just... the irreducible representations are given by an angle.

And moreover, for this cutoff projections in the Selecta paper, I had used the theory which was known, which is the theory due to Slepian and several other people, which allows you to compute this angle, and to compute the eigenvalues, so it allows you to understand the situation. But in the situation there is something amazing which happens, which is that when we look the projections $\mathcal{P}$ and $\widehat{\mathcal{P}}$,

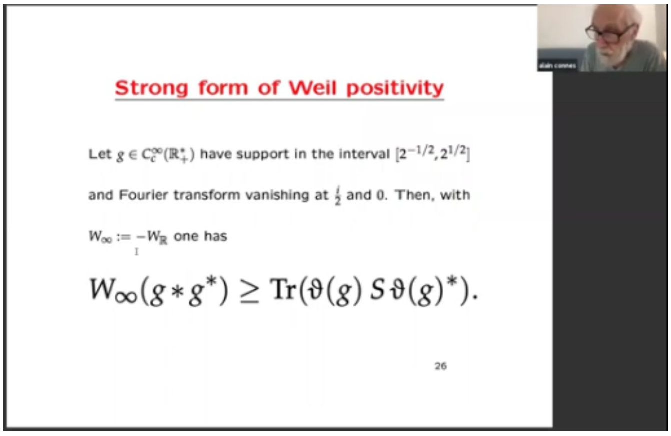


in $L^2(\mathbb{R})$ so let's take even functions, so $\mathcal{P}$ is a projection on functions which have support between -1 and 1, and $\widehat{\mathcal{P}}$ is the Fourier transform, now it turns out that it's not true that $\mathcal{P}$ and $\widehat{\mathcal{P}}$ generate the whole Hilbert space. There is a subspace which is orthogonal to both $\mathcal{P}$ and $\widehat{\mathcal{P}}$. This subspace is well known, it's called the Sonin's space and this Sonin's space is formed of ℓ^2[11] functions that vanish identically on [-1, 1] and whose Fourier transform also vanishes identically on [-1, 1]. And it's infinite dimensional. Okay. So you look, you look, and what do you find ?

[11] ? ou bien L^2 ?



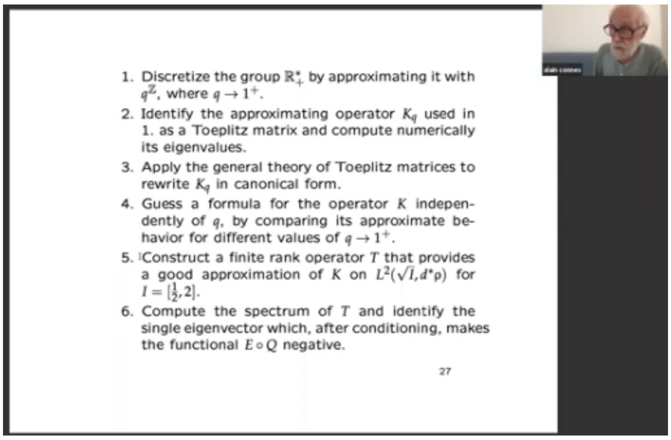
You find that in fact, this Sonin's space was perfectly there already in the decomposition of this local factor U_∞ in the splitting of the quantized calculus, namely it turns out that this Sonin's space is just the kernel of the $(U_\infty)_{22}$. So I remind you that if we had had a triangular operator, the kernel of $(U_\infty)_{22}$ would have been a ??[12], because $(U_\infty)_{22}$ would have be a co-isometry and its kernel here, it turns out that it's a simple exercize that this is exactly the Sonin's space.



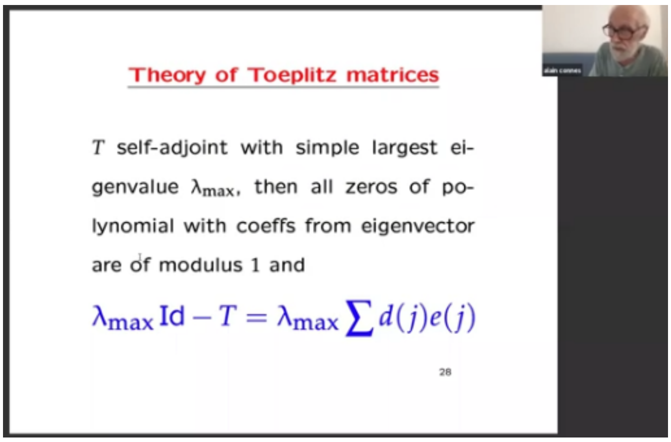
And what we proved with Katia in the june paper is that in fact there is a much stronger form of the Weil positivity that tells you that when you take the Weil functional, which you want to prove to be positive, and when you evaluate it on the positive elements $g * g^*$, then it turns out to be larger than, I mean under some boundary conditions, than the trace of $\theta(g) \times$ the Sonin's space $\times\, \theta(g)^*$. And of course the right hand side is obviously positive. But the moral of the story is that the root of the Weil positivity is given by the Sonin's space.

I don't want to spend time because I have a little time left but you see what happens is that the proof of this inequality is very very involved in the sense that the first thing that you do is

[12] key ????

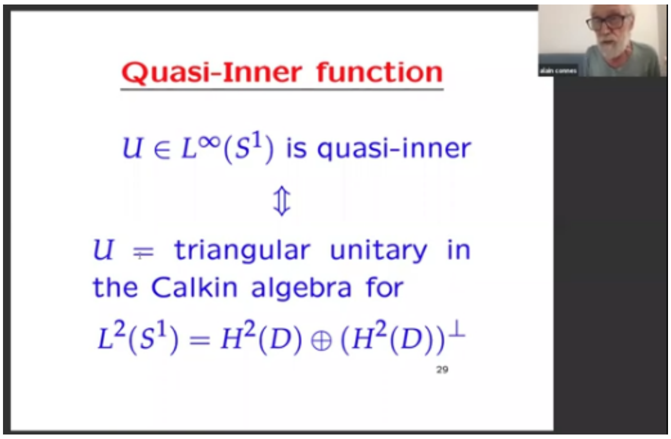


that you use the pair of projections to move the little square Δ into the big square. And then you find that what remains is the Sonin's space. And then, when you work with what remains, you have to prove the inequality, and to prove the inequality, what you do is that you use a technique which has a lot of conceptual meaning, that is to replace $\mathbb{R}_+^*$ first by discretizing it, by replacing by q^z, and you let q tend to 1^+. You know this is the story of $\mathbb{F}_q$ tending to $\mathbb{F}_1$ and so on. But then, what you find out is that you can approximate the operator that you don't know by an approximating operator which depends on q but that happens to be a Toeplitz matrix. Now, by a great fortune, at the time when we were doing that with Katia, I was collaborating with Walter, you know, on generalizing the non-commutative geometry to the much broader setup of operator space and so on, and we were focusing in particular on Toeplitz matrices. So I learned the theory of Toeplitz matrices just at that time, and it turned out that this theory of Toeplitz matrices was a sort of taylor-made to apply in our case with Katia which is that we were able thanks to this general theory of Toeplitz matrices to guess from the case of q tending to 1 what was the closed form of an operator that would actually be of finite rank and which was a perfect approximation to the operator K. What I should say also as far as the virtues of quantized calculus and so on and so forth is concerned, is that the role of Δ is like an infinitesimal difference between Weil and something which we know to be positive. So the idea that compact operators and infinitesimals play a crucial role here because you say "well okay you know if this small discrepancy and so on would be 0, we would be done" ; we are not done but we are dealing with something compact. So okay, I don't want to involve on that but then what we found out if you want

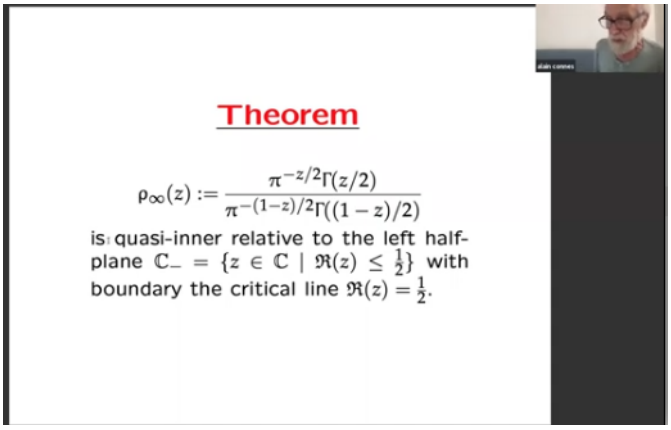


Yes, I should say one word on those Toeplitz matrices that should tell you a lot, which is that

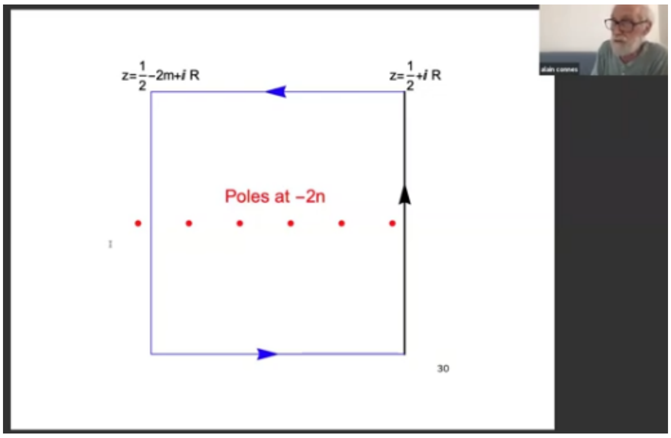
(laughing a little), there is a sort of baby version of RH for Toeplitz matrices which is the following. If you take a self-adjoint Toeplitz matrix, and if you take its largest eigenvalue, well, assume it is isolated, then, it turns out that there is an associated polynomial to the vector which represents this largest eigenvalue. And *(a bit laughing)*, it's a general theorem a bit amazing that all the zeroes of this polynomial are of modulus 1. So this is a fact which we used, in our work with Katia, to approximate, to find a limit when q tends to 1, and so on and so forth.
But if you want, the main



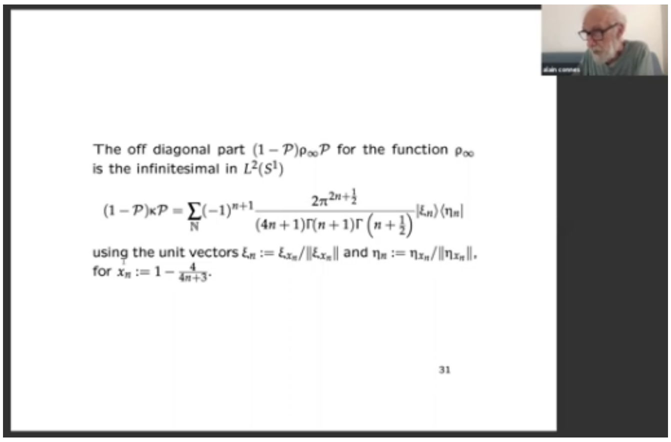
thing that we discovered can be encapsulated in a general definition. What we found is that in fact even though this unitary U_∞ is not an inner function, it is a quasi-inner function in the following sense : in the sense that when you look at the corresponding matrix, the unitary matrix, it's not triangular, but when you look in the Calkin algebra, it becomes triangular. Now *(laughing)* I mean, if you know operators, this is saying that the corresponding Haenkel operator is compact. That's equivalent to saying that. Now *(laughing)* of course, that function couldn't be inner, and it couldn't be inner because it's given by this formula



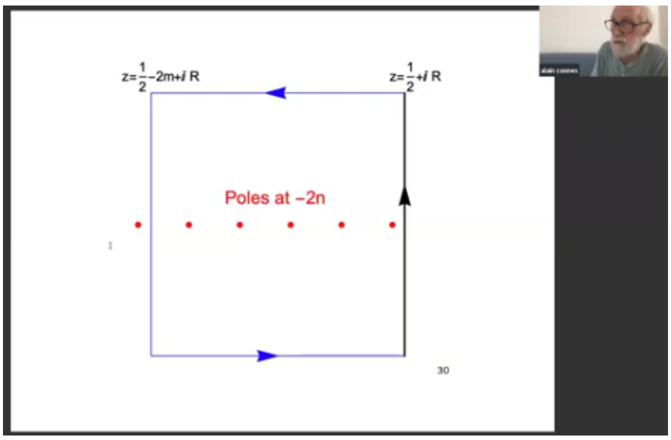
and when you look at this formula, this function has many poles in the place where it should be holomorphic.



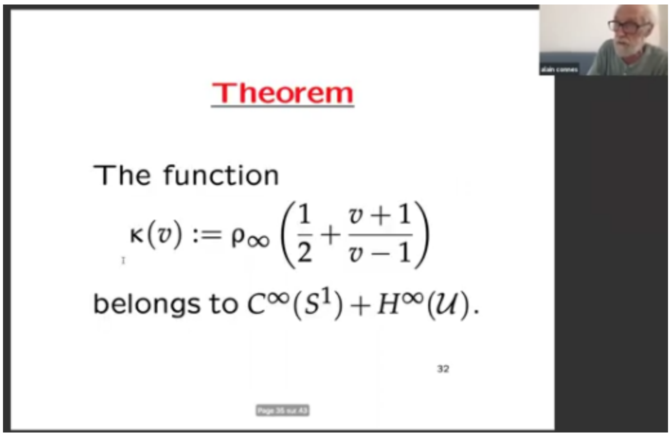
So it cannot be an inner function. It's not holomorphic. But if you work on it, if you work sufficiently on it, then you find out that you can use the Cauchy formula if you want to compute the negative Fourier coefficients that normally should be 0 if it were inner. So you do that. You use the Cauchy formula to compute these Fourier coefficients, and with the help of this Cauchy formula, what you find is that there is a beautiful expession, a closed expression, for the part that should be 0 if the function was inner.



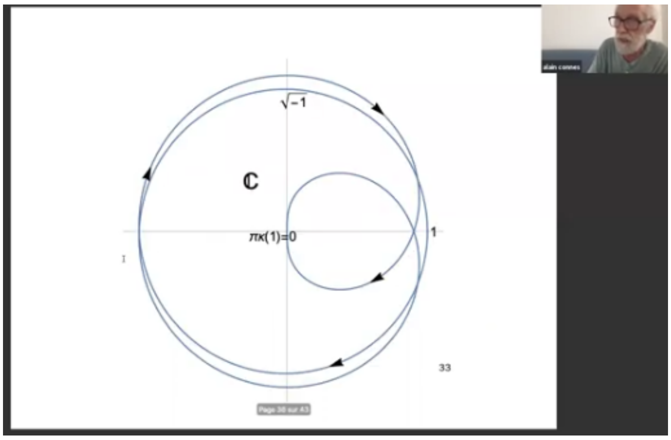
And look at this formula for this operator, I mean. So the operator is $(1-\mathcal{P})\kappa\mathcal{P}$ (κ is because I changed the variable). But look at the sum I mean. On the right hand side, you have a rank 1 operator, which is the Dirac bra and ket if you want, ξ_n and η_n which are both unit vectors but look at the coefficient.



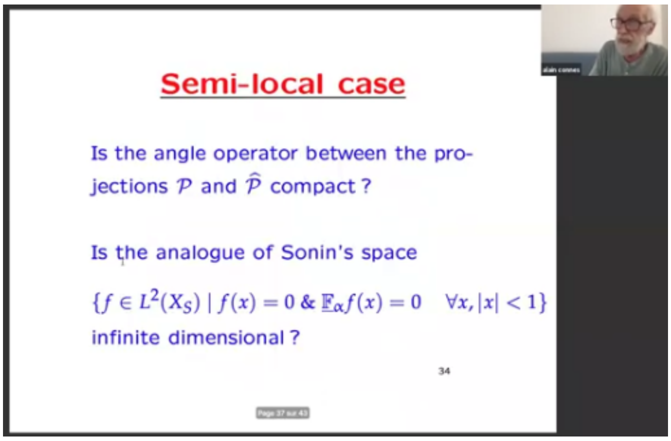
The coefficient is tending to 0 at a fantastic speed, because it's like 1 over Γ times Γ (i.e. $\dfrac{1}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)}$). Okay, it's a product of two gamma functions. So this tells you not only that this is an infinitesimal, I mean, it could have been an infinitesimal decaying like an exponential, no, it's an incredibly faster decaying infinitesimal, it's microscopic, almost nothing, okay. So now the question arises and I'm reaching the end, the question that arises is "What happens... ?" *(putting another slide)*



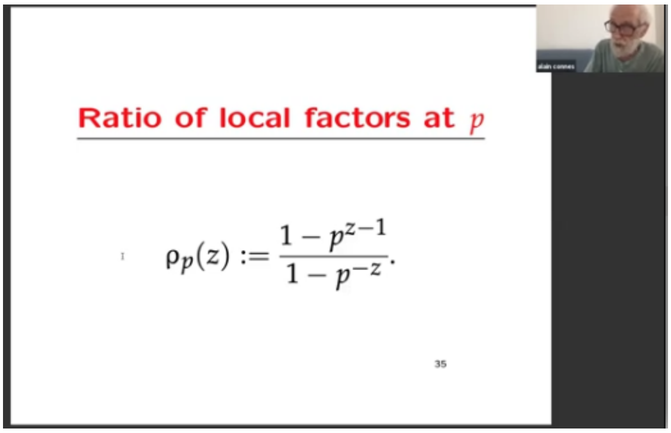
By the way, I just want to mention that there are general theories of Haenkel operators and so on, and we are able to decompose this function as a sum of a smooth function and a holomorphic function. That's ok, it's known by everyone, and ok, it has a certain form.



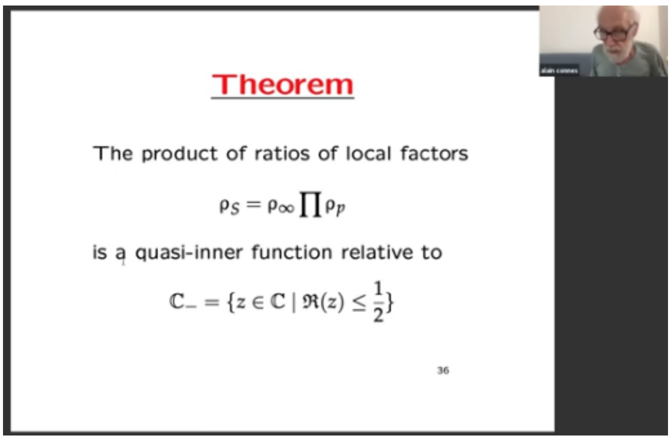
But the obvious question is "okay, all you have done is only for the archimedean place. So, what happens in the semi-local case ?".



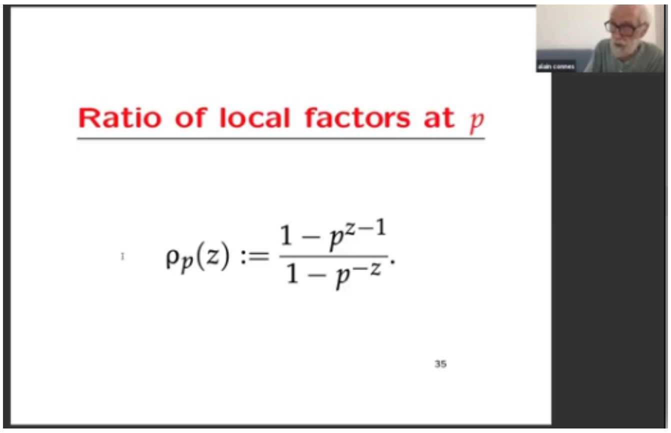
And there are two obvious questions. The first question, of course, the projections $\mathcal{P}$ and $\widehat{\mathcal{P}}$, they continue to exist. The first question is "is it still true that the angle operator between the projections $\mathcal{P}$ and $\widehat{\mathcal{P}}$ is compact ?" because if it is true then you will be able to imitate the method which was used in the archimedean case. And second question, well, there is an obvious analogue of Sonin's space that is that you can take functions in $L^2(X_S)$ which vanish in the unit interval, well the unit interval is a little bit more delicate to define, you have to use the modulus, and whose Fourier transforms also vanish there and this is true that this space is infinite-dimensional. There are only two questions, there, in front of us. And what we proved with Katia...



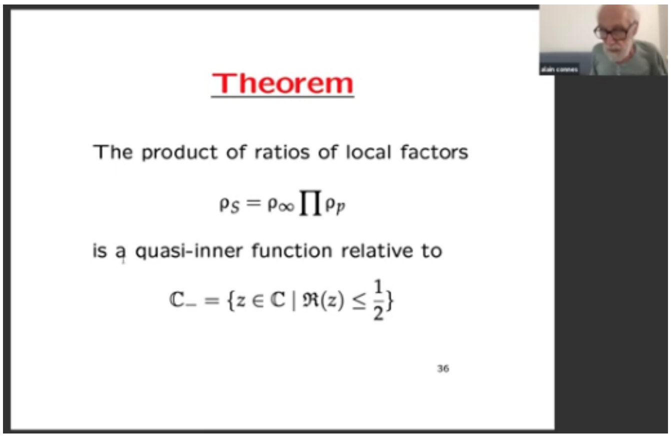
By the way, the local factors at primes are much simpler than the local factors at the archimedean place. They are given by this formula,



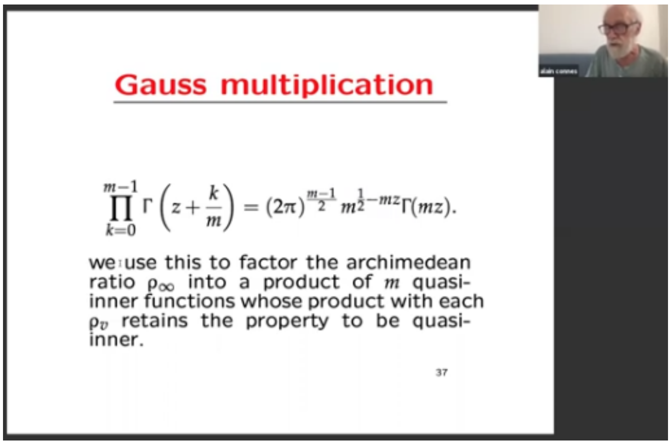
and the theorem that we proved with Katia, we proved that if you take a local factor



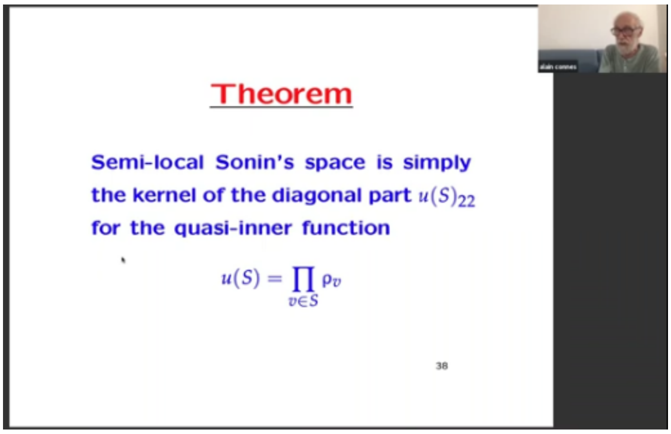
at a prime, it's not true that this function is quasi-inner. So that's a point, when you see that, you say "Oh my God, they are going to spoil everything", no, they don't, because when you take their product,



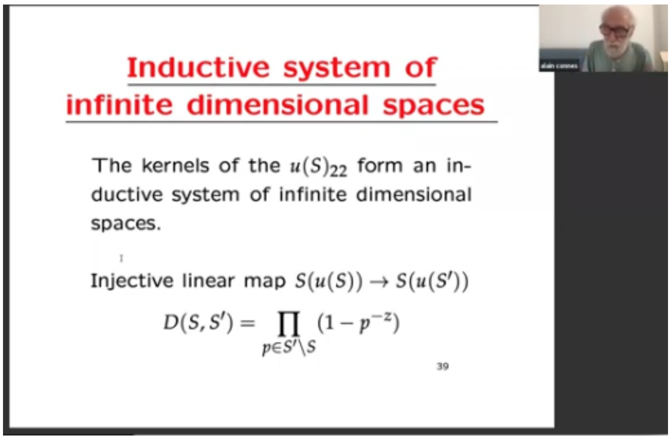
by the local factors at infinity, so if you take the product of the local factors at infinity by a finite number (of local factors) of ratio of local factors at finite places, then this is quasi-inner. And the way this is proved, it's very tricky, because the non-commutativity of the space is entering and the way it enters is that as soon as you have primes. If you have one prime, no problem. But when you get two primes, what happens is that when you look at the poles, the fact that powers of 2 can be very closed to powers of 3 (you know $2^{19} \approx 3^{12}$), this enter to spoil everything when you try to make the sum over the poles. So the way we proved it, we used the Gauss multiplication theorem



to factor the archimedean ratio ρ_∞ into a product of m quasi-inner functions which are very much the same taste as what you get for ρ_∞ and then, we distribute these quasi-inner functions to each of the primes and because the product of quasi-inner functions is still quasi-inner, we get the result.



So I will finish with the two statements. So the first statement is that, you know, as I said, there is the definition of the semi-local Sonin's space, but it turns out *(laughing)* that this semi-local Sonin's space is simply the kernel of the $2,2$ part of the quasi-inner function which is given by this product. Remember that if the unitary had been triangular, then the $2,2$ part could have been a co-isometry. So now, it's only quasi, if you want, like that, but still, we can look at this $u(S)_{22}$. This turns out to be the analog of the semi-local Sonin's space. But now the question arises, "okay, this is fine, is this space infinite-dimensional ?". This is not obvious at all. If the unitary was triangular, yes, but it's only triangular in the Calkin algebra, so is it true that this kernel of this $u(S)_{22}$ is infinite-dimensional ? So I will finish on that. That's what we found during this summer and it turns out that there is a remarkable map which turns to the Sonin's space, defined as above into a colimit, into an inductive system



and the injective linear map which goes when you enhance the set of places, for instance, if you have one prime, there is a kind of completely obvious map *(nearly laughing)* which turns out to map the Sonin's space for S to the Sonin's space for the enhanced S'. And this map is simply multiplication by the product of the $(1 - p^{-z})$. These are not the local factors, they are the Hilbert spaces for the local factors, for the elements that don't belong to S but that belong to S'. So, if you want, now, we are in the situation with Katia in which our task is to be able to do, for the semi-local case, what we did for the local case. We are not saying, you know, it's doable, it will certainly be hard, but somehow we have the analytical ingredients ready for that and what we plan to do is to put into action the geometric part, the geometric part that was so valuable in the case of the archimedean place, because it gave us the little square, and the big square, and it gave us if you want the understanding of the discrepancy and the control of the discrepancy, and so on and so forth. But I hope that I gave you taste of the fact that you know, quantized calculus, even though that one could try to reduce it to the calculus, no, it's much more powerful, and there is an incredible tight, link, between this calculus and the Weil problem, just because the formula for the Weil functional equation can be so simply and directly expressed in terms of the quantized calculus.

Okay, I'm here.

(Virtual applause).