ON THE DOMAIN OF VARIABILITY OF FOURIER COEFFICIENTS OF POSITIVE HARMONIC FUNCTIONS

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Introduction

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1. The following work deals with investigations I have pursued in Band LXIV of the Mathematischen Annalen[1] in connection with Picard's Theorem. Since this publication, I have not only obtained new results, but also rendered the entire presentation in a more perspicuous form. Because of this, it seems to me useful to fashion the following work independently of earlier results.

There, amongst other things, it was argued that the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \tag{1}$$

can only represent a harmonic function $U(r, \theta)$, which is *regular* and *positive* for $r < 1$, if the coefficients a_n and b_n are bounded by known limits. In particular the point in $2n$-dimensional space with the coordinates

$$a_1,\ b_1,\ a_2,\ b_2,\ \ldots,\ a_n,\ b_n$$

must necessarily be in the interior or, at least, on the boundary of the smallest convex body K_{2n} lying in this space that contains the curve with the coordinates

$$\cos\theta,\ \sin\theta,\ \cos 2\theta,\ \sin 2\theta,\ \ldots,\ \cos n\theta,\ \sin n\theta,$$

where θ denotes a parameter.

2. I had already proven a converse of this theorem, in which I showed that if the numbers

$$a_1,\ b_1,\ a_2,\ b_2,\ \ldots,\ a_n,\ b_n$$

represent the coordinates of a point which does not lie outside the body K_{2n}, at least one harmonic function exists which is regular and positive on the unit disc and whose expansion (1) begins with the given $2n$ coefficients. p.194

[1] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen* [Mathematische Annalen, Bd. LXIV (1907), S. 95-115].

Rend. Circ. Matem. Palermo, t. XXXII (2° sem. 1911). — Stampato il 16 agosto 1911.

In this work I shall now give another converse of the same theorem, which is significant for applications, in that I show that the following theorem holds:

If the $2n$ *coefficients*

$$a_1,\, b_1,\, a_2,\, b_2,\, \ldots,\, a_n,\, b_n$$

of the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

represent the coordinates of a point which lies in the interior of the particular convex body K_{2n} *for every* n, *then this series represents a regular positive harmonic function on the unit disc.*

3. For the boundary of the body K_{2n} I had deployed a parametric representation which completely described the boundary through trigonometric polynomials. Mr Toeplitz has now succeeded in discovering a very beautiful and very simple representation of this boundary in determinantal form, through which all theorems discovered can be rendered in an algebraic form.[2]

The transformation of the parametric representation of the body K_{2n} to Toeplitz's representation has led us to a system of equations, which has been considered multiple times in the literature. Namely, one faces the same algebraic problem when putting binary forms of odd rank into canonical form,[3] in mechanical quadrature,[4] and in the theory of continued fractions.[5]

This observation has allowed us to convey Toeplitz's result without making use of the theory of quadratic forms of finitely many variables.

As well as Toeplitz's results, a purely algebraic representation of my earlier theorems is achieved by Mr E. Fischer,[6] to whom I also owe the above literature.

Mr Friedrich Riesz has given a method based on Stieltjes's integral, which leads to
p.195 entirely new proofs for many results.[7]

4. The considerations in the following work remain predominantly geometric, just as in my earlier work. I begin by dealing with convex bodies in n-dimensional space by

[2] O. Toeplitz: *a*) *Über die* Fourier'*sche Entwickelung positiver Funktionen* [Rendiconti del Circolo Matematico di Palermo, Bd. XXXII (2. Semester 1911), S. 191–192]; *b*) *Zur Theorie der quadratischen Formen von unendlichvielen Veränderlichen* [Nachrichten von der Kgl. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, Jahrgang 1910, S. 489–506]

[3] F. Faà di Bruno, *Einleitung in die Theorie der binären Formen* (deutsch bearbeitet von Dr. Th. Walter) (Leipzig, Teubner, 1881), S. 94.

[4] E. Heine, *Handbuch der Kugelfunctionen, Theorie und Anwendungen* (Berlin, Reimer), Bd. II (1881), Kap. I.

[5] Hermite et Stieltjes, *Correspondance d'*Hermite *et de* Stieltjes (Paris, Gauthier-Villars, 1905); Ch. Possé, *Sur quelques application des fractions continues algébriques* (St. Pétersbourg, 1886), S. 93.

[6] E. Fischer, *Über das* Carathéodory'*sche Problem, Potenzreihen mit positiven reellen Teil betreffend* [Rendiconti del Circolo Matematico di Palermo, Bd. XXXII (2. Semester 1911), S. 240-256 (Sitzung 11. Juni 1911)].

[7] F. Riesz, *Sur certains systèmes singuliers d'équations intégrales* [Annales Scientifiques de l'École Normale Supérieure (Paris), Ser. III, Bd. XXVIII (1911), S. 33-62].

analogy with Minkowski and pre-occupy myself particularly with the smallest convex body which contains a given closed set of points $\mathfrak{M}$. Here it is very advantageous to consider the points of these shapes as centres of mass of positive assignments of mass to $\mathfrak{M}$, an observation I owe to a communication from Mr G. Herglotz by word of mouth.

In particular, the general results will then be applied to determine the properties of the smallest convex body of $2n$-dimensional space

$$x_1,\, y_1,\, x_2,\, y_2,\, \ldots,\, x_n,\, y_n,$$

which contains the curve with equation

$$x_1 = \cos\theta,\, y_1 = \sin\theta,\, x_2 = \cos 2\theta,\, y_2 = \sin 2\theta,\, \ldots,\, x_n = \cos n\theta,\, y_n = \sin n\theta.$$

The converse theorems about positive harmonic functions mentioned at the beginning of the introduction are then obtained very easily using geometric results.

§I.
The smallest convex domain which contains a closed set of points

5. The theorems which we wish to establish will be very perspicuous and easy to understand if we avail ourselves of the language of geometry in n-dimensional space. However, one must not forget here that the geometry of higher spaces *cannot* be dealt with intuitively, analogously to the theorems of ordinary geometry, but instead that a rigorous arithmetisation is feasible here, and carried out. In order to remind ourselves of this, we shall briefly explain the most elementary concepts:

A point $\mathfrak{x}$ of n-dimensional space is given if one knows its coordinates

$$x_1,\, x_2,\, \ldots,\, x_n.$$

The distance E_{xy} between two points $\mathfrak{x}$ and $\mathfrak{y}$ is given by the formula

$$E_{xy} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}. \tag{1}$$

Equations like

$$x_1 = \phi_1(t),\, x_2 = \phi_2(t),\, \ldots,\, x_n = \phi_n(t),$$

whose right side depends on a parameter t, represent a curve; if the $\phi(t)$ are all linear functions of t, and so of the form

$$x_k = m_k t + n_k,$$

then the curve is a line. The quantities m_k are called the direction coefficients of the line. The angle θ between two lines $x_k = m_k t + n_k$ and $x_k = m'_k t + n'_k$ is given by the *p.196* formula

$$\cos\theta = \frac{\sum_{k=1}^{n} m_k m'_k}{\left(\sum_{k=1}^{n} m_k^2 \cdot \sum_{k=1}^{n} {m'_k}^2\right)^{\frac{1}{2}}}. \tag{2}$$

If $\sum m_k m'_k = 0$, then the two lines are called orthogonal to each other.

The concepts *distance* and *angle* are absolute invariants under every *orthogonal transformation* of n-dimensional space, as an easy calculation shows. If three arbitrary points are given, then one can always find such a transformation which sends these points into the 2-dimensional plane of x_1 and x_2 (in which we therefore have $x_3 = 0, x_4 = 0, \ldots, x_n = 0$). If one chooses this last plane as a plane of projection, than one can consider the triangle whose corners are the transformed points and all metric properties of this triangle are the same as those of the original.

By a *hyperplane* of $\mathbb{R}^n$ we mean an $(n-1)$-dimensional linear manifold, and so a shape that is given by the equation

$$u_1x_1 + u_2x_2 + \cdots + u_nx_n + d = 0. \tag{3}$$

The u_k are constants which should not all vanish, and one can always assume that they satisfy the condition

$$u_1^2 + u_2^2 + \cdots + u_n^2 = 1.$$

We then say that the equation of the hyperplane is in *normal form.* Two hyperplanes whose equations are in normal form are *parallel* if the corresponding coefficients of x_k in both equations are the same or opposite.

The distance p of a point $(a_1, a_2, \ldots, a_n)$ from the hyperplane (3) is given by the formula

$$p = u_1a_1 + u_2a_2 + \cdots + u_na_n + d. \tag{4}$$

Every hyperplane divides space into two regions, which encompass all of those points of space for which p has the same sign.

By a p-dimensional linear manifold P we mean the set of points of $\mathbb{R}^n$ which comprise the intersection of $(n-p)$ linearly independent hyperplanes with non-empty intersection. Every hyperplane which is parallel to a hyperplane containing all the points of P is called *parallel to* P. One can find at least one hyperplane parallel to P through every point $\mathfrak{a}$ which is not a point of P, because it is certain that amongst the $(n-p)$ hyperplanes which define P, there is one which does not contain $\mathfrak{a}$.

If two different points

$$\begin{aligned} \mathfrak{a}&: \quad a_1,\, a_2,\, \ldots,\, a_n \\ \mathfrak{b}&: \quad b_1,\, b_2,\, \ldots,\, b_n \end{aligned}$$

p.197 are given, then the line

$$x_k = a_k t + (1-b)b_k \qquad (k=1,2,\dots,n) \tag{5}$$

contains both points $\mathfrak{a}$ and $\mathfrak{b}$. The piece of this line which one obtains if one lets t vary between nought and one, if therefore

$$0 \leqslant t \leqslant 1,$$

is called the *segment* $\overline{\mathfrak{ab}}$ connecting the points $\mathfrak{a}$ and $\mathfrak{b}$.

6. A *convex domain*[8] $\mathfrak{K}$ of $\mathbb{R}^n$ is a set of points which has the following properties:

A. It is closed, i.e., it contains its accumulation points.

B. If two points $\mathfrak{a}$ and $\mathfrak{b}$ belong to the domain $\mathfrak{K}$, then the same should hold for the whole segment $\overline{\mathfrak{ab}}$ which connects these points.

From both these properties it follows immediately that the set of points $\mathfrak{K}$ must be *perfect* (i.e., each of its points must be an accumulation point) so long as it contains at least two distinct points and, furthermore, is bounded.

If a convex domain $\mathfrak{K}$ and a line in space is given, then the two shapes have in common either no point, only one point, or a segment connecting two points; a fourth possibility, where the points in common between the two shapes are isolated or separated, is ruled out by the definition.

There exist convex domains: every segment $\overline{\mathfrak{ab}}$ is, e.g., such a domain.

Every point which is not contained in the set of points $\mathfrak{K}$ is called an *exterior* point of $\mathfrak{K}$.

By the *boundary* of a convex domain we understand the set of points which are accumulation points of exterior points. In particular, the endpoints of the segment which a line has in common with a domain $\mathfrak{K}$ are points of the boundary of $\mathfrak{K}$.

Now let $\mathfrak{a}$ be an exterior point of $\mathfrak{K}$; because the set of points $\mathfrak{K}$ is closed and the distance between two points is a continuous function of its $2n$ arguments, by the familiar theorem of Weierstrass, the lower bound of the distance of $\mathfrak{a}$ to the points of $\mathfrak{K}$ is attained for a particular point

$$\mathfrak{p}: \quad p_1, p_2, \dots, p_n.$$

We consider the hyperplane

$$(p_1 - a_1)(x_1 - a_1) + (p_2 - a_2)(x_2 - a_2) + \cdots + (p_n - a_n)(x_n - a_n) = 0, \tag{6}$$

which one obtains, if one sets the inner product of the n-dimensional vectors $(\mathfrak{p} - \mathfrak{a})$ p.198

[8]The theory of convex domains and bodies has been rendered in great generality by H. Minkowski. What follows is a liberal adaptation of some of his work. Vgl.: H. Minkowski, *Geometrie der Zahlen* (Leipzig, Teubner, 1896), Kap. I und II; H. Minkowski, *Volumen und Oberfläche* [Mathematische Annalen, Bd. LVII (1903), S. 447-495]; H. Minkowski, *Gesammelten Schriften* (Leipzig, Teubner), Bd. II (1911), S. 131 u. f.

and $(\mathfrak{x}-\mathfrak{a})$ equal to zero. This hyperplane contains the point $\mathfrak{a}$ and every line which is contained in (6) and goes through $\mathfrak{a}$ is orthogonal to the segment $\overline{\mathfrak{a}\mathfrak{p}}$. From this it follows, however, that the hyperplane (6) lies completely outside $\mathfrak{K}$: namely, if a point $\mathfrak{q}$ of (6) belongs to the domain $\mathfrak{K}$, then this must also be the case for the whole segment $\overline{\mathfrak{p}\mathfrak{q}}$. Considering the triangle $\mathfrak{a}\mathfrak{p}\mathfrak{q}$ that, as explained above, we can map into the projection plane, leads to a contradiction, however: indeed, this triangle is right-angled at $\mathfrak{a}$ and its hypoteneuse $\overline{\mathfrak{p}\mathfrak{q}}$ consists of points of $\mathfrak{K}$; the lower bound of the distance from $\mathfrak{a}$ to $\mathfrak{K}$ will hence not be attained at the point $\mathfrak{p}$. From this fact it follows further that one of the two regions into which the space $\mathbb{R}^n$ is divided by the hyperplane (6) cannot contain a single point of $\mathfrak{K}$; if two points $\mathfrak{p}$ and $\mathfrak{r}$ of $\mathfrak{K}$ lie on different sides of the hyperplane (6), then the segment $\mathfrak{p}\mathfrak{r}$ which belongs to $\mathfrak{K}$ must contain a point $\mathfrak{q}$ of (6), which, as we have seen, is not possible.

Every hyperplane which, like (6), does not separate the points of $\mathfrak{K}$ and does not contain a single point from $\mathfrak{K}$, is called a *shelf* of the convex domain.

Every point $\mathfrak{b}$ of the boundary is, by assumption, an accumulation point of points which lie outside $\mathfrak{K}$. If one considers a countable set of points $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n, \ldots$, which have $\mathfrak{b}$ as an accumulation point, then one can construct a shelf of the convex domain through each of these points, whose equation

$$u_{k1}x_1 + u_{k2}x_2 + \cdots + u_{kn}x_n + d_k = 0$$

is to be thought of as written in normal form. From these infinitely many equations one can select a subsequence for which the limit values

$$\lim_{k=\infty} u_{kj} = u_j \quad \text{and} \quad \lim_{k=\infty} d_k = d \quad {\scriptstyle (j=1,2,\ldots,n)}$$

exist; then the hyperplane

$$u_1x_1 + u_2x_2 + \cdots + u_nx_n + d = 0 \tag{7}$$

has the following properties: *a*) it contains the point $\mathfrak{b}$; all points of $\mathfrak{K}$ lie on one and the same side of the hyperplane (7). Every such hyperplane is called a *supporting hyperplane* of the domain $\mathfrak{K}$, following Minkowski.

Our results may be summarised in the following way: *every exterior point of a convex domain is contained in a shelf and every point of its boundary is contained in a supporting hyperplane.*

7. We now consider an *arbitrary* bounded closed set of points $\mathfrak{M}$ and again call hyperplanes which do not separate the points of $\mathfrak{M}$ and either contain points of $\mathfrak{M}$ or do not, respectively either *supporting hyperplanes* or *shelves* of $\mathfrak{M}$.

The points of $\mathbb{R}^n$ are then split into two classes, according to whether one can place a shelf of $\mathfrak{M}$ through them, or not. The totality $\mathfrak{K}$ of points through which one *cannot* place a shelf of $\mathfrak{M}$ forms a convex domain, as we shall prove:

a) The set of points $\mathfrak{K}$ is bounded; namely, if A denotes the maximum distance of the points of $\mathfrak{M}$ from the origin $\mathfrak{o}$, then it follows from the fact that every point whose distance from $\mathfrak{o}$ is greater than A is not a point of $\mathfrak{K}$, that every hyperplane whose distance from $\mathfrak{o}$ is greater than A is a shelf.

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b) The set of points $\mathfrak{K}$ is closed. Let $\mathfrak{a}$ be an accumulation point of the points $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots$, which all belong to the set of points $\mathfrak{K}$. If $\mathfrak{a}$ is not a point of $\mathfrak{K}$, then a shelf s of $\mathfrak{M}$ goes through $\mathfrak{a}$, whose distance from this closed set of points is equal to p. One can place a shelf of $\mathfrak{M}$ through every point whose distance from $\mathfrak{a}$ is less than p, namely the parallel hyperplane to s; hence the point $\mathfrak{a}$ cannot be an accumulation point of points which all belong to $\mathfrak{K}$.

c) If $\mathfrak{a}$ and $\mathfrak{b}$ are two distinct points of $\mathfrak{K}$ and $\mathfrak{c}$ is a point of the segment $\overline{\mathfrak{a}\mathfrak{b}}$, then $\mathfrak{c}$ belongs to $\mathfrak{K}$. Indeed, if it were possible to place a shelf s of $\mathfrak{K}$ through $\mathfrak{c}$, then, by assumption, the hyperplane s would contain neither $\mathfrak{a}$ nor $\mathfrak{b}$; these points then lie on different sides of s; and of the two hyperplanes parallel to s going through $\mathfrak{a}$ and $\mathfrak{b}$, one of them would have to be a shelf, which is impossible.
The set of points is therefore a convex domain, which, moreover, contains all points of $\mathfrak{M}$. Let $\mathfrak{K}'$ be a second convex domain which has the same property. One can place a shelf of $\mathfrak{K}'$, and so of $\mathfrak{M}$, through every point $\mathfrak{a}'$ which lies exterior to $\mathfrak{K}'$. Hence the point $\mathfrak{a}'$ also lies exterior to $\mathfrak{K}$, by which we have proven that all points of $\mathfrak{K}$ must be contained in $\mathfrak{K}'$.

The domain $\mathfrak{K}$ is, in other words, the smallest convex domain which contains the set of points $\mathfrak{M}$.

If the entire set $\mathfrak{M}$ lies in a p-dimensional linear manifold L_p, $p < n$, then this is also the case for $\mathfrak{K}$. This is since one can place at least one hyperplane parallel to L_p through every point exterior to L_p, and this hyperplane will be a shelf of $\mathfrak{M}$.

8. Let $\mathfrak{b}$ be some point on the boundary of $\mathfrak{K}$ and s a supporting hyperplane of $\mathfrak{K}$ through this point. The hyperplane s *necessarily* contains points of $\mathfrak{M}$; indeed, otherwise s would be a shelf of $\mathfrak{M}$ and $\mathfrak{b}$ could not be a point of $\mathfrak{K}$.

We denote by $\mathfrak{M}'$ the closed collection of points which is formed by the totality of the points of $\mathfrak{M}$ which lie on s and by $\mathfrak{K}'$ the smallest convex body which contains $\mathfrak{M}'$ and so lies entirely in s. $\mathfrak{K}'$ naturally forms a part of $\mathfrak{K}$; we shall, however, show conversely that all points of $\mathfrak{K}$ which lie on s are contained in $\mathfrak{K}'$.

Indeed, let $\mathfrak{a}$ be a point of s which lies exterior to $\mathfrak{K}'$; then there is a shelf of $\mathfrak{K}'$ through $\mathfrak{a}$.

If one uses the abbreviations t, s for the left-hand sides of the equations

$$\sum_{k=1}^{n} v_k x_k = 0, \quad \sum_{k=1}^{n} u_k x_k = 0,$$

which are written in normal form and represent the hyperplanes t, s, one can further stipulate that $s \geqslant 0$ for all points of $\mathfrak{M}$ and $t < 0$ for all points of $\mathfrak{M}'$. The totality of points of $\mathfrak{M}$ for which $t \geqslant 0$ forms a closed set of points $\mathfrak{M}''$ which cannot contain a single point of s. Let h be the distance between $\mathfrak{M}''$ and s; we denote the largest distance of a point of the set of points $\mathfrak{M}''$ from $t = 0$ by k. Let us consider the expression $(s - \frac{h}{2k} t)$; for the points of $\mathfrak{M}''$ we have *p.200*

$$s \geqslant h > 0, \quad t < k$$

and consequently

$$s - \frac{h}{2k}t > \frac{h}{2} > 0;$$

this expression is positive and non-zero for the remaining points of $\mathfrak{M}$, for which $t < 0$ and $s \geqslant 0$. The hyperplane

$$s - \frac{h}{2k}t = 0$$

is therefore a shelf of $\mathfrak{M}$, by which we have shown that $\mathfrak{a}$ is an exterior point of $\mathfrak{K}$.

9. The points of the smallest convex domain $\mathfrak{K}$ which contains a closed set $\mathfrak{M}$ can all be given as centres of mass of the set $\mathfrak{M}$ under assignments of mass in which all masses in consideration are positive and the total mass is 1.

For one-dimensional space the theorem is self-evident: one assigns the extreme points of the set $\mathfrak{M}$ with the masses t and $(1-t)$; then the centre of mass of these two masses traces out the whole domain $\mathfrak{K}$, including the limits, as t varies from zero to one.

We now assume that the theorem holds for $(n-1)$-dimensional space (or, equivalently, for $(n-1)$-dimensional linear manifolds in n-dimensional space), and consider $\mathfrak{K}$ and $\mathfrak{M}$ in n-dimensional space. Let $\mathfrak{c}$ be a point of $\mathfrak{K}$ and $\mathfrak{m}$ an arbitrary point of $\mathfrak{M}$, which we connect to $\mathfrak{c}$ by a line; we extend, if necessary, the segment $\mathfrak{mc}$ to the point $\mathfrak{b}$ where it intersects the boundary of $\mathfrak{K}$. We place through $\mathfrak{b}$ a supporting hyperplane of $\mathfrak{K}$ which contains the subset $\mathfrak{M}'$ of $\mathfrak{M}$. By assumption, we can make an allocation of positive mass to $\mathfrak{M}'$ of total mass one such that the centre of mass lies at $\mathfrak{b}$; if one multiplies all of these masses by a positive factor t, then one does not change the centre of mass; if one now adds to these masses a mass $(1-t)$ to $\mathfrak{m}$, so that $\mathfrak{c}$ lies in the segment $\mathfrak{mb}$ (or on one of its endpoints), then one sees that for a suitable value of t between zero and one, the centre of mass lies at $\mathfrak{c}$. Hence, our proof has shown that *every* point of $\mathfrak{K}$ can be viewed as the centre of mass of at most $(n+1)$ positive masses which are all points of the set $\mathfrak{M}$.

p.201 Conversely, the centre of mass $\mathfrak{c}$ of any allocation of positive mass to $\mathfrak{M}$ is a point of the domain $\mathfrak{K}$. If this were not the case, one would be able to place a shelf through $\mathfrak{c}$ whose distance d from $\mathfrak{M}$ was non-zero. The centre of mass of every discrete or continuous allocation of mass would then however also lie on the same side of the shelf as $\mathfrak{M}$ and its distance from this hyperplane would be at least d; it could therefore never coincide with $\mathfrak{c}$, contrary to assumption.

10. If a convex domain K of n-dimensional space contains $(n+1)$ points, which do not all lie in an $(n-1)$-dimensional hyperplane, then K is called a *convex body.* A convex body is characterised by the fact that it contains *interior points.*

A point i is called an interior point of K if a certain neighbourhood of i consists exclusively of points of the body K, or, equivalently, if the lower bound of the distance between i and the exterior points of K is non-zero.

Let $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be $(n+1)$ points of $\mathfrak{M}$ which do not all lie in an $(n-1)$-dimensional hyperplane: I shall prove that the centre of mass $\mathfrak{c}$ of $(n+1)$ positive masses, where none vanish and which are concentrated in these points, coincides with an interior point of K. Let s be an arbitrary hyperplane through $\mathfrak{c}$; at least one of the points $\mathfrak{a}$, for example

$\mathfrak{a}_0$, does not lie in s; $\mathfrak{c}$ coincides with an *interior* point of the segment connecting $\mathfrak{a}_0$ with the centre of mass of the masses which are concentrated in $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$. This segment however lies entirely in K and passes through s to the other side; consequently, s is neither a supporting hyperplane nor a shelf of K, whereby it is proven that $\mathfrak{c}$ can neither lie on the boundary of K, nor exterior to it.

Having proven the existence of an interior point, it is now easy to show that the points of the boundary of a convex body in n-dimensional space split this space into two regions, respectively consisting exclusively of interior points and exclusively of exterior points.

§II.
The smallest convex body K_{2n} which contains a spherical standard curve of $2n$-dimensional space

11. The considerations of the preceding extract lead to particularly simple shapes if one chooses a standard curve of $2n$-dimensional space as the set of points $\mathfrak{M}$.

As is well-known, an algebraic curve of the k-th degree is called a standard curve if it contains $(k+1)$ points which do not all lie in a $(k-1)$-dimensional linear manifold of the space $\mathbb{R}^n$. So, for example, non-degenerate conics or non-planar curves are standard curves of degree three.

In the spaces $\mathbb{R}^{2n}$ of the right dimension, there are standard curves whose image is closed and bounded, and which entirely lie on an $(n-1)$-dimensional sphere. The equation of this curve can, if one considers it under a change of basis and a rotation of $2n$-dimensional space, be rendered in the form: *p.202*

$$\begin{cases} x_1 = \cos\theta,\ x_2 = \cos 2\theta,\ \ldots,\ x_n = \cos n\theta, \\ \overline{x}_1 = \sin\theta,\ \overline{x}_2 = \sin 2\theta,\ \ldots,\ \overline{x}_n = \sin n\theta. \end{cases} \tag{8}$$

This curve is of degree $2n$, as one can see immediately if one sets

$$\cos k\theta = \frac{\alpha^k + \alpha^{-k}}{2},\ \ \sin k\theta = \frac{\alpha^k - \alpha^{-k}}{2i}$$

$$\alpha = e^{i\theta}$$

and introduces α as a parameter. Every equation of the form

$$u_1 \cos\theta + \overline{u}_1 \sin\theta + \cdots + u_n \cos n\theta + \overline{u}_n \sin n\theta = d, \tag{9}$$

in which at least some of the coefficients $u_k, \overline{u}_k$ do not vanish, is of degree at most $2n$ in α and therefore will have no more than $2n$ real roots if it is not identically zero. In other words: the $(2n-1)$-dimensional hyperplane

$$u_1 x_1 + \overline{u}_1 x_1 + \cdots + u_n x_n + \overline{u}_n \overline{x}_n = d$$

has at most $2n$ points in common with the curve if it does not contain the entire curve. The latter is, however, ruled out, because one can deduce that an arbitrary trigonometric polynomial vanishes everywhere from the fact that each of its coefficients is zero.

The curve (8) is therefore a standard curve; in what follows, we will now consider the smallest convex body which contains this curve. Our considerations can also easily be transferred, with minor modifications, to every other standard curve, for example to the curve

$$x_1 = t,\ x_2 = t^2,\ \dots,\ x_n = t^n$$

or also to pieces of these curves.

Because, as we have seen, every hyperplane has at most $2n$ points in common with the curve (8), every arbitrary hyperplane — and consequently also every supporting hyperplane — is tangent to the curve at *at most n points*. Conversely, the theorem that every hyperplane which is tangent to the curve at n separate points must be a supporting hyperplane holds here. Indeed, if this hyperplane met the curve in a point other than the tangent points, then the equation (9) would posses more than $2n$ (simple and double) roots, and the same would be the case if the curve penetrated the hyperplane at one of its tangent points, whereby the corresponding root would count three times.

12. From these simple facts one can assemble the following properties of the smallest convex domain K_{2n} which contains the curve (8):

a) The domain K_{2n} is a convex *body*, because $(2n+1)$ arbitrary points of the standard
p.203 curve (8) can never lie in a $(2n-1)$-dimensional hyperplane.

In particular, the origin is an *interior* point of K_{2n} because the curve (8) intersects every hyperplane

$$u_1 x_1 + \overline{u}_1 \overline{x}_1 + \cdots + u_n x_n + \overline{u}_n \overline{x}_n = 0$$

and such a hyperplane can never be a supporting hyperplane. This is easiest to see if one forms the expression

$$f(\theta) = \sum_{k=1}^{n} u_k \cos k\theta + \overline{u}_k \sin k\theta$$

and remarks that

$$\int_0^{2\pi} f(z)\, dz = 0.$$

b) Every centre of mass $\mathfrak{b}$ of p positive masses which lie on the curve (8) is a point of the boundary of K_{2n}, so long as $p \leqslant n$.

Indeed, if one adds $(n-p)$ distinct points to the p points where the masses lie, then there is an $(n-1)$-dimensional hyperplane which contains all n points (the original ones and the additional ones) and also contains the tangents to the standard curve at these points; this plane is a supporting hyperplane of K_{2n} and contains the point $\mathfrak{b}$. However, this point is a point of K_{2n} which lies on the boundary, because, as a centre of mass of a distribution of positive masses to (8), it belongs to a supporting hyperplane.

c) Every point $\mathfrak{b}$ of the boundary of K_{2n} can be viewed as a centre of mass of p positive masses on the curve (8), where $p \leqslant n$. Indeed, there is at least one supporting hyperplane through $\mathfrak{b}$, which by above, has at most n tangent points with the standard curve. The point $\mathfrak{b}$ however belongs to the smallest convex domain which these tangent points determine.

d) The representation of a point of the boundary of K_{2n} by a assignment of positive masses to (8) is *unique*, i.e. two assignments of mass of total mass one, of which one consists of at most n points, while the other is arbitrary, can only have the same centre of mass $\mathfrak{b}$ if they are identical.

Indeed, let $\mathfrak{a}$ be some point of the second distribution of masses, which does not agree with one point of the first; we can then always place a supporting hyperplane through the centre of mass $\mathfrak{b}$ of the first distribution of masses, which does not contain the point $\mathfrak{a}$: by this we only need to add to the $p \leqslant n$ points of the first aggregate $(n-p)$ points which are distinct from $\mathfrak{a}$ and to consider the supporting hyperplanes of K_{2n} which do not contain these n points. Let $m_{\mathfrak{a}}$ now be the mass concentrated in $\mathfrak{a}$ and $s_{\mathfrak{a}}$ the distance of $\mathfrak{a}$ to the hyperplane s; then the centre of mass of the second distribution of masses is a distance of at least $m_{\mathfrak{a}} s_{\mathfrak{a}}$ from s.

The two distributions of masses must therefore consist of precisely the same p points, where $p \leqslant n$. However, one can then always place through $(p-1)$ of these points a supporting hyperplane to the standard curve which does not contain the p-th; the p points therefore do not lie in a $(p-1)$-dimensional linear manifold. Every point b of *p.204* the smallest convex domain which contains this aggregate of p points, therefore can only be represented in one way as the centre of mass of non-negative masses with total mass one which lie in this aggregate, as follows from §9 of this work. This proves the uniqueness of the representation.

e) From this last circumstance it now follows that the centre of mass of *more* than n positive masses which lie on the standard curve necessarily lies in the *interior* of the body K_{2n}.

The same holds for every continuous distribution of masses, whose centre of mass, as we know, also belongs to K_{2n}.

In particular, if one assumes that the density ρ of a distribution of mass to the whole curve is constant, then the centre of mass coincides with the origin $\mathfrak{o}$, which provides a new proof that $\mathfrak{o}$ lies in the interior of K_{2n}.

13. We can now collect our results into formulas and give a parameterisation of the boundary of K_{2n}. Indeed, if allocate the masses $\lambda_1, \lambda_2, \ldots, \lambda_p$ to the p points of the curve (8), which correspond to the values $\theta_1, \theta_2, \ldots, \theta_p$ of the parameter θ, where $p \leqslant n$, then the coordinates $a_1, \overline{a}_1, a_2, \overline{a}_2, \ldots, a_n, \overline{a}_n$ of the centre of mass are of the following form:

$$a_k = \sum_{j=1}^{p} \lambda_j \cos k\theta_j, \quad \overline{a}_k = \sum_{j=1}^{p} \lambda_j \sin k\theta_j \qquad {\scriptstyle (k=1,2,\ldots,n)};$$

we must however assume here that the λ_j are all positive and have a sum of one. These formulas are identical to the following, in which p does not occur directly:

$$\begin{cases} a_k = \sum_{j=1}^n \lambda_j \cos k\theta_j, \quad \overline{a}_k = \sum_{j=1}^n \lambda_j \sin k\theta_j \\ \lambda_1 \geqslant 0,\, \lambda_2 \geqslant 0,\, \dots,\, \lambda_n \geqslant 0 \\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1; \end{cases} \tag{10}$$

here one has set $\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_n = 0$.

§III.
Algebraic representation of the boundary of K_{2n}

14. Let

$$a_1,\, \overline{a}_1,\, a_2,\, \overline{a}_2,\, \dots,\, a_n,\, \overline{a}_n \tag{11}$$

be the coefficients of an arbitrary point $\mathfrak{p}$ of the $2n$-dimensional space $\mathbb{R}^{2n}$. We connect this point with the origin $\mathfrak{o}$ using a line segment which we extend to the point $\mathfrak{q}$ where it intersects the boundary of K_{2n}.

This point $\mathfrak{q}$ can be viewed in a unique way as the centre of mass of p masses ($p \leqslant n$) which lie on the standard curve (8). Hence $\mathfrak{p}$ can be represented in a unique way by the formula:

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$$a_k = \sum_{j=1}^n \lambda_j \cos k\theta_j, \quad \overline{a}_k = \sum_{j=1}^n \lambda_j \sin k\theta_j,$$

where the θ_j are real and the λ_j are real and non-negative, but in general have a sum which does not equal one: the quantity

$$\sum_{j=1}^n \lambda_j = (1 - \lambda_0) > 0$$

namely represents the ratio $\frac{\mathfrak{po}}{\mathfrak{qo}}$ of the segment, and is only equal to one, if p finds itself on the boundary of K_{2n}, i.e. if $\mathfrak{p}$ and $\mathfrak{q}$ coincide. The quantity λ_0, which is negative when the point lies outside the body K_{2n}, can be viewed as the mass which is concentrated at the origin. Then the point is the centre of mass of the $(p+1)$ masses $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_p$.

15. In order to determine the parameters λ and θ from the quantities (11), for expedience we introduce the notation:

$$\begin{cases} a_k + i\overline{a}_k = \alpha_k,\ a_k - i\overline{a}_k = \alpha_{-k} \\ \alpha_0 = 1 - \lambda_0,\ e^{i\theta_j} = \cos\theta_j + i \sin\theta_j = \sigma_j. \end{cases} \tag{12}$$

We obtain the system of equations

$$\text{(13)} \qquad \begin{cases} \sum_{j=1}^{n} \lambda_j \sigma_j^n = \alpha_n \\ \sum_{j=1}^{n} \lambda_j \sigma_j^{n-1} = \alpha_{n-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{j=1}^{n} \lambda_j \sigma_j = \alpha_1 \\ \sum_{j=1}^{n} \lambda_j = \alpha_0 = 1 - \lambda_0 \\ \sum_{j=1}^{n} \lambda_j \sigma_j^{-1} = \alpha_{-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{j=1}^{n} \lambda_j \sigma_j^{-n} = \alpha_{-n} \end{cases}$$

of $(2n+1)$ equations in the $(2n+1)$ unknowns $\lambda_0, \lambda_1, \dots, \lambda_n; \sigma_1, \sigma_2, \dots, \sigma_n$. *p.206*

This system of equations is entirely analogous to that which one obtains in the canonicalisation of binary forms of non-linear orders, and one can formally find the minimal solution using a method Sylvester has developed.

We will moreover show, using our earlier results, that the formal solution of the system of equations (13) always leads to the desired outcome.

Indeed, we know that this system has a solution, in which the first p λ_j are non-zero and positive and are uniquely determined by the first p σ_j; the remaining $\lambda_{p+1}, \lambda_{p+2}, \dots,$ λ_n are thereby all zero; the quantities $\sigma_1, \sigma_2, \dots, \sigma_p$ are complex, have an absolute value one, and no two of these quantities are equal to each other; the whole number p is non-zero and not larger than n.

We now assume that the values $\sigma_1, \sigma_2, \dots, \sigma_p$ have been determined and further introduce $(n-p)$ complex values $\sigma_{p+1}, \sigma_{p+2}, \dots, \sigma_n$ of absolute value one, such that none of the n numbers $\sigma_1, \sigma_2, \dots, \sigma_n$ are equal to each other; then the determinant

$$\Delta = \begin{vmatrix} \sigma_1^{n-1} & \sigma_2^{n-1} & \dots & \sigma_n^{n-1} \\ \sigma_1^{n-2} & \sigma_2^{n-2} & \dots & \sigma_n^{n-2} \\ \dots & \dots & \dots & \dots \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \\ 1 & 1 & \dots & 1 \end{vmatrix},$$

which we express using powers of σ_1, may be written in the following way

$$\text{(14)} \qquad \Delta = A_0 + A_1\sigma_1 + A_2\sigma_1^2 + \dots + A_{n-1}\sigma_1^{n-1},$$

so long as the minor A_0 is non-zero.

If we now think of these values of σ_j substituted into the system of equations (13), there is certainly a system of quantities λ_j, which satisfy all these equations. We can eliminate the quantities $\lambda_2, \lambda_3, \dots, \lambda_n$ from the first n equations of the system (13) and obtain:

$$\sigma_1\lambda_1\Delta = A_0\alpha_1 + A_1\alpha_2 + \dots + A_{n-1}\alpha_n;$$

we remove the first equation in (13) and eliminate $\lambda_2, \dots, \lambda_n$ from the first n equations of the system reduced in this way, and get:

$$\lambda_1\Delta = A_0\alpha_0 + A_1\alpha_1 + \dots + A_{n-1}\alpha_{n-1};$$

we proceed in this way and so obtain a system of $(n+2)$ equations, namely as follows:

$$(15)\qquad \begin{cases} \sigma_1\lambda_1\Delta = A_0\alpha_1 + A_1\alpha_2 + \cdots + A_{n-1}\alpha_n \\ \quad \lambda_1\Delta = A_0\alpha_0 + A_1\alpha_1 + \cdots + A_{n-1}\alpha_{n-1} \\ \sigma_1^{-1}\lambda_1\Delta = A_0\alpha_{-1} + A_1\alpha_0 + \cdots + A_{n-1}\alpha_{n-2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sigma_1^{-n}\lambda_1\Delta = A_0\alpha_{-n} + A_1\alpha_{-(n-1)} + \cdots + A_{n-1}\alpha_{-1}. \end{cases}$$

We now multiply each two the following equations of this system respectively by one
p.207 and $-\sigma_1$ and add them together; then we obtain, if we introduce the notation,

$-\sigma A_0 = B_0,\ A_0 - \sigma_1 A_1 = B_1,\ A_1 - \sigma_1 A_2 = B_2,\ \dots,\ A_{n-2} - \sigma_1 A_{n-1} = B_{n-1},\ A_{n-1} = B_n$

$$(16)\qquad \begin{cases} 0 = B_0\alpha_0 + B_1\alpha_1 + B_2\alpha_2 + \cdots + B_n\alpha_n \\ 0 = B_0\alpha_{-1} + B_1\alpha_0 + B_2\alpha_1 + \cdots + B_n\alpha_{n-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0 = B_0\alpha_{-n} + B_1\alpha_{-(n-1)} + \cdots + B_n\alpha_0. \end{cases}$$

Because the quantity B_0, as a product of two non-vanishing quantities, is certainly non-zero in this system of linear equations for the B_j, the determinant

$$(17)\qquad D_n = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_{-1} & \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\ \alpha_{-2} & \alpha_{-1} & \alpha_0 & \dots & \alpha_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{-n} & \alpha_{-(n-1)} & \alpha_{-(n-2)} & \dots & \alpha_0 \end{vmatrix}$$

vanishes.

This determinant is a so-called *recurrent* determinant, i.e. the elements in every parallel to the main diagonal are the same as one another; moreover, two arbitrary elements which lie symmetrically across the main diagonal are complex conjugates. From this, it follows that the determinant (17) is determined if one knows the elements of the first row. In what follows, we will denote determinants which are formed in the same way as (17) by their first row, and so in the following way

$$D_n(\alpha_0, \alpha_1, \dots, \alpha_n).$$

16. By our reckoning, we are led to the equation

$$(18)\qquad D_n(\alpha_0, \alpha_1, \dots, \alpha_n) = 0,$$

which, if we substitute for α_0 its value $(1-\lambda_0)$, transforms into an $(n+1)$-th degree equation for λ_0.

This equation

$$(19)\qquad D_n((1-\lambda_0), \alpha_1, \dots, \alpha_n) = 0$$

has only real roots, as a characteristic equation of a Hermitian form; for us, however, only *one* of these roots comes into consideration, because we know that the solution of (13) which we are seeking, is *uniquely* determined.

In order to determine this special root, we must probe the geometric meaning of (19). Indeed, the point $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $2n$-dimensional space is only found on the boundary of K_{2n} if $\lambda_0 = 0$. The equation

$$D_n(1, \alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \tag{20}$$

must therefore hold for every point on the boundary of K_{2n}.

The boundary of K_{2n} consists of one part of a single algebraic surface, whose equation is given by (20). *p.208*

This important result, which was discovered by Toeplitz, can be completed by a property of K_{2n} which this author has also proven. Indeed, one can show that the determinant D_n is always positive and non-zero in the interior of the convex body K_{2n}.

Indeed, let

$$\mathfrak{a} : a_1,\, \overline{a}_1,\, a_2,\, \overline{a}_2,\, \ldots,\, a_n,\, \overline{a}_n$$

be a point of the interior of K_{2n}. We consider the intersection of the convex *body* K_{2n} with the 2-dimensional linear manifold that one obtains if one fixes the coordinates

$$x_1 = a_1,\, \overline{x}_1 = \overline{a}_1,\, \ldots,\, x_{n-1} = a_{n-1},\, \overline{x}_{n-1} = \overline{a}_{n-1} \tag{21}$$

and lets $x_n, \overline{x}_n$ vary freely.

This intersection, by its nature, is a closed convex curve which contains the point $\mathfrak{a}$ in its interior. On the other hand, the intersection of the 2-dimensional plane (21) with the surface (20), which contains the whole boundary of K_{2n}, can be written

$$D_n(1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, x_n + i\overline{x}_n) = \begin{vmatrix} 1 & \alpha_1 & \ldots & \alpha_{n-1} & (x_n + i\overline{x}_n) \\ \alpha_{-1} & 1 & \ldots & \ldots & \alpha_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{-(n-1)} & \alpha_{-(n-2)} & \cdots & \cdots & \alpha_1 \\ (x_n - i\overline{x}_n) & \alpha_{-(n-1)} & \cdots & \cdots & 1 \end{vmatrix}$$

or if one expands

$$(x_n^2 + \overline{x}_n^2)L + x_n M + \overline{x}_n \overline{M} + N = 0. \tag{22}$$

This is the equation of a circle, which consequently must coincide with our convex curve. For an interior point of this circle, D_n cannot possibly vanish, whereby it is simultaneously proven that D_n is non-zero for every point of the interior of K_{2n}. Because, on the other hand,

$$D_n(1, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

is a continuous function, and has the value

$$D_n(1, 0, 0, \ldots, 0) = 1 > 0$$

for the origin $\mathfrak{o}$, we obtain our claim that D_n is positive and non-zero for interior points of K_{2n}.

One notices that, for $p < n$, the convex body K_{2p} represents the projection of K_{2n} to a $2p$-dimensional manifold and that interior points of K_{2n} always project to interior points of K_{2p}. From this it follows that the n conditions

$$D_p(1, \alpha_1, \alpha_2, \dots, \alpha_p) > 0 \qquad {\scriptstyle (p=1,2,\dots,n)} \tag{23}$$

p.209 must be satisfied for interior points of K_{2n} and, furthermore, that the conditions

$$D_p(1, \alpha_1, \alpha_2, \dots, \alpha_p) \geq 0 \qquad {\scriptstyle (p=1,2,\dots,n)}$$

hold for every point of K_{2n} including the boundary, which consist of boundary points and interior points.

The conditions (23), which we have shown to be necessary for the internal points of K_{2n}, are also sufficient: indeed, let us assume that the claim is already established for $(2n-2)$-dimensional space and $a_1, \overline{a}_1, a_2, \overline{a}_2, \dots, a_n, \overline{a}_n$ is a point for which the inequality (23) holds. Then $a_1, \overline{a}_1, \dots, a_{n-1}, \overline{a}_{n-1}$ is a point of the convex body K_{2n-2} and there is at least one point of K_{2n} which projects onto $a_1, \dots, \overline{a}_{n-1}$. If we substitute these values into equation (22), then all the points of the interior or the border of the circle represented by this equation will correspond to points of the interior or of the boundary of K_{2n}; because now $L = -D_{n-2} < 0$, the points which do not lie outside the circle (22) coincide with those for which $D_n \geqslant 0$, whereby our claim is proven that the point $a_1, \dots, \overline{a}_n$ which we have considered belongs to the body K_{2n}.

In the above proof, we have *only* used the non-vanishing of the determinant D_{n-2}. It also therefore shows that the conditions

$$\begin{aligned} D_p(1, \alpha_1, \dots, \alpha_p) &> 0 \qquad {\scriptstyle (p=1,2,\dots,(n-2))} \\ D_{n-1}(1, \alpha_1, \dots, \alpha_{n-1}) &\geqslant 0 \\ D_n(1, \alpha_1, \dots, \alpha_n) &\geqslant 0 \end{aligned}$$

are sufficient for the point $a_1, \overline{a}_1, \dots, a_n, \overline{a}_n$ to lie in the interior or the boundary of the body K_{2n}.

If, however, one of the earlier determinants vanishes, in no way can one deduce that the given point lies in K_{2n} from the non-negativity of D_p for $p = 1, 2, \dots, n$. For $n = 5$, one obtains e.g., if one has $\alpha_1 = 1, \alpha_2, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 5$, that $D_1 = D_2 = D_3 = 0$ and $D_4 = D_5 = 1$. On account of this, the given point lies outside the body K_{10}, because the real parts of α_3 and α_5 exceed the number 1.

In order to give a condition which is sufficient in every case, we note that it is enough to show that, for every $\nu > 1$, the point

$$\frac{a_1}{\nu}, \frac{\overline{a}_1}{\nu}, \dots, \frac{a_n}{\nu}, \frac{\overline{a}_n}{\nu}$$

lies in the *interior* of K_{2n}. Thereby it is enough, as will be shown in detail in the next section, that the equation in ν

$$D_n(\nu, \alpha_1, \alpha_2, \dots, \alpha_n) = 0$$

has no root which is greater that $+1$.

If for an arbitrary point of K_{2n}, the determinant $D_p = 0$ and $p < n$, then the point

$$a_1,\, \overline{a}_1,\, a_2,\, \overline{a}_2,\, \ldots, a_p,\, \overline{a}_p$$

must lie on the boundary of K_{2p} and cannot be viewed as a projection of a point of the interior of $K_{2q} (q > p)$. From the fact that the given point lies in K_{2n} and that $D_p = 0$, the equations *p.210*

$$D_{p+1} = 0,\, D_{p+2} = 0,\, \ldots,\, D_n = 0$$

therefore follow.

17. Now we may return to our original problem and determine the value of the parameter λ_0 which must satisfy the equation (19).

To this end, we note that, by the exposition in §14, the point with coordinates

$$\frac{a_k}{1-\lambda_0}, \frac{\overline{a}_k}{1-\lambda_0} \qquad {\scriptstyle (k=1,2,\ldots,n)} \tag{24}$$

must lie on the boundary of K_{2n}. If one lets λ_0 increase from $-\infty$ to $+1$, then this point describes a radius vector whose *first* intersection point with the curve $D_n = 0$ must lie on the boundary of K_{2n}. Hence, we must take the smallest algebraic root of the equation

$$D_n\left(1, \frac{\alpha_1}{1-\lambda_0}, \frac{\alpha_2}{1-\lambda_0}, \ldots, \frac{\alpha_n}{1-\lambda_0}\right) = 0,$$

or, equivalently, of the equation (19)

$$D_n[(1-\lambda_0), \alpha_1, \ldots, \alpha_n] = 0 \tag{19}$$

for λ_0.

Thus we have proven that, in all cases, this equation has a root which is smaller than $+1$.

18. After λ_0 is determined, we can determine the number p of non-zero λ_j in the parametric representation. For this we consider the series

$$D_k(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k) \qquad {\scriptstyle (k=1,2,\ldots,n)},$$

where the value $(1-\lambda_0)$ just determined is substituted for α_0. Let D_p be the first of this series which vanishes. That means, by the results of §16, that the point with the coordinates (24) falls in the interior of the body K_{2p-2}, but on the boundary of K_{2p}. By our exposition in §12, this point will therefore be represented by the centre of mass of exactly p distinct discrete masses on the standard curve (8).

In other words, we only need yet to determine the $2p$ quantities $\lambda_1, \ldots, \lambda_p$; $\sigma_1, \ldots, \sigma_p$ to obtain the solution of the system of equations (13).

19. If we introduce the notation

$$A_n = -\sigma_1^{n+1} \lambda_1 \Delta$$

in the first $(n+1)$ equations of the system of equations (15), then they assume the form:

$$\begin{aligned}
0 &= A_0\alpha_1 + A_1\alpha_2 + \cdots + A_{n-1}\alpha_n + A_n\sigma_1^n \\
0 &= A_0\alpha_0 + A_1\alpha_1 + \cdots + A_{n-1}\alpha_{n-1} + A_n\sigma_1^{n-1} \\
&\dots \\
0 &= A_0\alpha_{-n+1} + A_1\alpha_{-n+2} + \cdots + A_{n-1}\alpha_0 + A_n,
\end{aligned}$$

p.211 whereby it follows that σ_1 must satisfy the equation

$$\begin{vmatrix}
\sigma_1^n & \alpha_1 & \alpha_2 & \dots & \alpha_n \\
\sigma_1^{n-1} & \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\
\sigma_1^{n-2} & \alpha_{-1} & \alpha_0 & \dots & \alpha_{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\sigma_1 & \alpha_{-(n-2)} & \alpha_{-(n-3)} & \cdots & \alpha_1 \\
1 & \alpha_{-(n-1)} & \alpha_{-(n-2)} & \cdots & \alpha_0
\end{vmatrix} = 0.$$

One can see in exactly the same way that all the other σ_j satisfy the same condition.

We therefore obtain for all σ_j a single equation

$$\begin{vmatrix}
\sigma_1^n & \alpha_1 & \alpha_2 & \dots & \alpha_n \\
\sigma_1^{n-1} & \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\
\sigma_1^{n-2} & \alpha_{-1} & \alpha_0 & \dots & \alpha_{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\sigma_1 & \alpha_{-(n-2)} & \alpha_{-(n-3)} & \cdots & \alpha_1 \\
1 & \alpha_{-(n-1)} & \alpha_{-(n-2)} & \cdots & \alpha_0
\end{vmatrix} = 0. \tag{25}$$

In the general case, where $D_{n-1}(\alpha_0, \alpha_1, \dots, \alpha_n) > 0$, the coefficient of σ^n in this equation is non-zero; then there are at most n distinct numbers which satisfy this equation. On the other hand, we know from the exposition of §18 that actually n distinct σ_j must exist, and so the equation (25) moreover determines *simple* roots and we can determine all our σ_j.

If the σ_j are determined, then one can obtain the corresponding λ_j by solving the linear equations (13) for n unknowns, as follows from the proof of uniqueness §12 d).

If $D_{n-1}(\alpha_0, \dots, \alpha_n) = 0$, then the equation (25) is *imaginary* and all its coefficients are identically zero; in fact, it must be satisfied by arbitrary values of σ for which $|\sigma| = 1$.

Then let D_p be the first vanishing expression in the series

$$D_k(\alpha_0, \alpha_1, \dots, \alpha_k) \qquad (k=1,2,\dots,n);$$

by §18 only $2p$ quantities λ_j and σ_j are then to be determined, and we can solve the question using the earlier method in which we removed from system (13) the $(n-p)$ first and the $(n-p)$ last equations, and set $\lambda_{p+1}, \dots, \lambda_n$ equal to zero from then on.

20. The surface $D_n(1, \alpha_1, \alpha_2, \dots, \alpha_n) = 0$, which contains the boundary of K_{2n}, has the noteworthy quality of being *rational*. Indeed, if we place a line whose direction coefficients are denoted by $c_k, \bar{c}_k$ through the point with the coordinates $x_k = 1, \overline{x}_k = 0$ and introduce the notation $\gamma_k = c_k + i\bar{c}_k, \gamma_{-k} = c_k - i\bar{c}_k, \gamma_0 = 0$, then the point of intersection of this line with the surface is given by the equation *p.212*

$$\begin{vmatrix} 1+\gamma_0 t & 1+\gamma_1 t & 1+\gamma_2 t \dots & 1+\gamma_n t \\ 1+\gamma_{-1} t & 1+\gamma_0 t & 1+\gamma_1 t \dots & 1+\gamma_{n-1} t \\ \dots & \dots & \dots & \dots \\ 1+\gamma_{-n} t & 1+\gamma_{-(n-1)} t & \dots \quad \dots & 1+\gamma_0 t \end{vmatrix} = 0.$$

If we subtract the previous row from every row, this equation may be written:

$$t^n \begin{vmatrix} 1+\gamma_0 t & 1+\gamma_1 t & \dots & 1+\gamma_n t \\ \gamma_{-1}-\gamma_0 & \gamma_0-\gamma_1 & \dots & \gamma_{n-1}-\gamma_n \\ \dots & \dots & \dots & \dots \\ \gamma_{-n}-\gamma_{-(n-1)} & \gamma_{-(n-1)}-\gamma_{-(n-2)} & \dots & \gamma_0-\gamma_1 \end{vmatrix} = 0$$

besides one n-fold root $t = 0$, it only has one other root, which can be expressed as a rational function of the direction coefficients. The value of this last root can be written in a particularly elegant and short way if one introduces the following notation:

$$\delta_k = 2\gamma_k - \gamma_{k+1} - \gamma_{k-1},$$

which applies for both positive and negative k. Indeed, we then obtain

$$t = -D_{n-1}(\delta_0, \delta_1, \dots, \delta_{n-1}) : D_n(\gamma_0, \gamma_1, \dots, \gamma_n),$$

as an easy calculation shows.

The point $a_k = 1, \bar{a}_k = 0$, which lies on the standard curve (8), is by this an n-fold point of the surface $D_n = 0$. We claim that the whole standard curve is an n-fold curve of this surface. In order to prove this, we shall show that there is a continuous one-parameter group of *rotations* in $\mathbb{R}^{2n}$ which leave the standard curve and the expression D_n invariant, as well as the boundary of K_{2n}.

These rotations may be characterised by the formula

$$x_k + i\overline{x}_k = (x_k^0 + i\overline{x}_k^0)e^{ik\tau} \qquad (k = 1, 2, \dots, n), \tag{26}$$

where τ denotes the parameter of the group. One can easily see that under these transformations distances are preserved, so that one truly is dealing with rotations whose unique fixed point is the origin.

If one performs the transformation (26) on the α_k in D_n, one obtains:

$$\begin{vmatrix} \alpha_0 & \alpha_1\zeta & \alpha_2\zeta^2 \dots & \alpha_n\zeta^n \\ \alpha_{-1}\zeta^{-1} & \alpha_0 & \alpha_1\zeta \dots & \alpha_{n-1}\zeta^{n-1} \\ \dots & \dots & \dots & \dots \\ \alpha_{-n}\zeta^{-n} & \alpha_{-n+1}\zeta^{-n+1} & \dots \quad \dots & \alpha_0 \end{vmatrix},$$

where we have written ζ instead of $e^{i\tau}$. This expression however turns into $D_n(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ if one multiplies the k-th column by ζ^{n-k+1} and the k-th row by ζ^{-n+k-1} and performs this operation for $k = 1, 2, \ldots, n$. *p.213*

The expression D_n, and likewise the surface $D_n = 0$, therefore stays invariant under all these rotations.

21. As a final observation, which can be useful, we shall investigate the largest and smallest distance of the origin from the boundary of K_{2n}. The maximum distance E_n must coincide with the distance of the spherical standard curve from the origin; the surface of the ball with radius $\sqrt{n}$ contains the standard curve, and, as the boundary of a convex body, can contain no interior points of K_{2n}. Therefore

$$E_n = \sqrt{n}.$$

To calculate the minimum e_n is quite complicated, but one can easily find an upper bound for this quantity. In fact the line

$$x_k = t,\ \overline{x}_k = 0$$

intersects our convex body at two points, which correspond to the values $t = 1$ and $t = -\frac{1}{n}$ of the parameter; it is therefore certain that

$$e_n \leqslant \frac{1}{\sqrt{n}};$$

this number is generally too large; already for $n = 2$ one finds that

$$e_2 = \frac{\sqrt{7}}{4} < \frac{1}{\sqrt{2}};$$

it however shows that while

$$\lim_{n\to\infty} E_n = \infty,$$

we have

$$\lim_{n\to\infty} e_n = 0.$$

§IV.
Positive harmonic functions

22. Let $U(r, \theta)$ be a positive harmonic function of two variables, with r and θ the polar coordinates of the plane given by these variables. We assume that $U(r, \theta)$ is regular and positive in the circle $r < 1$ and equal to $\frac{1}{2}$ at the origin. Then the following expansion holds

$$U(r, \theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + \overline{a}_n \sin n\theta), \tag{27}$$

which converges absolutely and uniformly for every $r \leqslant \rho$ (where ρ denotes an arbitrary
p.214 positive fixed constant, for which $0 < \rho < 1$).

Therefore the following formulas hold for every $r < 1$:

$$\frac{1}{\pi}\int_0^{2\pi} U(r,\theta)\,d\theta = 1, \tag{28}$$

$$\begin{cases} \frac{1}{\pi}\int_0^{2\pi} U(r,\theta)\cos k\theta\,d\theta = a_k r^k, \\ \frac{1}{\pi}\int_0^{2\pi} U(r,\theta)\sin k\theta\,d\theta = \overline{a}_k r^k. \end{cases} \tag{29}$$

The left side of the first $2n$ of the equations (29) can be interpreted (if one considers (28) along with the property of $U(r,\theta) \geqslant 0$) as coordinates of centres of mass of distributions of mass with density $\frac{1}{\pi}U(r,\theta)$ on the spherical standard curve

$$\cos k\theta, \quad \sin k\theta \qquad {\scriptstyle (k=1,2,\ldots,n)},$$

of $\mathbb{R}^{2n}$ which we investigated in the earlier paragraphs.

From here it follows that the point with the coordinates

$$a_k r^k,\ \overline{a}_k r^k \qquad {\scriptstyle (k=1,2,\ldots,n)}$$

lies in the interior of the smallest convex body K_{2n} which contains this curve; because now this property continues to hold for every value of $r < 1$, it then follows that every point with coordinates $a_k, \overline{a}_k$, which we shall call the n-th geometrical representative of U, must lie in the interior *or on the boundary of* K_{2n}.

By the results of the earlier paragraphs, we now see that as a consequence of the conditions that

$$\Delta U = 0,\ U(r,\theta) \geqslant 0 \text{ and is regular for } r < 1,\ U(0) = \frac{1}{2} \tag{30}$$

a series of inequalities for the coefficients $a_k, \overline{a}_k$ must *necessarily* be satisfied, namely the following:

$$D_n(1, a_1 + i\overline{a}_1, a_2 + i\overline{a}_2, \ldots, a_n + i\overline{a}_n) \geqslant 0 \qquad {\scriptstyle (n=1,2,\ldots,\text{ ad inf.})}. \tag{31}$$

23. We shall invert the result spoken of above in two ways and show that:

1°) If the first $2n$ coefficients

$$a_1,\ \overline{a}_1,\ a_2,\ \overline{a}_2,\ \ldots,\ a_n,\ \overline{a}_n$$

are given and the corresponding geometrical representative lies in the interior or on the p.215
boundary of K_{2n}, a function $U(r,\theta)$ can always be found which satisfies the conditions (30) and whose expansion (27) begins with the given coefficients.

2°) If infinitely many numbers

$$a_1,\ \overline{a}_1,\ a_2,\ \overline{a}_2,\ \ldots,\ a_n,\ \overline{a}_n,\ \ldots \text{ ad inf.}$$

are given arbitrarily, and if the corresponding geometrical representative lies in the interior or on the boundary of K_{2n} for every n, then the formal expansion (27) represents a function which satisfies all the conditions (30); it is harmonic, regular, and positive for $r < 1$.

In order to prove the first part of this theorem, we remark initially that there are functions whose geometrical representative lies on the standard curve for every n and, moreover, satisfy all the conditions (30). If we consider in fact the expansion

$$\epsilon(\phi; r, \theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n (\cos n\phi \cos n\theta + \sin n\phi \sin n\theta),$$

whose geometrical representative lies on the standard curve for every n and which converges absolutely for $r < 1$. We can write:

$$\begin{aligned}
\epsilon(\phi; r, \theta) &= \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\phi - \theta) \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{r^n}{2} \left(e^{in(\phi-\theta)} + e^{-in(\phi-\theta)} \right) \\
&= \frac{1}{2} \left\{ 1 + \frac{re^{i(\phi-\theta)}}{1 - re^{i(\phi-\theta)}} + \frac{re^{-i(\phi-\theta)}}{1 - re^{-i(\phi-\theta)}} \right\} \\
&= \frac{1}{2} \frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2}
\end{aligned}$$

from which it follows that $\epsilon(\phi; r, \theta)$ is a *positive* harmonic function.

It further follows from the results obtained above that for non-negative λ_0, λ_1, …, λ_n, which satisfy the condition $\sum_{j=0}^{n} \lambda_j = 1$, the function

$$\frac{1}{2}\lambda_0 + \sum_{j=1}^{n} \lambda_j \epsilon(\zeta_j; r, \zeta)$$

satisfies all the conditions (30). Its n-th geometrical representation has coordinates

$$a_k = \sum_{j=1}^{n} \lambda_j \cos k\zeta_j,\ \ \overline{a}_k = \sum_{j=1}^{n} \lambda_j \sin k\zeta_j$$

and, as we know, can represent with these formulas any given point on the interior or the boundary of K_{2n} for an appropriate choice of $(2n + 1)$ values

$$\lambda_0 \geqslant 0,\ \lambda_1 \geqslant 0,\ \lambda_2 \geqslant 0,\ \ldots,\ \lambda_n \geqslant 0,\ \theta_1,\ \theta_2,\ \ldots,,\theta_n$$

along with the condition

$$\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1.$$

We have, in other words, a function associated to every point of our convex body K_{2n} which has this point as its n-th geometrical representative and which satisfies the conditions (30).

p.216 In order now to prove the second part of our theorem, we remark that, given infinitely many numbers

$$a_1,\, \overline{a}_1,\, a_2,\, \overline{a}_2,\, \ldots,\ \text{ ad inf.}$$

the point $a_1, \overline{a}_1, \ldots, a_n, \overline{a}_n$ lies in K_{2n} for every n, only if for every k

$$|a_k| \leqslant 1,\ |\overline{a}_k| \leqslant 1 \qquad {\scriptstyle (k=1,2,\ldots,\text{ ad inf.})}.$$

Indeed, the cube

$$|x_1| = 1,\ |\overline{x}_1| = 1,\ \ldots,\ |x_n| = 1,\ |\overline{x}_n| = 1$$

circumscribes the standard curve (8) and therefore contains the convex body K_{2n}.

The series

$$\frac{1}{2} + \sum r^n (a_n \cos n\theta + \overline{a}_n \sin n\theta) \tag{32}$$

is therefore absolutely convergent for all $r < 1$ and represents a harmonic function $U(r, \theta)$. We now form the series of functions

$$U_1(r,\theta),\ U_2(r,\theta),\ \ldots,\ U_n(r,\theta),\ \ldots \text{ ad inf.} \tag{33}$$

such that the first n coefficients of $U_n(r,\theta)$ coincide with $a_1, \overline{a}_1, \ldots, a_n, \overline{a}_n$ and such that $U_n(r,\theta)$ satisfies the conditions (30), i.e. for $r < 1$ it is harmonic and positive. We can, e.g. for this purpose repeat the construction used above with the help of $\epsilon(\phi, r, \theta)$. We remark that all the coefficients of the functions likewise have absolute value smaller than one, as a consequence of (33).

From here it follows that the remainder $R_p^{(n)}$ of the expansion of U_n for every point of the interior of the circle $r \leqslant \rho < 1$ is smaller than

$$\sum_{k=p}^{\infty} \rho^k \{|a_{k,n}| + |b_{k,n}|\} \leqslant \frac{2\rho^p}{1-\rho}$$

and that consequently for $n > p$ we must have

$$|U_n(r,\theta) - U(r,\theta)| < \frac{4\rho^p}{1-\rho},$$

for every point of the circle $r < \rho$.

In other words, (because one can choose ρ to be arbitrarily close to one) for every value $r < 1$, we have

$$U(r,\theta) = \lim_{n=\infty} U_n(r,\theta) \geqslant 0.$$

24. We shall now finally prove that if the n-th geometrical representative of a function, which for $r < 1$ is positive, harmonic, and begins with constant term $\frac{1}{2}$, comes to lie on the *boundary* of K_{2n}, then this function is uniquely determined by its first $(2n+1)$ coefficients and is of the form $\sum \lambda_j \epsilon(\theta_j; r, \theta)$.

Indeed, let $U(r, \theta)$ be an arbitrary function which has the properties described; its *p.217* $(n+p)$-th geometrical representative will lie on the boundary of $K_{2(n+p)}$ and will be represented as a centre of mass of n discrete masses on the standard curve. Therefore the coefficients $a_{n+p}, \overline{a}_{n+p}$ are *uniquely* determined by $a_1, \overline{a}_1, \ldots, a_n, \overline{a}_n$. Thus there is only one function which satisfies our conditions, and it must coincide with the one we have set out.

Breslau, March 1911.

C. CARATHÉODORY

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