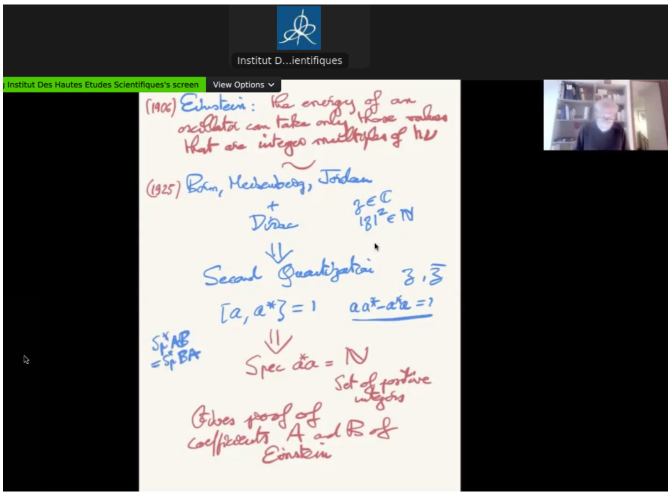
Renormalisation et théorie de Galois
Alain CONNES
18.11.2020
À l'occasion des 60 ans de Dirk Kreimer

Dear Dirk, I wish of course a happy celebration of your 60th during this week and I really want to congratulate you for your great discoveries. So this talk will be, indeed you know, a great occasion for me to pay tribute to Dirk whose encounter was a key turning point in my own understanding, both I would say about physics and mathematics and relation between both.

I will start by saying that I've always been fascinated by the courage with which physicists hand seemingly unpractical mathematical problems. I mean the one I would discuss today would be renormalization and then, it would become clearer, what I am trying to emphasize.



I would start from, you know, the birth of Quantum Field Theory which after Planck, of course, after Planck's discovery in 1900, there is this clear statement in Einstein's paper of 1906 that the energy of an oscillator can take only those values which are integer multiples of $h\nu$. So this is quantization, it's quantization which is a sort of a wishful thinking. And it was put on solid grounds by two papers : there is a paper of Born-Heisenberg and Jordan I think in 1925, I'm not completely sure, and then of course, the paper of Dirac in 1930. So what is done there, it is something really quite amazing that, you know, you want, because of this statement of Einstein, you want a kind of very strange condition on a complex number ; you want complex numbers z, which belong to complex numbers, but you want to subject them to the condition that the absolute value squared is an integer. Now, as it stands, you know, it's really something which looks totally impossible, totally unnatural but why not. But this is what you want for the coefficients which will appear you know in the Fourier expansion of a wave, okay.

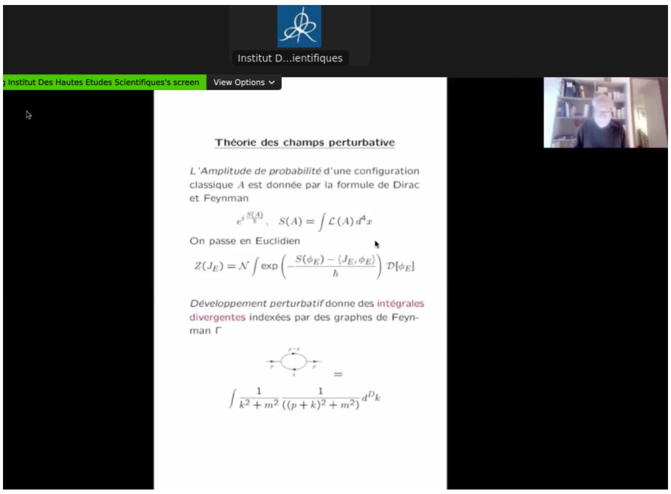
Conférence donnée à distance lors du colloque "*Algebraic structures in perturbative quantum field theory*", organisé par l'IHÉS.
Video visionnable ici https://www.youtube.com/watch?v=bshH2$_i6whc$.
Fichier associé au diaporama ici : https://indico.math.cnrs.fr/event/4834/attachments/2600/3291/AlainCONNES.pdf
Transcription Denise Vella-Chemla, novembre 2020.

So the amazing ideas, the amazing ansatz, which comes from both papers, I mean in Born-Heisenberg-Jordan, they studied the oscillator and they found the corresponding operator and then Dirac used it in the second quantization, in the first example of second quantization. And what is a miracle is that if you take, not a complex number but an operator, and this operator you know it's like if z was not commuting with $\bar{z}$ so I mean the replacement of the $\bar{z}$ is the adjoint of the operator, so you have two operators A and A^*, adjoined of each other, and they fulfill the condition that their commutator, they mean $AA^* - A^*A$, you know, is equal to 1.

Okay, this is extremely simple and just by this formula, it immediately implies that when you take A^* which is modulus of that squared, it will be an integer : the reason is very simple, you know, the reason is that in general, you have spectrum of AB is equal to spectrum of BA, except possibly from the presence of zero, the point zero in the spectrum. And so, I mean, if you have this relation, this means that you can descend, you can descend from an element, first of all, the spectrum is positive, because the operator is positive, and if you take a number which is in the spectrum, you know you will descend it by one and so on provided you... the only way in which you can descend to something negative which is of course absurd is that you land on zero so this side in the spectrum is formed of positive integers.

Now with this and that, you know, Dirac was able to prove physically, mathematically speaking, the formulas that Einstein had guessed by thought experiments about the constants A and B, coefficients of absorption and emissions of radiation by an atom ; so this was a fantastic success in 1930 and it was really the birth of Quantum Field Theory.

Now, what I want to show is a kind of a joke[1], but which is very very comforting somehow, you know, so this is this. I don't know how real it is, I mean I don't know if Einstein really said that, but we don't care, okay, so what he says here is that : "do not worry about your difficulties in mathematics I can assure you that mine are still greater". Well, you know, this is quite an amazing statement.



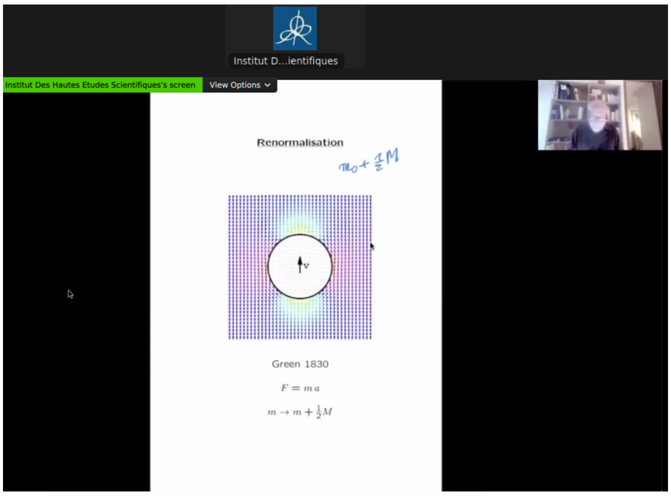
Now indeed, you know the mathematical difficulties which you reach extremely soon, when you handle Quantum Field Theory, are encapsulated by one formula, which is due to Feynman, you

[1]photo of Einstein

know there is a saying in a more physicist probably of the past which was that you know "Schwinger brought Quantum Field Theory to an art[2] and Feynman brought it to the masses". Well the reason why he brought into the masses is that, you know, you have some principle, which is incredibly simple to formulate, which is that the probability amplitude (remember the probability amplitudes are like square roots of probabilities), probability amplitude of a configuration is given by this formula[3], okay, which is the imaginary exponential of the action in units of $\hbar$. Well, of course, the action is defined like this so, I mean, this is extremely delicate because, you will write functional integral and if you would write this functional integral in Minkowski space not in Euclidean, then you would immediately meet the difficulties that the propagator has singularity and you don't know how to handle this.

Now, it is handled by what one calls the Feynman i(epsilon) prescription but that means essentially that you are passing to Euclidean. And so, what you do is that you compute, if you want, the source, so you compute the functional integral and what is sort of coming out very quickly is that, first of all, you don't know what is this integration measure at all. And the only thing you can do is actually take the free theory or free field and perturb around the free field. When you do that, you have to integrate by parts under gaussian which okay, anybody can do, and then you get some expressions which are giving you the perturbative expansion. But what you find out, almost immediately, is that when you go beyond the tree level, so beyond the level at which Dirac was working, you find that the integrals that you get, the expressions that you get, are in fact divergent integrals. So, I mean, on the face of it, you know you get something which is meaningless.

Now, this type of meaningless result has an old ancestor actually. And this old ancestor, I remember, you know, a talk by Sidney Coleman in 1978 in which he was giving an example which is a slight variant of the following.



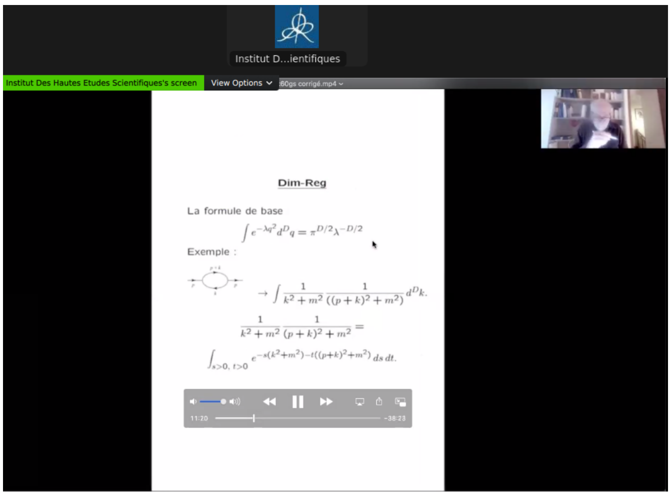
I mean, you know, he was giving the example of a balloon filled with helium and you compute the initial acceleration of the balloon when it's sort of left going up. But you find something ridiculously different from the observed. And I mean you can also take an example of a ping-pong ball in the water. And so, the archimedean principle if you want, the fact that you know the force will be

[2] ?

[3] surrounding $e^{i\frac{S(A)}{\hbar}}$

corresponding to the volume of the mass of the water that goes up and so on, doesn't work at all : it gives you a result which is in contradiction with experiment. And I mean, it was observed by Green in 1830, actually, that there is a beautiful explanation to that. And the explanation is that when you compute actually the mass that should enter in the Newton law, you find that it's not the original mass m_0 if you want, that you would have for the balloon or for the ping-pong ball, but you have to add to this a correction term which is actually one half of the mass of the water contained in the ping-pong ball, if it were in water, or it's a mass of the air and so on. And what you find out then is that the initial acceleration cannot exceed 2 ($2g$). And I mean the reason behind this is that you know the ping-pong ball or the balloon is actually immersed into a fluid. And in the motion, you know, what happens is that it creates a disturbance in the fluid and when you compute the energy of this disturbance, this actually adds an additional term to the effective mass that you are handling. So somehow you know in the case of the balloon or in the case of the ping-pong ball, you can actually find out what is m_0.

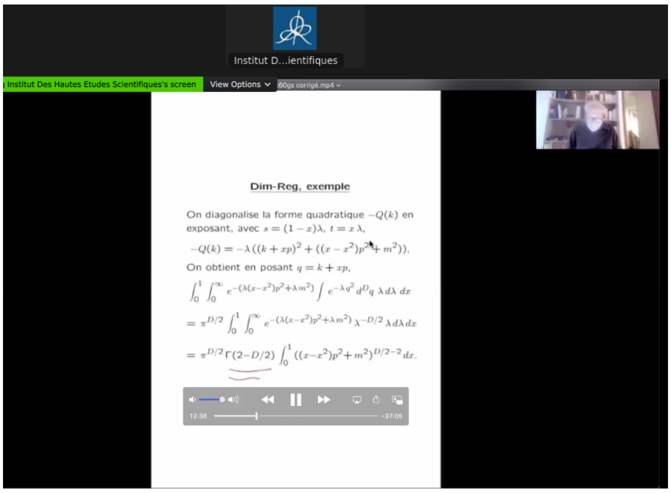
What you do is you take the ping pong-ball out of water, that's it, you can weigh it. But what physicist understood very very soon is that this is not the case for the electron because if you have the electron, you cannot put it out of the electromagnetic field whatever you do. So you will never be able to find what is called the bear mass, for instance, of the electron. So this resulted in a lot of fight, a lot of thinking, and amazing, how to say, you know, development which is called renormalization, and which led physicists to slowly understand what was going on, as I said, you know, of course, so in the hands of Schwinger, Feynman, Dyson, and then you know, Bogoliubov-Parasiuk-Hepp-Zimmermann came up actually with a very good way, in fact, I think this dim-reg actually due to 't Hooft and Veltman, in order to understand and get rid of these divergences, from the physical principle that you know, for instance, the bear mass is different from the effective mass and similarly for the charges, similarly for the field strength. You have to use the regularization process.



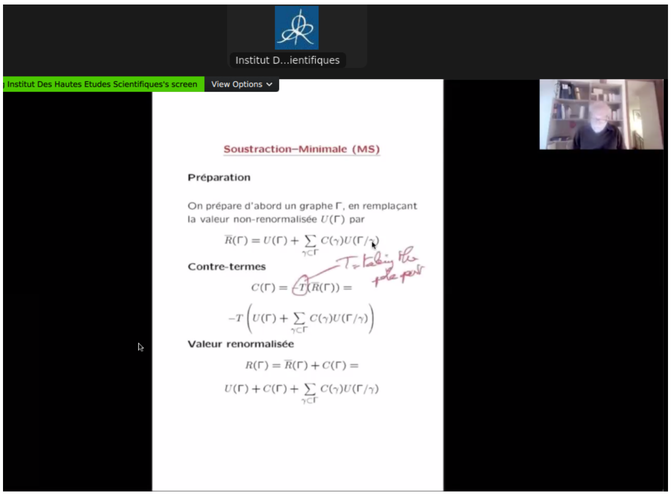
So the actually most efficient regularization process is what is called dim-reg. So the idea of dim-reg is simply this formula[4]. So this formula tells you that if you have to integrate a gaussian in d dimensions, you don't have to worry whether d is an integer : you can put a definition, and this

[4]surrounding the first formula on the page.

definition is that the integral of this gaussian in dimension d is given by this formula. Now what you do is that you take one of these divergent integrals you were getting from Feynman graphs, and you handle it by passing to what are called the Schwinger parameters. Namely you rewrite it, you know, you rewrite the integrand as a sum of gaussians, I mean, in gaussian expressions, and then you compute. Okay, you compute with this and, I mean when, you compute with this on an example,

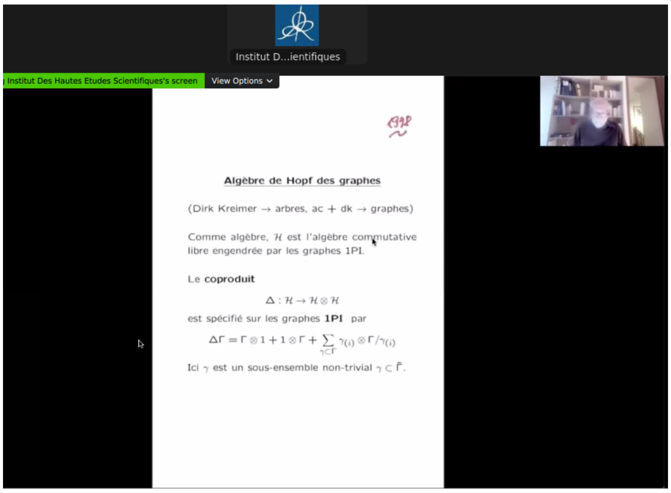


you can find out what you get and typically what you will get, in simplest examples, will be gamma functions. And these gamma functions will have... because of the divergency of the integral, the bad taste if you want to have a pole at the dimension you are interested in. For instance, you know if you are in dimension four then this expression when d equals four, it will have the pole of gamma at $z = 0$, multiplied by something that you can compute. So what the physicists have invented, all these years, of fight and of understanding and so on, is a process which is a process which is combinatorial, which is called minimal substraction, and which allows you at the end of the day to get a finite result.



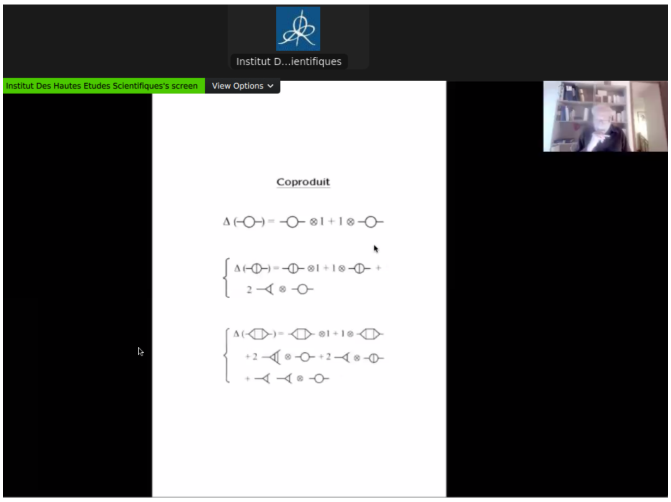
So first of all, you have the preparation, and this preparation comes from the fact that you know, you have to take into account, when you work with higher loops and so on, of what you did before. So these are called the sub-divergences and you have to prepare a graph by accounting for the terms that you had computed before by a certain formula. So this will yield for you, will give you what are called the counter terms. And in this counter terms, what is really important is if you want to take the pole part so T is taking the pole part, so the part which is, you know, divergent part. So I mean of course not only one epsilon but one over epsilon squared. And so on and so forth.

Okay. So you take these counter terms and you define the renormalized value by minimally substracting if you want the divergent terms. So the renormalized value is given by this formula. Okay, so this is a combinatorial recipe, it's very complicated and when you see it, as a mathematician, at first, you say "okay, well, it's hopeless !", you know, because okay I mean, one understands why in physics you have to do that but mathematically speaking, you know it's very difficult to imagine that this could have mathematical meaning, not that it's not rigorous, it's perfectly rigorous, no ! But you know, I mean conceptual meaning, okay ? And this is what we found with Dirk Kreimer in our collaboration.



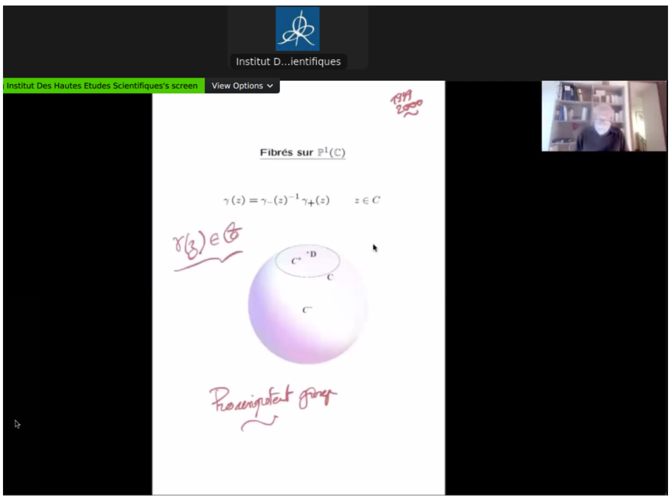
And I mean, we began, I think it was in 1998. Yes, we began to work together in 1998 and the idea, the key idea came from Dirk. The idea of Dirk was that, you know, when you look at this graph in fact at first, he was working with rooted trees, there is Hopf algebra structure behind the scene. Now at the time when I met Dirk I was working with Henri Moscovici. And we also were working on Hopf algebra. So I was sort of you know perfectly ready to absorb the discovery of Dirk.

So, it turns out that when you formulate this Hopf algebra in terms of graphs, it's a beautiful thing, namely, you have graphs, you take the free commutative algebra generated by graphs. So you take linear combinations of graphs and so on, and products, formal products. And you define a co-product. And this co-product is sort of specified on the graphs which are one particle irreducible by this formula. Well, these are the sub-divergences, I mean, are the subgraphs if you want which would be corresponding to sub-divergences. So you define this formula.

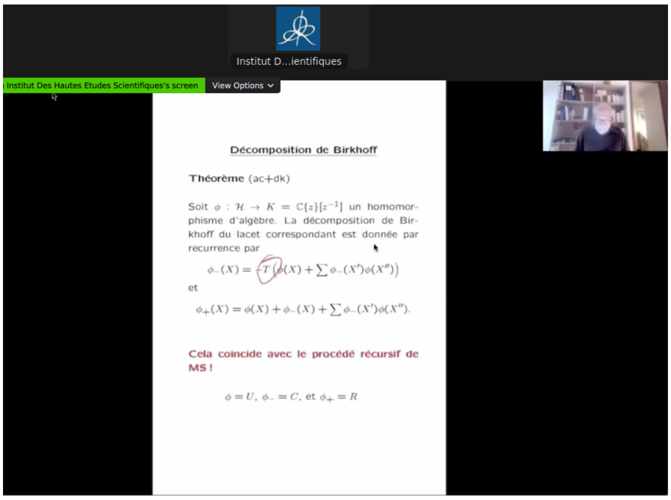


You play with it and I mean it's quite amazing that you know the co-product that you have defined like this is actually a co-associative you know it's a morphism of algebra and so on. And, I mean, this co-product goes from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$ for h and I mean, after, when you think about it, after the fact, you know after a long process, you find out that the right analogy between... What is the kind of Hopf algebra ? This type of Hopf algebra which is commutative but not co-commutative. Because co-product is not co-commutative in general, you know, because you have terms like this.

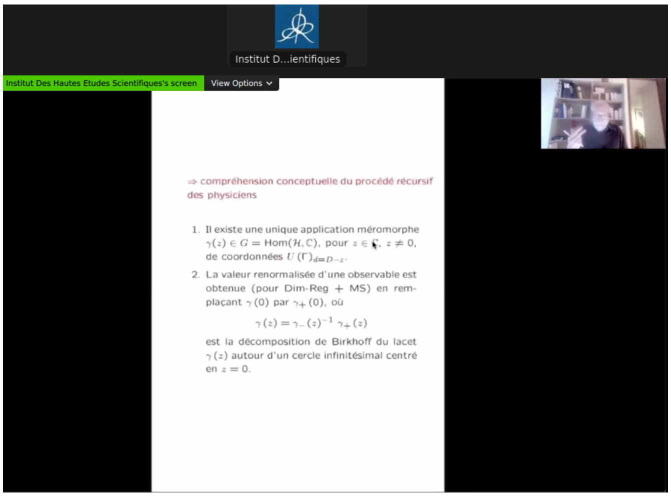
I mean, it's an underlying group, a formal group. Now the clues you can think of which is in fact very very closely related to that one is the Hopf algebra that you would get if you look at Taylor expansions of diffeomorphisms ; so in fact, you know, the correct name which we cooked up was diffeo-graphisms because of the graphs. So I mean this is the way you have to think about it ; you have to think that it's an underlying group : this group is a composition of things which are like diffeomorphisms and they are given by their Taylor expansion which corresponds to a perturbative expansion.



So we have this co-product and now the great discovery that we made, I think to, either in 1999, and forward in 2000, but this was absolutely a fantastic moment is that in fact this procedure, this combinatorial procedure of physicists is in fact nothing, but something which is known in mathematics and which is related to a geometric problem. And this geometric problem is the problem of understanding the bundles on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and in order to understand such bundles, what you do is, they are given by a gluing data, gluing data, because if you want, if you look at (I'm talking about holomorphic bundles) so because if you want the part which will occur on the upper hemisphere or the lower hemisphere, okay I mean these parts will be easily understandable so, but the part which is non-trivial is how we glue them. So a lot of work was done in mathematics on that, Grothendieck, for instance, worked on that, and it turned out that in physics, because of the nature of the problem which is a perturbative problem, instead of considering a bundle which has values in a group like $GL(n, C)$ so it would be a vector bundle with values in $GL(n, C)$, this is the group Grothendieck was working on, instead, we shall be working with bundles, whose structure group is a pro-unimpotent group. Okay. So it's a pro-unimpotent group and that makes things much simpler in the sense that you don't have, if you want, global obstructions, to trivialize the bundle but, when you compute when you understand... what it means, that you sort of trivialize the bundle, then you apply it to the following loop, you see, when we're talking about the Hopf algebra, it turns out that when you look at the values of the graphs when the dimension is not the dimension which is critical, you know, like equals four, and so on, so forth. Well then you can give it a meaning and this meaning, what does it tell you ? It tells you that what you have is the following : you have a loop $\gamma(z)$ okay, whose values in this group, attached to the Hopf algebra, but this $\gamma(z)$ is like a gluing data. And you don't know how to evaluate it at this dimension d because there, you know, it's singular. So what you do is that you apply the method which allows you to trivialize this bundle, and this method is called Birkhoff decomposition and what does it do ?

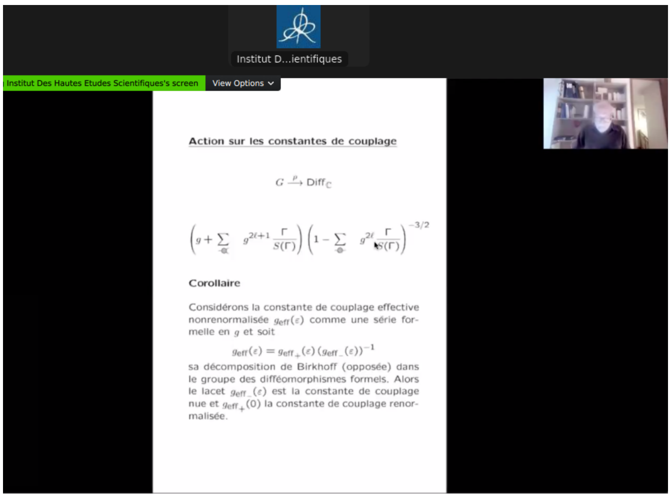


It writes this loop with values in the group, the group is highly non-abelian group, it's not commutative group at all. But you write it as a ratio of two loops, one which will be quite singular but one which will be perfectly regular in this $(C, +)$ which is this $\gamma_+(z)$ and the amazing result that we proved with Dirk is that when you look at the mathematical uniquely defined, if you want, Birkhoff decomposition of the loop corresponding to the data which are computed by dim-reg, and so on, then you find by induction that it's given by this formula where T is the same as before, it's the extraction of the pole part, okay and so amazingly, I mean this was an amazing moment that this process exactly coincides with the recursive process with the combinatorial recipe that was given in minimal substraction, okay. So we made the translation from one to the other. So this was you know an absolutely key moment and well, what does it mean ?

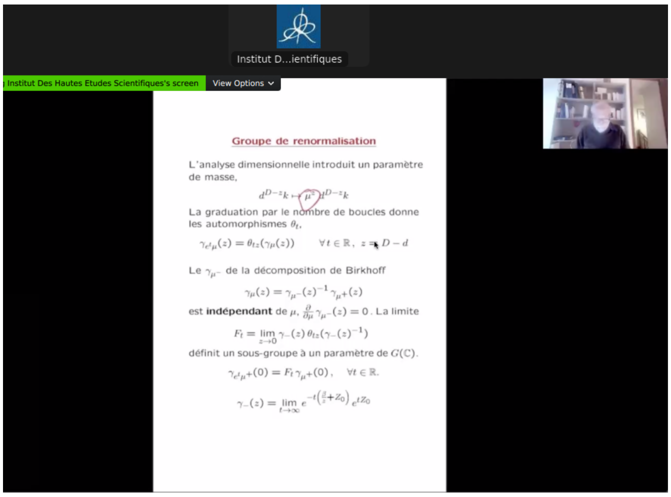


It means that one has, you know, a conceptual understanding of this recursive process of physicists. So in other words, you know, we have a unique meromorphic map which goes to this group associated to the Hopf algebra, okay. And when you take, now, the renormalized values of an observable, and so on. What do you do ? for the dim-reg + MS scheme ? Then what you do is that you ignore the divergence by replacing $\gamma(0)$ by $\gamma_+(0)$ in this non-commutative decomposition process which is the Birkhoff decomposition. So what does it mean ? It means you know that if in the middle of the night, somebody would come and put a gun on your head and would tell you what is renormalization, this would be my answer. My answer would be "okay, well, look, I mean, you know, it's a Birkhoff decomposition of the loop and you take the part of the loop which makes sense, and you ignore the other one". Okay, so now it turns out that, you know, there is much more to that, there is much more stuff behind the scene in this data, and what is behind the scene is related to Galois theory. I will come to Galois theory much later.

But somehow, I will now describe results which were obtained you know, in collaboration with Matilde Marcolli,



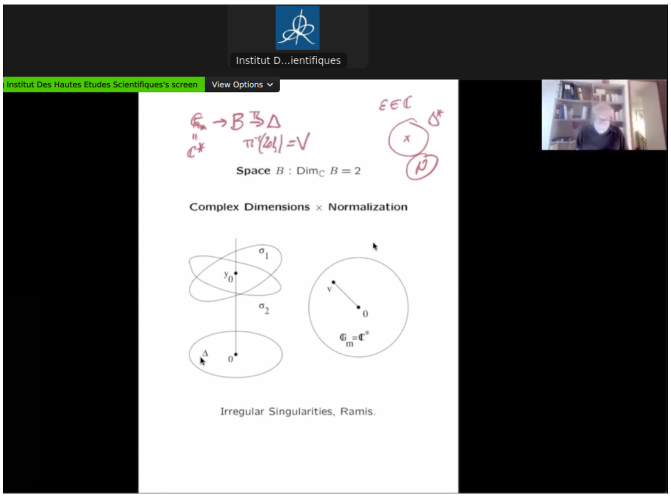
and I mean, before I do that, I would like to say that, you know, the work that we did with Dirk, we had as a corollary of what was going on, the way that the group was acting on coupling constants ; namely, there is a natural morphism from the group associated to graphs to diffeomorphisms as I was saying, you know, this group should be thought as diffeo-graphisms. So it's related to the diffeomorphisms and the way it's related to diffeomorphism is by the way it's acting on coupling constants. So it's acting on coupling constants by, if you want, the image of this series. But because of the Birkhoff decomposition, is a sort of functorial, what happens is that you can get if you want what is the effective coupling constant from the Birkhoff decomposition. So this is, I mean, you can get if you want the finite, the renormalized, coupling constant from the Birkhoff decomposition. So this is the corollary of what we have done before.



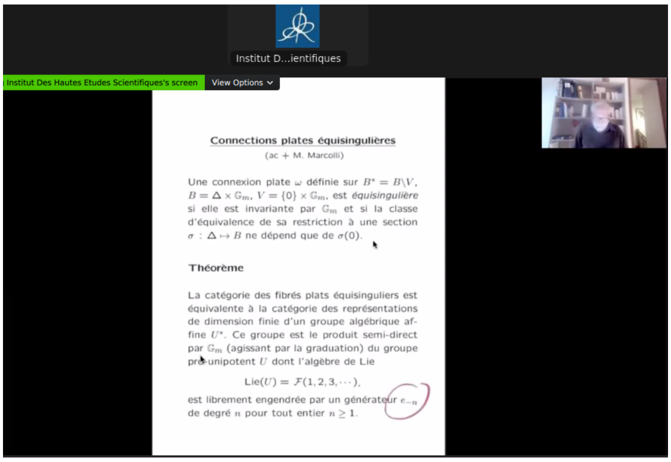
So as I said, you know, I continued working on this and in what we had done with Dirk, we had understood the renormalization group, and this was coming from essentially the fact that, you know, when you look at dimensional analysis, when you do integration in dimension $D - z$, you have to introduce a dimension full parameter which has the dimension of a mass which we call μ, okay, and which you put in formulas.

And the amazing fact, which is a fact of life, you know is that which was known before, is that when you take the negative piece (now we say in the Birkhoff decomposition but, okay, I mean in physics terms, you know, this was in the DPH method), this negative piece in the Birkhoff decomposition is actually independent of μ so from that, you know there is a one parameter group of subgroup of the group associated to the Hopf algebra which appears, completely naturally and then, in my work with Matilde Marcolli, what we did was to, if you want, understand the link between all these facts that I mentioned before and Galois theory.

I mean differential Galois theory, but differential Galois theory in a situation which is much wilder than when you look at you know the Picard ratio or differential Galois for regular singular differential equations. And I mean, fortunately, you know the Picard ratio theory, which was, you know, very beautiful which applies very well for regular singular differential equations, after a long time, you know, for a long time, it stayed a little bit silent, because you know there was an essential result which was that the Galois group was the Zariski closure of the monodromy. But, then under the hands of Martinez-Ramiz-Malgrange... Deligne, and also Ecalle, it became, you know considerably, how to say, a sophisticated theory that applies to singular situations. In the work with Matilde, what we have found is that we have applied if you want the Tannakian formalism which was at first, you know, formulated by Grothendieck and then that had been developed by many other people, in particular by Deligne, so what we have found is how... if you want, there is a natural Tannakian category of... how to say... of differential systems or if you want of connections and modules and which is associated to the renormalization problem and which embodies all the previous properties which I talked about. Now, the main idea is the notion of an equi-singular flat connection.

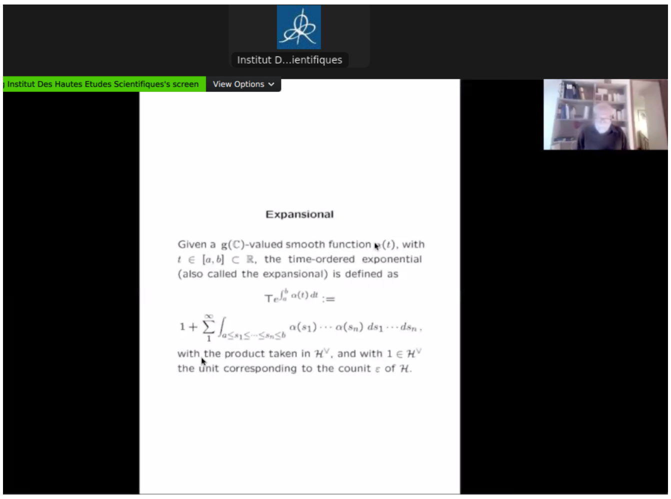


So for that, I have to make a little bit of geometric thinking and you know, you have to think that somehow, when I was telling you that there was the epsilon which was a complex number, say very close to zero, but you don't want epsilon equals zero so what you do is you take a punctured disk that you call Δ^*. But there is also this μ, this μ parameter, and this μ parameter, when you combine it with the epsilon, what you get is a space of dimension two, complex dimension two, and in this space of complex dimension two, which essentially you know it fibers by the multiplicative group G_m which is $\mathbb{C}^*$, if you want, it fibers over the disc Δ but you want to remove the part which is above zero.

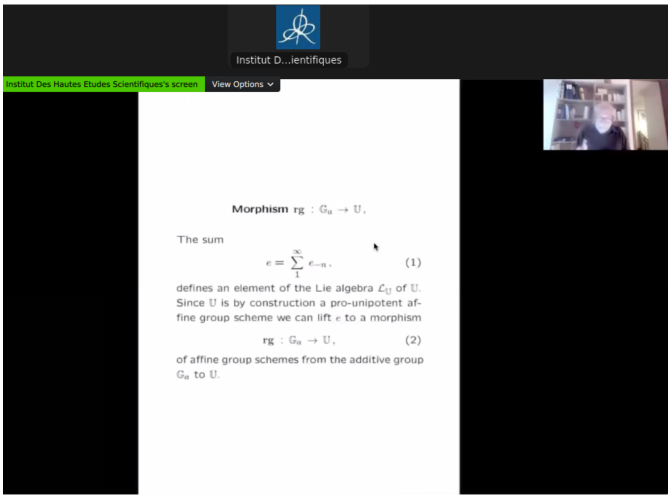


And the part which is above zero, I call it $\pi^{-1}(0)$. I call it and I will talk later about the meaning of $\pi^{-1}(Z)$, you know, in terms of the constant, the Planck constant $\hbar$, but what happens is that because of this independence of the negative part in the Birkhoff decomposition for such loops, what you have is that they are associated, in fact, to what are called equisingular connections. So the equisingular connections are flat connections, which are invariant under the multiplicative group, but which are such that, when you restrict them to a section, from Δ to B, so this is one section, another section, then you know the singularity when you get to the point zero are the same. I guarantee. So then what we have found is that applying the Tannakian formalism which is a beautiful thing, you know, what it tells you is that if you have what one called the Tannakian category, so it's an abelian category, but it also has a like a tensor product, and what you assume now is that this category has what is called a fiber functor. So it has a functor which goes to ordinary for instance vector spaces, when you are covering a field, and then you look... one can prove abstractly under certain conditions that it defines an algebraic affine group which is given if you want as a functor from an arbitrary commutative ring to the groups. And the corresponding group is like the automorphisms of the fiber functor, when you take it over the ring. So what we prove there is that if we take the category or equisingular flat bundles, then it turns out to be equivalent to the category of representations of finite dimensions of a certain algebraic affine group which is uniquely determined. And it turns out that this group is a semi-direct product by the multiplicative group which acts by the way by the loop graduation, by the variation by the loop number of a certain unipotent group. And this unipotent group is uniquely determined and it is the unipotent group whose Lie algebra is generated freely by a generator, ε_{-n} in each degree n for every integer n.

Now okay I should mention just briefly in passing that similar group *does* appear in motivic Galois theory but not in a canonical manner. So it's very illusive to understand the relation. Now to go a little bit deeper in what happens, as I said, you know we are dealing with irregular singularities. So I mean, it's very connected to the theory of Ramis, I mean of the exponential torus of Ramis...

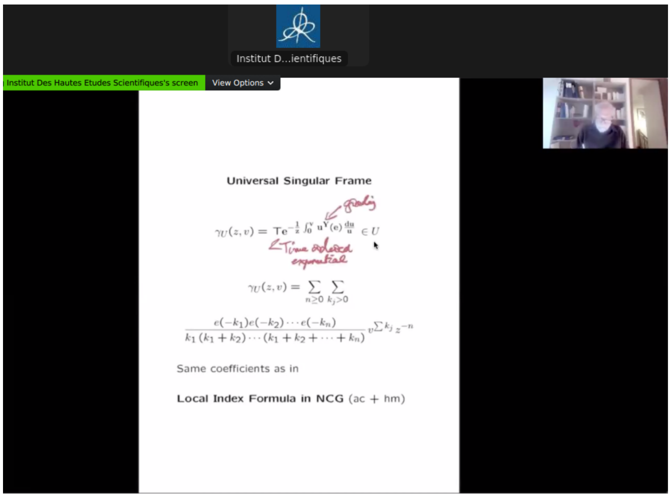


and what happens is that one needs, in order to write formulas, to use a device which is called the expansional or time ordered exponential, and there is by the way a beautiful paper of Araki, going back to the 70's, in which you know I think it's in the Annales of École Normale Supérieure, in which he gives a beautiful general theory for this expansional. So this is something which is well understood and it's a time-ordered exponential and it's very useful in order to write solutions to differential equations. Okay. So it makes sense in Hopf algebras.

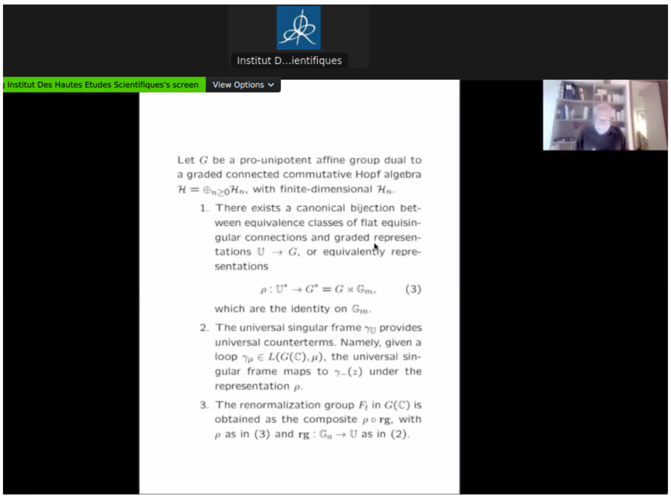


and it turns out that behind the scene, in what I told you before, there is a certain canonical morphism from the additive group to the group to the pro-unimpotent group $\mathbb{U}$ which I defined and which is, you know, underlying the previous result, this theorem here[5]. So this is the underlying group $\mathbb{U}$ and I mean this group, it's defined by this formula, and as we shall see a little bit later, it embodies exactly what is called the renovation group, which is just a subgroup of the group we are dealing with, which is much richer because it's a highly non-abelian group.

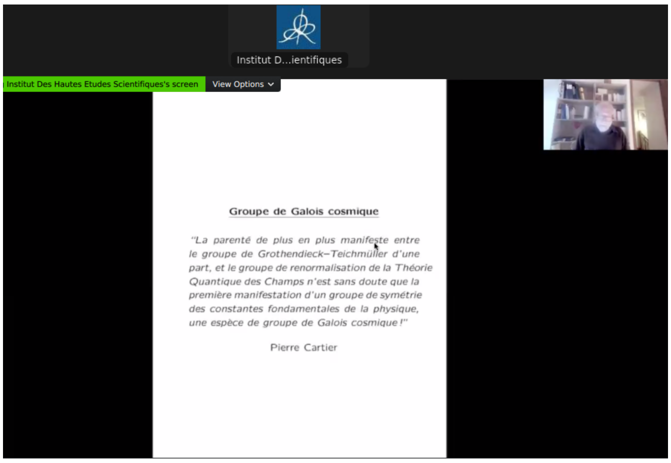
[5] Theorem on the page concerning equisingular flat connections.



So it turns out that there is an object which is defined by a time order, so this is a time ordered exponential, the y which appears here is the grading by the loop number, okay. So all this stuff makes sense and the funny thing which at the moment has no real good explanation is that when you expand this universal singular frame, I will explain what is its role, you know, you get the same coefficients as in the local index formula that we have with Henri Moscovici with which we were working a few years before. But this has not yet found the conceptual explanation.



Now the main result is the following : it is that if you take a pro-unimpotent affine group, dual to a graded connected commutative algebra, exactly what happens in physics, thanks to the Hopf algebra of Dirk, then first of all, there exists a canonical bijection between the equivalence classes of flat equisingular connections and graded representations of this universal stuff that I defined okay to the group G (or, equivalently, of course, you can make cross product by the multiplicative group G_m corresponding to the grade). Now the universal singular frame provides universal counterterms, this is a fantastic fact. And I mean, it's related, you know, to what are called the Gross-'t Hooft relations, namely, given a loop universal singular frame maps automatically through the representation ρ to the negative piece of the Birkhoff decomposition. And finally the renormalization group, which physicists love, you know, which is a one parameter subgroup of the group assigned to the Hopf algebra, is obtained as a composition of the representation with the **rg** which was defined before as a morphism from the additive group to the $\mathbb{U}$.



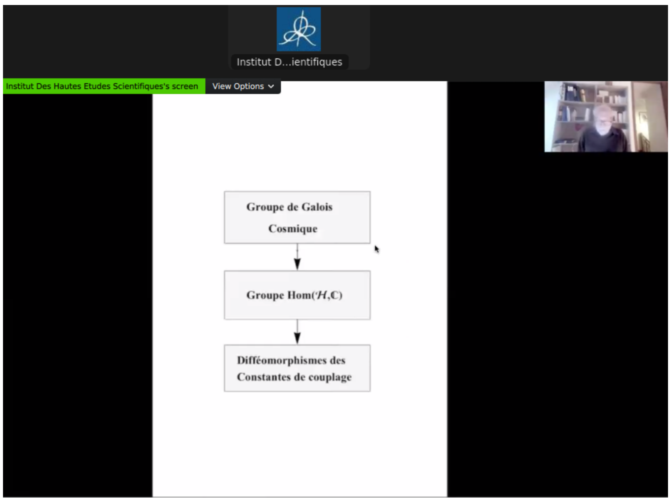
Now you see, one cannot refrain from quoting Cartier because perhaps Cartier had a slightly different motivation, but anyway, he had the right vision, in the sense that what he wrote is that :

> *La parenté de plus en plus manifeste entre le groupe de Grothendieck-Teichmüller d'une part,...*

That was another inspiration because it came from Number theoretic stuff,

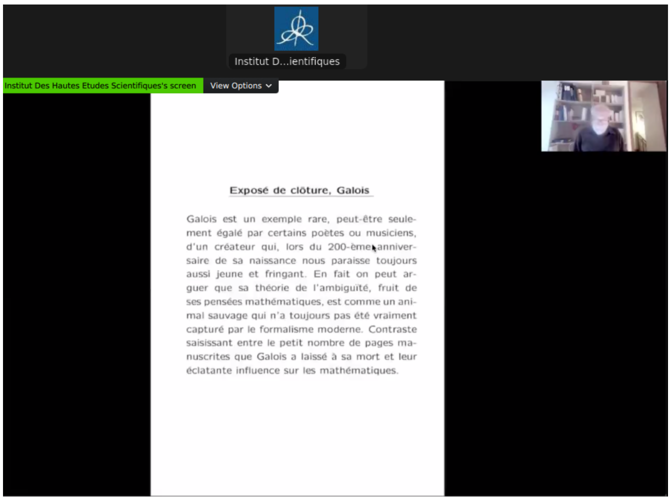
> *et le groupe de renormalisation de la Théorie Quantique des Champs n'est sans doute que la première manifestation d'un groupe de symétrie des constantes fondamentales de la physique, une espèce de groupe de Galois cosmique !"*

So when we found with Matilde, you know, this group, I mean this group which was coming from the Tannakian category and so on, we couldn't refrain from calling it, you know, the cosmic Galois group.



It's really what it is, because as I said before, you know, from the work with Dirk, when we're acting on the coupling constants, this group actually maps to the group of the given theory. And in turns it maps to the different morphisms of the coupling constants. So in fact this cosmic Galois group is acting exactly as Cartier was sort of envisaging, it really acts on the fundamental constants, of course as you know very well you know, I mean the fundamental constants of physics, I mean they are not constants, I mean, they are functions, they depend on the energy scale, so this is exactly what's going on here.

Now all this leads me to Galois ? Because the idea behind the renormalization group, the idea behind all of this, is that you know, when you do physics, you find out that there is something very elusive in the renormalization process which is that there is still some ambiguity and this ambiguity relates to fundamental ideas of ambiguity which is the idea of Galois. I had the occasion to give a talk about Galois and I mean, when I talked about Galois, I said the following :

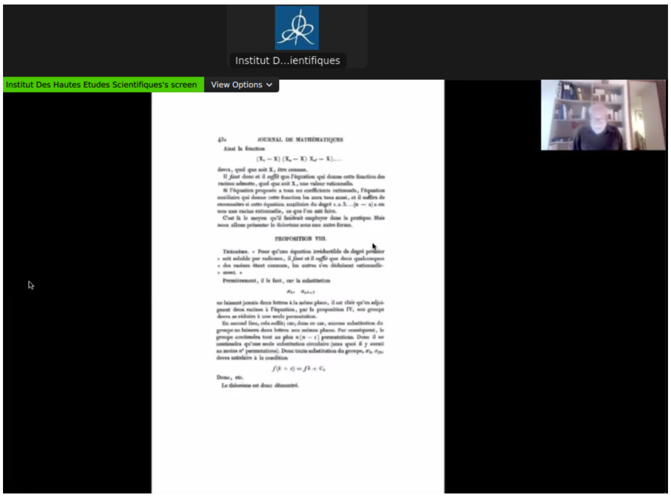


> *"Galois is a rare example, perhaps only equated by some poets or musicians, of a creator which, on the 200th anniversary of its birth, appears still so young and fringant[6], I don't know how to translate into english. So what I continue by saying is that one can assert is that his theory of ambiguity which is fruit of his own mathematical source is like a savage animal you know it's like a wild animal which has never been captured by the modern formalism."*

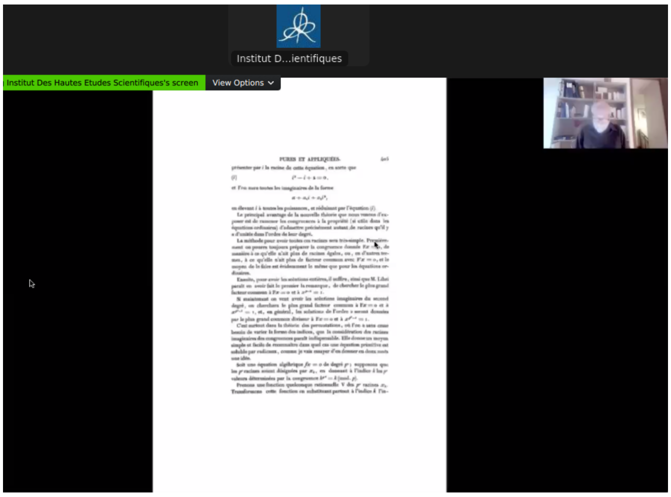
Grothendieck was very very close to capture it with, you know, Tannakian formalism, and so on, but I mean this is a striking contrast between the small number of pages that Galois left at his death and the incredible influence on mathematics. Now, you know, there are misgivings about Galois because many people think that what Galois did was to invent the Galois group, and to understand symmetries, and so on. But this is very far remote from the reality of what he did. What he did, if you want, there is always this contrast between the formal things and the things which are very concrete, which are behind.

Of course, this contrast is very present in renormalization. But it's equally present in the work of Galois.

[6] dashing.



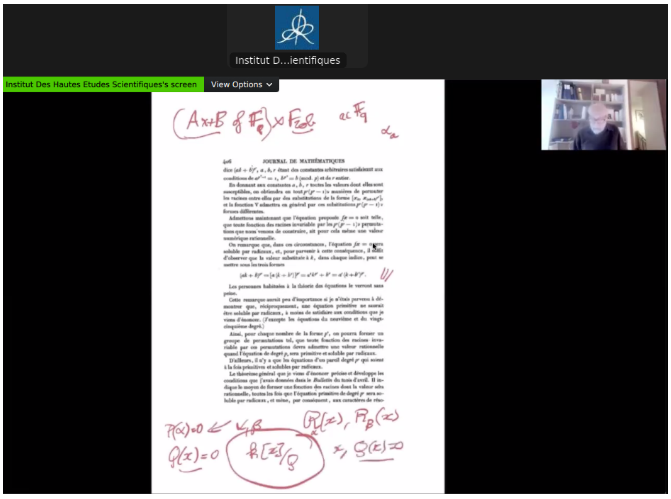
And what one has to know in order to appreciate Galois is that when he was 17 or 18 he wrote a paper in which he defined a finite field which in anglo-saxon countries are called Galois fields and in France, they are not, because you can't I mean it's a little bit, how to say, it's leading to call “le corps de Galois”[7] You immediately think about this death when you say that but what is amazing is that when he was 17 or 18, he enontiated an amazing theorem which even now, if you try to prove it, you will have trouble even though you think you know Galois theory.



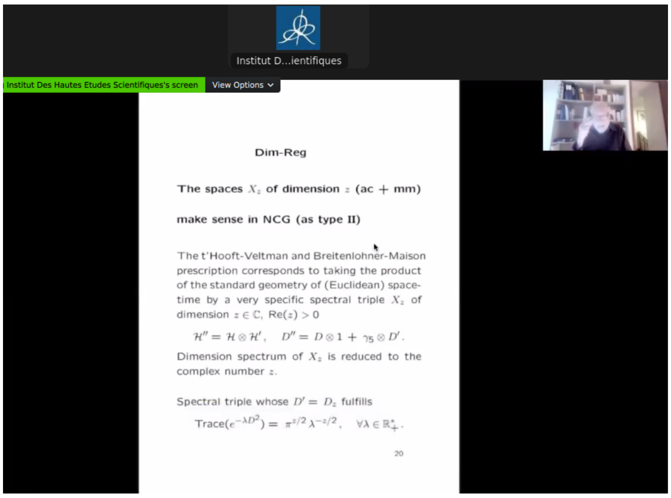
What is the theorem that he enounced, I mean, the theorem is enounced around here, it's like it's saying that if you take what he called a primitive equation then for this equation to be solvable, it's necessary and sufficient that you can index its roots by a finite field F_q, okay, so if you want, you label the roots by α_a where a belongs to F_q and the Galois group has to be a subgroup of what ? of the affine group, okay, the $ax + b$ group of F_q, okay, but cross-producted by the Frobenius automorphisms, powers of the Frobenius. So this is absolutely mind-blowing. I mean, it's amazing that he could find this result at that age, and moreover you know when you look at the carefully at the paper of Galois, you find out that : what was his motivation ? His motivation was not that of Lagrange finding invariant and so on, no, no no no no, no no, no no : his motivation was to find all relations that hold between the roots of a given equation and I mean, what you will find out if you go deeper, you will find out that the way he did it, it is by finding an auxiliary equation which is a much higher degree such that the root of your given equation, the roots like α, β and so on, they

[7] the body of Galois.

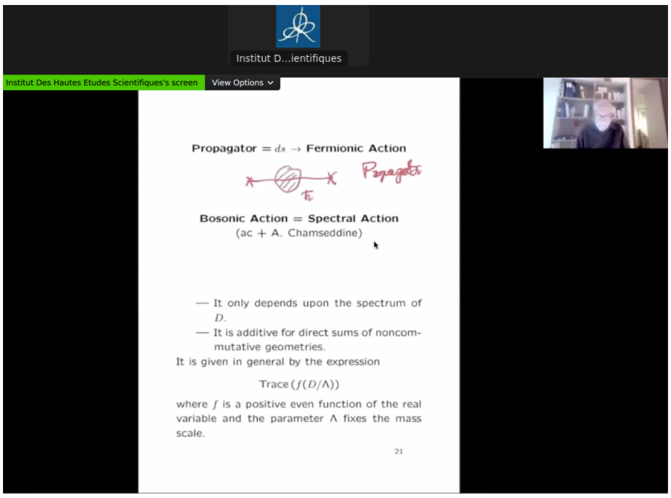
are all rational functions of the roots x, you know, of the additional equation.



So, for instance, now, alpha is an alpha of x, beta is a beta of x and they are all rational functions, so polynomials otherwise, they have to be polynomials because x would be the solution of some equation. And now, you know you say "okay, that's very nice but how do I know x ?" Well, how do you know x ? Well x is the solution of another equation, so so these were solutions of an equation $p(x) = 0$, sorry $p(\alpha) = 0$, okay. And this x is solution of another equation of much higher degree. So the equation is like $Q(x) = 0$. So you wonder "okay, well, you have replaced this one by this one ?! But what did you gain ?" Well you gained something tremendous, because how do you solve an equation "Q(x)=0" ? Now I am doing in the case of Galois, the pupil of Picard-Vessiot : how do you solve it ? Well, you just take or you just take, you know, all polynomials over the field you are interested in so you take all polynomials $k[x]$ okay and you divide it by q. Now when you do that, of course x is a solution, so x satisfies $q(x) = 0$. Well, that's fine, yes, but now you know that all the roots are rational functions of this x so if you want to know if there is any relation you can imagine, rational relation between the roots, you plug it in, and it will hold (the computer will tell you if it holds or not) not up to epsilon, I mean, to tell you exactly, because what you do is you take this rational function of these roots and you put it here and you wonder whether or not you know it's zero in the quotient, maybe if it's a multiple of q. Now this is an amazing powerful thing so this completely formal way of solving an equation becomes a center, you know, of a power which is absolutely amazing in the hands of Galois. And in fact, what Galois writes was his problem was to find all rational relations between roots of an equation, it was not to find invariant functions or whatever as well. You know, you have, of course, you have the obvious relations which are the symmetric functions but what we found is that in general, there are other relations and it is this which led him to the Galois group. Okay, ? Okay !



So now, since I have very little time left what I would like to do, you know, is to give you end[8] by essentially some open questions. So there is a fact you know which is that, okay, when you look at this universal singular frame what it tells you roughly is that you know when we are letting the epsilon go to zero in the dim-reg stuff, well, we shouldn't try to land in the geometry world usefully. I mean the universal singular frame is telling us that we should follow this universal singular frame and in fact we should correct the geometry that we have so that you know the renormalization is actually taken into account by the geometry. Now a little step towards this has been taken in the book with Matilde, okay, well, what we have done is that we have given an incarnation of the space of dimension z. It's very tricky because it's a type 2 stuff but what we have found is that at the one loop level, it works perfectly well. Namely you know when you do the t'Hooft-Veltman stuff of renormalization and so on, you have to deal, because of Gauge theory and chiral properties (chiral anomaly) you have to deal with the chirality and there is a recipe which is called the Breitenlohner-Maison prescription. And at the one loop level, this prescription corresponds to taking the product of the standard geometry by a very specific spectral triple. So here I am alluding to what is non-commutative geometry, I don't want to spend time on it, I mean you know,

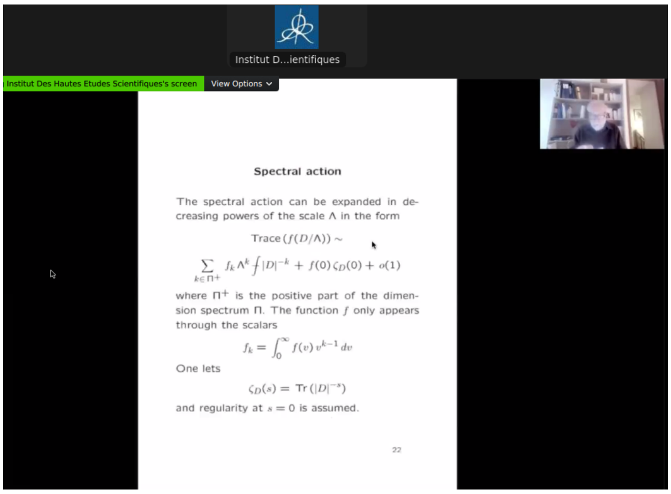


but what it is is a geometry which is based on... the geometry itself is defined by the propagator of fermions. Okay... properties propagator, and it is this which is the inverse of the Dirac which defines

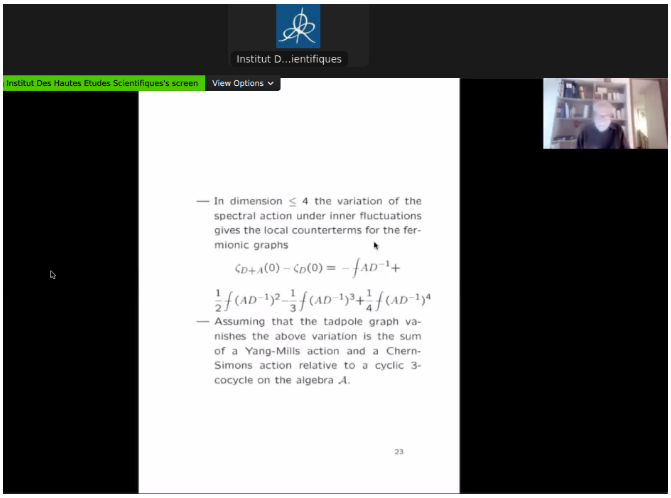
[8]correctly heared ?

the geometry. And the beauty is that Quantum Field Theory can already be taken into account, at the one loop level, because this propagator gets dressed by the common fields. So there are formal series in $\hbar$. Now what we have done, if you want, in developing this geometry, with Chamseddine and also with Walter Van Suijlekom, is that we have developed an action.

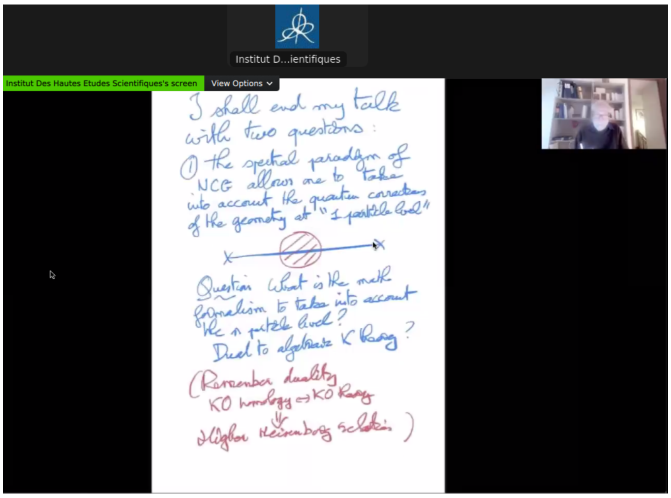
So now, I am not no longer, you know, dealing with any ? I want to give you a very very specific one and this action actually depends on the geometry only by the spectrum of this operator I mean which is the inverse of the propagator. So it's given by what is called spectral action and so on and so forth.



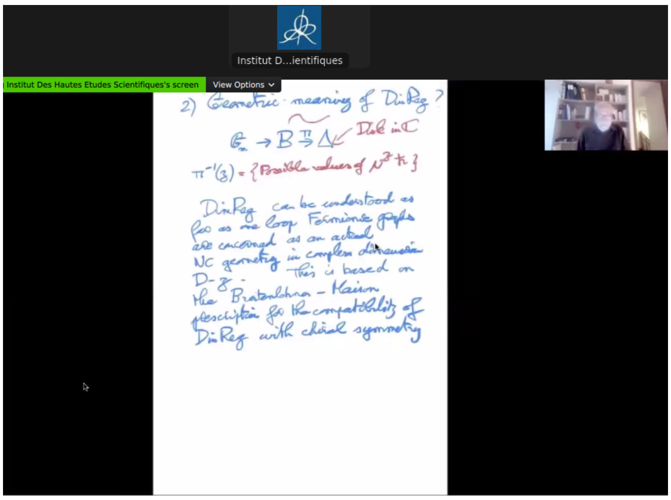
This spectral action, I mean, gives you in its expansion, gives you the important terms which occur in the action.



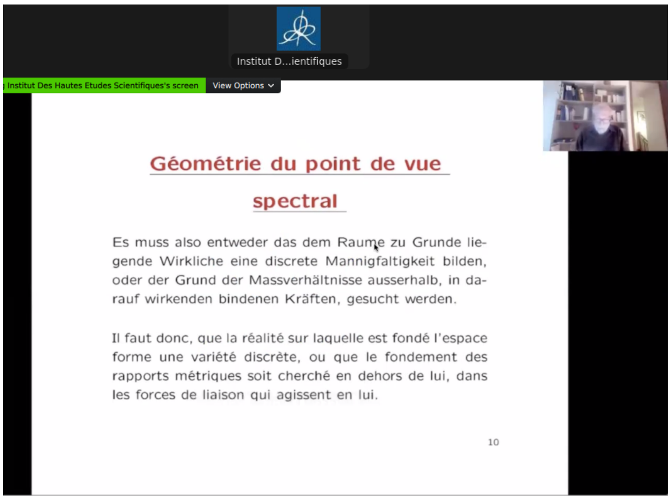
And I mean moreover I mean it allows you to start computing, when you look at you know inner fluctuations, and so start computing the various terms that you would have as counter terms.



Now I want to end if you want this talk with two questions. So as I said, the spectral paradigme of non-commutative geometry allows one to take into account the quantum corrections, as I said, you know, at the one particle level. Now it turns out that Quantum Field Theory tells us to know that, of course, restricting oneself to the one-loop level or to the one-particle level is a little bit too naive. And there is a fundamental question whose answer I don't know, I just have some guess about it which is : "what is the mathematical formalism which will allow us to take into account the n-particle level. And then *my* guess is that it's probably dual to the algebraic K-theory of Quillen. So there is this theory which is very sophisticated. The reason why I say that, you know, it's because of Schwinger terms and so on. Moreover, you know, in the more recent times, with Chamseddine, Mukhanov and van Suijlekom, what we have done is that, you know, we have obtained by analyzing the K-homology and the duality between K-homology and KO-theory, we have obtained either Heisenberg relations, which really, I mean, made me completely happy, because I had the impression, you know, that there was no longer any problem with the understanding of the gauge group, and so on, which appear in the Standard Model, they are forced upon you by this duality. And what happens, if you want, is that instead of, you know, as being in a very very, how to say, you know, arbitrary stuff, I mean we are in the stuff which because of this duality and so on, is the one that can be encapsulated by the simplest non-trivial recipe. Now the second question is, as I said before, what is the geometric meaning of dim-reg ?



Here I mention, you know, that when you look at this fibration that I mentioned before, you know the fiber over z which is in the disk here, are the possible values, it's very tricky, it's not of μ, it's the possible values of μ to the power z times $\hbar$, where $\hbar$ is the Planck constant. Now, as I said, you know, dim-reg has been understood as far as one-loop family graph are concerned, and this is based on this dim-reg and so-on, but the dream that I have is that one will be able to reconcile the understanding of the Standard Model coupled to Gravity coming from pure gravity on a fine structure of the space time with renormalization and that this reconciling, if you want, would somehow, you know, correspond to the sort of optimal realization of what Riemann was saying in his inaugural talk,



namely that the true geometry would be entirely based on the forces which are involved in the very small.

Okay so I will end on this point and I thank you for your patience.