SOME FOURIER TRANSFORMS IN PRIME-NUMBER THEORY

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1. Introduction

IT has been shown that, if the Riemann hypothesis is true, then the relationship between the distribution of the non-trivial zeros of the Riemann zeta-function and that of the logarithms of the powers of prime numbers can be expressed in various ways closely connected with Fourier and Hankel transformations.†

In the present paper I discuss two pairs of Fourier cosine-transforms which illustrate one aspect of this relationship. I have previously discussed a related pair of Hankel transforms,‡ but it is of some interest to construct a result which only involves the simpler Fourier cosine kernel. In each of the present pairs of transforms one function has simple discontinuities of magnitude $(2\pi)^{\frac{1}{2}}/x$ when the argument x passes through a zero of $\zeta(\frac{1}{2}+ix)$, while the transform has simple discontinuities of magnitude $1/mp^{\frac{1}{2}m}$ when the argument passes through a value of $\log p^m$, where p^m is a positive integral power of a prime p.

I also derive an alternative proof of an infinite-series formula for $N(T)$, the number of zeros of $\zeta(s)$ in $0 < I(s) < T$.

All the results described above require the assumption of the Riemann hypothesis. Some simpler analogous results requiring no unproved hypothesis are given in the last section.

2. First pair of transforms

Suppose that the Riemann hypothesis is true, and let $\frac{1}{2}\pm i\gamma_n$ $(n = 1,2,3,...;\ 0 < \gamma_n \leqslant \gamma_{n+1})$ run through the non-trivial zeros of $\zeta(s)$.§ Then

$$\sum_{\gamma_n<x}\frac{1}{\gamma_n} = \int_1^x \frac{dN(t)}{t} = \frac{N(x)}{x}+\int_1^x N(t)\frac{dt}{t^2} \tag{2.1}$$

† A. Wintner, *Duke J. of Math.* 10 (1943), 99–105 (99), and A. P. Guinand, *Proc. Lond. Math. Soc.* (to appear shortly), referred to in the sequel as (A).

‡ (A), Theorem 1.

§ If $\frac{1}{2}+i\gamma_n$ is a multiple zero of $\zeta(s)$ of order $r+1$ then we put

$$\gamma_n = \gamma_{n+1} = \dots = \gamma_{n+r}.$$

since $$\gamma_1 = 14{\cdot}13 > 1.$$

Now† $$N(x) = \frac{x}{2\pi}\log\frac{x}{2\pi} - \frac{x}{2\pi} + \frac{7}{8} + R(x),$$

where $$R(x) = O(\log x).$$

Hence (2.1) becomes

$$\frac{1}{2\pi}\log\frac{x}{2\pi} - \frac{1}{2\pi} + \frac{R(x)}{x} + \int_1^x \left(\log\frac{t}{2\pi} - 1\right)\frac{dt}{2\pi t} + \int_1^x R(t)\frac{dt}{t^2} + \frac{7}{8x} + \frac{7}{8}\int_1^x \frac{dt}{t^2}$$
$$= \frac{1}{4\pi}\log^2 x - \frac{1}{2\pi}\log 2\pi\log x - \frac{1}{2\pi}(1+\log 2\pi) +$$
$$+ \int_1^\infty R(t)\frac{dt}{t^2} + \frac{7}{8} - \int_x^\infty R(t)\frac{dt}{t^2}$$
$$= \frac{1}{4\pi}\log^2 x - \frac{1}{2\pi}\log 2\pi\log x + k + O\left(\frac{\log x}{x}\right), \tag{2.2}$$

where
$$k = \int_1^\infty R(t)\frac{dt}{t^2} - \frac{1}{2\pi}(1+\log 2\pi) + \frac{7}{8}$$
$$= \lim_{x\to\infty}\left\{\sum_{\gamma_n<x}\frac{1}{\gamma_n} - \frac{1}{4\pi}\log^2 x + \frac{1}{2\pi}\log 2\pi\log x\right\}.$$

Now put
$$F(x) = (2\pi)^{\frac{1}{2}}\left\{{\sum_{\gamma_n\leqslant x}}'\frac{1}{\gamma_n} - \frac{1}{4\pi}\log^2 x + \frac{1}{2\pi}\log 2\pi\log x - k\right\},$$

where the dash indicates that the terms $\gamma_n = x$, if they occur, are to be halved. Then

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\gamma_N} F(t)\cos xt\,dt$$
$$= 2\sum_{n=1}^{N-1}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \ldots + \frac{1}{\gamma_n}\right)\int_{\gamma_n}^{\gamma_{n+1}}\cos xt\,dt -$$
$$-\frac{1}{2\pi}\int_0^{\gamma_N}\log^2 t\cos xt\,dt + \frac{1}{\pi}\log 2\pi\int_0^{\gamma_N}\log t\cos xt\,dt - 2k\int_0^{\gamma_N}\cos xt\,dt$$

† E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 4.

$$=\frac{2}{x}\sum_{n=1}^{N-1}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\ldots+\frac{1}{\gamma_n}\right)(\sin x\gamma_{n+1}-\sin x\gamma_n)-$$

$$-\frac{1}{2\pi x}\log^2\gamma_N\sin x\gamma_N+\frac{1}{\pi x}\int_0^{\gamma_N}\log t\sin xt\frac{dt}{t}+$$

$$+\frac{1}{\pi x}\log 2\pi\log\gamma_N\sin x\gamma_N-\frac{1}{\pi x}\log 2\pi\int_0^{\gamma_N}\sin xt\frac{dt}{t}-\frac{2k}{x}\sin x\gamma_N$$

$$=-\frac{2}{x}\sum_{n=1}^{N}\frac{\sin x\gamma_n}{\gamma_n}+$$

$$+\frac{2}{x}\sin x\gamma_N\left\{\sum_{n=1}^{N}\frac{1}{\gamma_n}-\frac{1}{4\pi}\log^2\gamma_N+\frac{1}{2\pi}\log 2\pi\log\gamma_N-k\right\}+$$

$$+\frac{1}{\pi x}\int_0^{\infty}\log t\sin xt\frac{dt}{t}-\frac{1}{\pi x}\log 2\pi\int_0^{\infty}\sin xt\frac{dt}{t}+O\left(\frac{\log\gamma_N}{\gamma_N}\right). \qquad (2.3)$$

Now, by (2.2), the expression in the brackets $\{\ \ \}$ is $O\left(\frac{\log\gamma_N}{\gamma_N}\right)$. Also

$$\int_0^{\infty}\log t\sin xt\frac{dt}{t}=\int_0^{\infty}\log u\sin u\frac{du}{u}-\log x\int_0^{\infty}\sin u\frac{du}{u}$$
$$=-\tfrac{1}{2}\pi(C+\log x),$$

where C is Euler's constant. Hence (2.3) becomes

$$-\frac{2}{x}\sum_{n=1}^{N}\frac{\sin x\gamma_n}{\gamma_n}-\frac{1}{2x}(C+\log 2\pi x)+O\left(\frac{\log\gamma_N}{\gamma_N}\right). \qquad (2.4)$$

Now it is known that†

$$-\frac{\zeta'(s)}{\zeta(s)}=\sum_{p^m\leqslant y}{}'\frac{\log p}{p^{ms}}-\frac{y^{1-s}}{1-s}-\sum_{q=1}^{\infty}\frac{y^{-2q-s}}{2q+s}+\sum_{\rho}\frac{y^{\rho-s}}{\rho-s},$$

where ρ runs through the non-trivial zeros of $\zeta(s)$; $s\neq 1, -2q, \rho$; and $y>1$. If we put $s=\frac{1}{2}$, $\rho=\frac{1}{2}\pm i\gamma_n$, $y=e^x$, then it follows, after some manipulation,‡ that the series

$$2\sum_{n=1}^{\infty}\frac{\sin x\gamma_n}{\gamma_n}$$

† E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 81.
‡ See (A) for details.

converges to the sum

$$-\sum_{m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}}+4\sinh\tfrac{1}{2}x+\tfrac{1}{2}\log\coth\tfrac{1}{4}x+ \\ +\arctan e^{-\frac{1}{2}x}-\tfrac{1}{2}C-\tfrac{1}{4}\pi-\tfrac{1}{2}\log 8\pi. \quad (2.5)$$

Substituting (2.5) in (2.4) and making N tend to infinity, we have, after rearrangement,

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^{\to\infty} F(t)\cos xt\,dt$$

$$=\frac{1}{x}\left\{\sum_{m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}}-4\sinh\tfrac{1}{2}x+\tfrac{1}{2}\log\left(\frac{\tanh\frac{1}{4}x}{\frac{1}{4}x}\right)+(\tfrac{1}{4}\pi-\arctan e^{-\frac{1}{2}x})\right\}$$

$$= G(x), \text{ say.}$$

That is, $G(x)$ is the Fourier cosine-transform of $F(x)$, and the integral converges in the ordinary sense.

Further, it follows from (2.2) that $F(x)$ belongs to $L^2(0,\infty)$. Hence $G(x)$ also belongs to $L^2(0,\infty)$, and, since $F(x)$ is of bounded variation in any finite interval excluding the origin, it follows by a theorem of Titchmarsh† that

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^{\to\infty} G(t)\cos xt\,dt.$$

Thus we have

THEOREM 1. *If the Riemann hypothesis is true, and*

$$F(x) = (2\pi)^{\frac{1}{2}}\left\{\sum_{\gamma_n\leqslant x}{}' \frac{1}{\gamma_n}-\frac{1}{4\pi}\log^2 x+\frac{1}{2\pi}\log 2\pi\log x-k\right\},$$

where k is chosen so that $\lim_{x\to\infty} F(x) = 0$, and

$$G(x)=\frac{1}{x}\left\{\sum_{m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}}-4\sinh\tfrac{1}{2}x+\tfrac{1}{2}\log\left(\frac{\tanh\frac{1}{4}x}{\frac{1}{4}x}\right)+\tfrac{1}{4}\pi-\arctan e^{-\frac{1}{2}x}\right\},$$

then, for $x > 0$,

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^{\to\infty} G(t)\cos xt\,dt,$$

and

$$G(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^{\to\infty} F(t)\cos xt\,dt.$$

† E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), Theorem 58.

3. Second pair of transforms

We use the following lemma:†

LEMMA. *If $f(x)$ and $g(x)$ belong to $L^2(0,\infty)$ and are a pair of Fourier cosine-transforms, then the functions*

$$f(x)-\int\limits_x^\infty f(t)\frac{dt}{t},\qquad g(x)-\frac{1}{x}\int\limits_0^x g(t)\,dt$$

are also a pair of Fourier cosine-transforms belonging to $L^2(0,\infty)$.

Now suppose that $\{\alpha_n\}$ is an increasing sequence of positive numbers tending to infinity, that $\{a_n\}$ is another sequence not necessarily increasing or positive, that $A(x)$ is everywhere differentiable, and that

$$f(x)=\frac{1}{x}\Big\{\sum_{\alpha_n\leqslant x}{}' a_n-A(x)\Big\} \tag{3.1}$$

belongs to $L^2(0,\infty)$. If we choose M so that $\alpha_{M-1}<x\leqslant\alpha_M$, and put $N>M$, $\alpha_N=T$, then

$$\int\limits_x^T f(t)\frac{dt}{t}=(a_1+a_2+...+a_{M-1})\int\limits_x^{\alpha_M}\frac{dt}{t^2}+$$

$$+\sum_{n=M}^{N-1}(a_1+a_2+...+a_n)\int\limits_{\alpha_n}^{\alpha_{n+1}}\frac{dt}{t^2}-\int\limits_x^T A(t)\frac{dt}{t^2}$$

$$=\Big(\frac{1}{x}-\frac{1}{\alpha_M}\Big)\sum_{n=1}^{M-1}a_n+\sum_{n=M}^{N-1}(a_1+a_2+...+a_n)\Big(\frac{1}{\alpha_n}-\frac{1}{\alpha_{n+1}}\Big)+$$

$$+\Big[\frac{A(t)}{t}\Big]_x^T-\int\limits_x^T A'(t)\frac{dt}{t}$$

$$=\sum_{n=M}^{N}\frac{a_n}{\alpha_n}-\frac{1}{\alpha_N}\sum_{n=1}^{N}a_n+\frac{1}{x}\sum_{n=1}^{M-1}a_n+$$

$$+\frac{A(T)}{T}-\frac{A(x)}{x}-\int\limits_x^T A'(t)\frac{dt}{t}$$

† This follows immediately from E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), Theorem 69.

$$= \frac{1}{x}\Big\{\sum_{\alpha_n \leqslant x}' a_n - A(x)\Big\} - \Big\{\sum_{\alpha_n \leqslant x}' \frac{a_n}{\alpha_n} - \int_1^x A'(t)\frac{dt}{t}\Big\} +$$

$$+\Big[\Big\{\sum_{\alpha_n < T} \frac{a_n}{\alpha_n} - \int_1^T A'(t)\frac{dt}{t}\Big\} - \frac{1}{T}\Big\{\sum_{\alpha_n < T} a_n - A(T)\Big\}\Big]. \quad (3.2)$$

Now the first expression in (3.2) is equal to $f(x)$. Hence, making T tend to infinity, we have

$$f(x) - \int_x^\infty f(t)\frac{dt}{t} = \sum_{\alpha_n \leqslant x}' \frac{a_n}{\alpha_n} - \int_1^x A'(t)\frac{dt}{t} - K,$$

where

$$K = \lim_{T\to\infty}\Big[\Big\{\sum_{\alpha_n < T} \frac{a_n}{\alpha_n} - \int_1^T A'(t)\frac{dt}{t}\Big\} - \frac{1}{T}\Big\{\sum_{\alpha_n < T} a_n - A(T)\Big\}\Big].$$

Using a similar notation for b_n, β_n, and $B(x)$, and putting

$$g(x) = \sum_{\beta_n \leqslant x}' \frac{b_n}{\beta_n} - \int_1^x B'(t)\frac{dt}{t} - L,$$

where L is a constant, then, if $g(x)$ belongs to $L^2(0,\infty)$, an argument similar to the above shows that

$$g(x) - \frac{1}{x}\int_0^x g(t)\,dt = \frac{1}{x}\Big\{\sum_{\beta_n \leqslant x}' b_n - B(x)\Big\}.$$

If we now put $\beta_n = \gamma_n$, $b_n = (2\pi)^{\frac{1}{2}}$, $L = k$, and

$$B(x) = (2\pi)^{\frac{1}{2}}\Big\{\frac{x}{2\pi}\log\frac{x}{2\pi} - \frac{x}{2\pi}\Big\},$$

then $g(x)$ is equal to $F(x)$ of Theorem 1, and

$$g(x) - \frac{1}{x}\int_0^x g(t)\,dt = \frac{(2\pi)^{\frac{1}{2}}}{x}\Big\{N_0(x) - \frac{x}{2\pi}\log\frac{x}{2\pi} + \frac{x}{2\pi}\Big\}, \quad (3.3)$$

where $$N_0(x) = \sum_{\gamma_n \leqslant x}' 1 = \tfrac{1}{2}\{N(x-0)+N(x+0)\}.$$

Further, if we put

$$\alpha_n = m\log p, \qquad a_n = \log p/p^{\frac{1}{2}m}, \qquad A(x) = 4\cosh\tfrac{1}{2}x - \phi(x),$$

and $$\phi(x) = \tfrac{1}{2}\log\Big(\frac{\tanh\frac{1}{4}x}{\frac{1}{4}x}\Big) + \tfrac{1}{4}\pi - \arctan e^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}, \quad (3.4)$$

then $f(x)$ is equal to $G(x)$ of Theorem 1, and, by the lemma, the Fourier cosine-transform of (3.3) is accordingly

$$f(x)-\int\limits_x^\infty f(t)\frac{dt}{t} = \sum_{m\log p\leqslant x}{}' \frac{1}{mp^{\frac{1}{2}m}} - 2\int\limits_1^x \sinh\tfrac{1}{2}t\frac{dt}{t} - K + \int\limits_1^x \phi'(t)\frac{dt}{t}, \tag{3.5}$$

where

$$K = \lim_{T\to\infty}\Bigg[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}} - 2\int\limits_1^T \sinh\tfrac{1}{2}t\frac{dt}{t} - \frac{1}{T}\Bigg\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}} - 4\cosh\tfrac{1}{2}T\Bigg\} + \int\limits_1^T \phi'(t)\frac{dt}{t} - \frac{\phi(T)}{T}\Bigg]$$

$$= \int\limits_1^\infty \phi'(t)\frac{dt}{t} + 2\int\limits_0^1 \sinh\tfrac{1}{2}t\frac{dt}{t} + \lim_{T\to\infty}\Bigg[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}} - 2\int\limits_0^T \sinh\tfrac{1}{2}t\frac{dt}{t} - \frac{1}{T}\Bigg\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}} - 2e^{\frac{1}{2}T}\Bigg\}\Bigg].$$

If we write l for the last limit above, then

$$K = \int\limits_1^\infty \phi'(t)\frac{dt}{t} + 2\int\limits_0^1 \sinh\tfrac{1}{2}t\frac{dt}{t} + l.$$

Substituting this in (3.5) we have

$$f(x)-\int\limits_x^\infty f(t)\frac{dt}{t} = \sum_{m\log p\leqslant x}{}' \frac{1}{mp^{\frac{1}{2}m}} - 2\int\limits_0^x \sinh\tfrac{1}{2}t\frac{dt}{t} - l - \int\limits_x^\infty \phi'(t)\frac{dt}{t}. \tag{3.6}$$

Now the function

$$\int\limits_x^\infty \phi'(t)\frac{dt}{t} \tag{3.7}$$

does not reduce to any simple expression, and it is more convenient to eliminate it. The Fourier cosine-transform of (3.7) is

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^\infty \cos xt\,dt\int\limits_t^\infty \phi'(u)\frac{du}{u}$$

$$= \frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\sin xt\int\limits_t^\infty \phi'(u)\frac{du}{u}\right]_{t=0}^{t=\infty} + \frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^\infty \phi'(t)\sin xt\frac{dt}{t}. \tag{3.8}$$

Differentiating (3.4) we find that

$$\phi'(x) = \frac{1}{4}\left(\operatorname{cosech}\tfrac{1}{2}x - \frac{2}{x}\right) + \tfrac{1}{4}\operatorname{sech}\tfrac{1}{2}x - 2e^{-\frac{1}{2}x}.$$

On substituting this in (3.8) the integrated terms vanish, and (3.8) becomes

$$\frac{1}{4x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \left\{\operatorname{cosech}\tfrac{1}{2}t - \frac{2}{t}\right\}\sin xt\frac{dt}{t} + \frac{1}{4x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \operatorname{sech}\tfrac{1}{2}t\sin xt\frac{dt}{t} - $$

$$-\frac{2}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\frac{1}{2}t}\sin xt\frac{dt}{t}$$

$$= \frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(\tfrac{1}{4}I_1 + \tfrac{1}{4}I_2 - 2I_3), \quad \text{say.} \tag{3.9}$$

Now,† if $R(z) > -1$,

$$\log\Gamma(1+z) = \int_0^\infty \left\{ze^{-t} - \frac{1-e^{-zt}}{e^t-1}\right\}\frac{dt}{t}.$$

Putting $z = -\frac{1}{2} \pm ix$ and taking the difference we get

$$\frac{1}{2i}\log\frac{\Gamma(\frac{1}{2}+ix)}{\Gamma(\frac{1}{2}-ix)} = \operatorname{am}\Gamma(\tfrac{1}{2}+ix)$$

$$= \int_0^\infty \left\{xe^{-t} - \frac{e^{\frac{1}{2}t}}{e^t-1}\sin xt\right\}\frac{dt}{t}$$

$$= -\frac{1}{2}\int_0^\infty \left\{\operatorname{cosech}\tfrac{1}{2}t - \frac{2}{t}\right\}\sin xt\frac{dt}{t} + \int_0^\infty \left\{xe^{-t} - \frac{\sin xt}{t}\right\}\frac{dt}{t}$$

$$= -\tfrac{1}{2}I_1 + x\int_0^\infty \left\{\frac{\sin t}{t} - \frac{\sin xt}{xt}\right\}\frac{dt}{t} - x\int_0^\infty \left\{\frac{\sin t}{t} - e^{-t}\right\}\frac{dt}{t}. \tag{3.10}$$

Now the second expression in (3.10) is a Frullani integral, and is equal to $x\log x$. Further, for $R(s) > -1$,

$$\int_0^\infty (t^{s-2}\sin t - t^{s-1}e^{-t})\,dt = -\Gamma(s-1)\cos\tfrac{1}{2}s\pi - \Gamma(s)$$

$$= \frac{\Gamma(s+1)}{1-s}\left\{1 - \frac{1-\cos\frac{1}{2}s\pi}{s}\right\}.$$

† This follows from Binet's first integral formula for $\log\Gamma(z)$. Cf. C. A. Stewart, *Advanced Calculus* (London, 1940), 493.

Letting $s \to 0$, we find that

$$\int_0^\infty \left\{\frac{\sin t}{t} - e^{-t}\right\}\frac{dt}{t} = 1,$$

and hence (3.10) gives

$$I_1 = -2\{\operatorname{am}\Gamma(\tfrac{1}{2}+ix) - x\log x + x\}, \tag{3.11}$$

where $\operatorname{am}\Gamma(\frac{1}{2}+ix)$ is defined by putting $\operatorname{am}\Gamma(\frac{1}{2}) = 0$ and continuing analytically along the line $\frac{1}{2}+ix$.

Now†

$$\int_0^\infty \operatorname{sech}\tfrac{1}{2}t\cos xt\,dt = \pi\operatorname{sech}\pi x.$$

Integrating with respect to x we have

$$I_2 = \int_0^\infty \operatorname{sech}\tfrac{1}{2}t\sin xt\frac{dt}{t} = \pi\int_0^x \operatorname{sech}\pi u\,du = \arctan(\sinh\pi x). \tag{3.12}$$

Further

$$I_3 = \int_0^\infty e^{-\frac{1}{2}t}\sin xt\frac{dt}{t} = \arctan 2x. \tag{3.13}$$

Substituting (3.11), (3.12), (3.13) in (3.9) we find that (3.7) and

$$\frac{1}{(2\pi)^{\frac{1}{2}}x}[-\{\operatorname{am}\Gamma(\tfrac{1}{2}+ix) - x\log x + x\} + \tfrac{1}{2}\arctan(\sinh\pi x) - 4\arctan 2x] \tag{3.14}$$

are a pair of Fourier cosine-transforms of $L^2(0,\infty)$. Adding (3.7) to (3.6) and (3.14) to (3.3) we obtain another pair of Fourier cosine-transforms of $L^2(0,\infty)$. These functions are also of bounded variation in any finite interval, and hence it follows as in Theorem 1 that the Fourier integrals concerned converge in the ordinary sense. The result is:

THEOREM 2. *If the Riemann hypothesis is true, and*‡

$$H(x) = \sum_{m\log p\leqslant x}{}' \frac{1}{mp^{\frac{1}{2}m}} - 2\int_0^x \sinh\tfrac{1}{2}t\frac{dt}{t} - l, \tag{3.15}$$

† E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), 177 (7.1.6).

‡ If necessary we may substitute

$$2\int_0^x \sinh\tfrac{1}{2}t\frac{dt}{t} = \operatorname{li}(e^{\frac{1}{2}x}) - \operatorname{li}(e^{-\frac{1}{2}x})$$

in (3.15).

where

$$l = \lim_{T\to\infty}\left[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}}-2\int\limits_0^T \sinh\tfrac{1}{2}t\frac{dt}{t}-\frac{1}{T}\left\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{2}T}\right\}\right], \tag{3.16}$$

and

$$K(x) = \frac{(2\pi)^{\frac{1}{2}}}{x}\left\{N_0(x)-\frac{1}{2\pi}\operatorname{am}\Gamma(\tfrac{1}{2}+ix)+\frac{x}{2\pi}\log 2\pi+ \right.$$
$$\left. +\frac{1}{4\pi}\arctan(\sinh\pi x)-\frac{2}{\pi}\arctan 2x\right\},$$

then, for $x > 0$, $$H(x) \doteq \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^{\to\infty} K(t)\cos xt\,dt,$$

and $$K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^{\to\infty} H(t)\cos xt\,dt. \tag{3.17}$$

4. The formula for $N(x)$

As before, putting $\alpha_n = m\log p$, $\alpha_N = T$, $a_n = \log p/p^{\frac{1}{2}m}$, the right-hand side of (3.17) is

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int\limits_0^{\to\infty}\left\{\sum_{\alpha_n\leqslant t}{}'\frac{a_n}{\alpha_n}-2\int\limits_0^t\sinh\tfrac{1}{2}u\frac{du}{u}-l\right\}\cos xt\,dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{N\to\infty}\left[\sum_{n=1}^{N-1}\left(\frac{a_1}{\alpha_1}+\frac{a_2}{\alpha_2}+\dots+\frac{a_n}{\alpha_n}\right)\int\limits_{\alpha_n}^{\alpha_{n+1}}\cos xt\,dt - \right.$$
$$\left. -2\int\limits_0^T\cos xt\,dt\int\limits_0^t\sinh\tfrac{1}{2}u\frac{du}{u}-\frac{l}{x}\sin xT\right]$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{N\to\infty}\left[\frac{1}{x}\sum_{n=1}^{N-1}\left(\frac{a_1}{\alpha_1}+\frac{a_2}{\alpha_2}+\dots+\frac{a_n}{\alpha_n}\right)(\sin x\alpha_{n+1}-\sin x\alpha_n) - \right.$$
$$\left. -\frac{2}{x}\int\limits_0^T\sinh\tfrac{1}{2}u(\sin xT-\sin xu)\frac{du}{u}-\frac{l}{x}\sin xT\right]$$

$$= \frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{N\to\infty}\left[-\sum_{n=1}^{N}\frac{a_n}{\alpha_n}\sin x\alpha_n + \right.$$
$$\left. +\sin xT\left\{\sum_{n=1}^{N}\frac{a_n}{\alpha_n}-2\int\limits_0^T\sinh\tfrac{1}{2}u\frac{du}{u}-l\right\}+2\int\limits_0^T\sinh\tfrac{1}{2}u\sin xu\frac{du}{u}\right].$$

By (3.16) this is equal to

$$\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{N\to\infty}\left[-\sum_{n=1}^{N}\frac{a_n}{\alpha_n}\sin x\alpha_n+\int_0^T e^{\frac{1}{4}u}\sin xu\frac{du}{u}+\right.$$
$$\left.+\frac{\sin xT}{T}\left\{\sum_{n=1}^{N}a_n-2e^{\frac{1}{4}T}\right\}\right]-\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\frac{1}{4}u}\sin xu\frac{du}{u}$$
$$=-\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lim_{T\to\infty}\left[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}}\sin(xm\log p)-\int_0^T e^{\frac{1}{4}u}\sin xu\frac{du}{u}-\right.$$
$$\left.-\frac{\sin xT}{T}\left\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{4}T}\right\}\right]-\frac{1}{x}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\arctan 2x,$$

and by (3.17) this expression is equal to $K(x)$. Substituting the value of $K(x)$ from Theorem 2 and rearranging the terms, we obtain the result:

THEOREM 3.† *If the Riemann hypothesis is true and* $x \geqslant 0$, *then*

$$\tfrac{1}{2}\{N(x-0)+N(x+0)\}-\left(\frac{x}{2\pi}\log\frac{x}{2\pi}-\frac{x}{2\pi}\right)$$
$$=-\frac{1}{\pi}\lim_{T\to\infty}\left[\sum_{m\log p<T}\frac{1}{mp^{\frac{1}{2}m}}\sin(xm\log p)-\int_0^T e^{\frac{1}{4}u}\sin xu\frac{du}{u}-\right.$$
$$\left.-\frac{\sin xT}{T}\left\{\sum_{m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{4}T}\right\}\right]+\frac{1}{2\pi}\{\operatorname{am}\Gamma(\tfrac{1}{4}+ix)-x\log x+x\}-$$
$$-\frac{1}{4\pi}\arctan(\sinh\pi x)+\frac{1}{\pi}\arctan 2x,$$

where $\operatorname{am}\Gamma(\frac{1}{4}+ix)$ *is defined by taking* $\operatorname{am}\Gamma(\frac{1}{4})=0$ *and continuing* $\Gamma(s)$ *along the line* $s=\frac{1}{4}+ix$.

Theorem 3 is, in a sense, analogous to (2.5), and the argument of this section can be reversed to deduce Theorem 2 from Theorem 3.

5. Simpler pairs of transforms

The pairs of transforms discussed in §§ 2, 3 have simpler analogues with regularly spaced discontinuities. For example, the functions

$$\frac{1}{x}\left\{\sum_{n\leqslant x}{}' 1-x\right\}, \qquad C-\left\{\sum_{n\leqslant x}{}'\frac{1}{n}-\log x\right\} \tag{5.1}$$

† See (A), Theorem 2, for an alternative proof and discussion of the result.

are a pair of transforms with respect to the kernel $2\cos 2\pi x$. Further, if $\chi(n)$ is a real primitive character *modulo* κ ($\kappa > 1$) and

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n} = S,$$

then the functions

$$\frac{1}{x}\sum_{n\leqslant x}' \chi(n), \qquad S - \sum_{n\leqslant x}' \frac{\chi(n)}{n}$$

are a pair of transforms with respect to the kernel $2\kappa^{-\frac{1}{2}}\cos(2\pi x/\kappa)$ if $\chi(-1) = 1$, or with respect to the kernel $2\kappa^{-\frac{1}{2}}\sin(2\pi x/\kappa)$ if

$$\chi(-1) = -1.$$

These results are easily proved by the method of § 2, using an ordinary Fourier series in the place of (2.5). The pair of transforms (5.1) can also be derived as a limiting case of an earlier result.†

† A. P. Guinand, *J. of London Math. Soc.* 14 (1939), 97–100. Let $s \to 1-0$ in (1).