Music of the spheres
Pierre Cartier and Edwige Kertès

Pierre Cartier : 1, 2, 3, let's start the game, 4, 5, 6 ... Let's continue the litany. To count is that. Grain, numbers, one behind the other. These numbers that the mathematician calls integer, exact. And yet they are not enough. Nature has played tricks on us.

Edwige Kertès : Yes, certain quantities and the whole problem is there, have no common measure. Their ratio cannot be calculated exactly.

Pierre Cartier : Take the calendar : Nature has provided us with two natural clocks, the Moon and the Sun. But they are not exactly in agreement.

Edwige Kertès : This lack of concordance questions the mathematician, who dreams of harmony. No concordance between the succession of day and night, the reappearance of the moons, the cycle of the tides, the return of the seasons. Celestial mechanics obeys complex laws.

Pierre Cartier : One year, everyone knows, it is 365 days. No, not entirely : 365 days a quarter. And even there, not a quarter, 1 divided by four, plus something like a tenth. And again, that would require to be specified. But just like small streams make big rivers, little things can sow great disturbances.

Edwige Kertès : Indeed, it only takes a little imprecision, in the evaluation of a year, to lead to a progressive shift of the seasons. Julius Caesar had established a year of 365 days a quarter, or three years of 365 days and a year of 366 days. But this day moreover, every four years weighed too heavy in the balance time since the solar year is slightly less than 365 a quarter days. It was not until the end of the 16th century century that the abolition of three leap days every four hundred years will be decreed.

Pierre Cartier : In music too, we encounter the same phenomena. The fifth, the octave, which are the basic intervals on which everything is built in music, do not stick exactly. We can't do an exact number of fifths which corresponds to an exact number of octaves, regardless of how you get there take, there is always a little leftover. The musicians call it a coma, barely audible.

Edwige Kertès : Barely audible, but if you mix two notes separated by a coma, you clearly perceive the beat between the two vibrations of neighboring frequencies. The musical scale has a mathematical structure. Pythagoras first, looked for the relationship between the length of a string, and the sound it emits. He discovered that if you split a string in half, you get a sound at the top octave. If you split a string two-thirds we get a sound at the top fifth. But going from fifth to fifth, so two-thirds to two-thirds, you never get to the same point as going from octave to octave, so from half to half. Because the powers of three never meet that of two, always pairs. Going through the cycle of so-called natural fifths, we see that the last fifth, the one that joins the octave cycle, is necessarily shorter. It is the fifth of wolf.

Pierre Cartier : Basically, in construction, both of calendars and ranges, the problem is the same : it is a question of distributing both the long and short intervals, the tones, the semitones, by example, and to best distribute the small differences that result from the incommensurability of the two periods.

Edwige Kertès : Coma is the small difference between the wolf's fifth and the natural fifth. To build a moderate range, this difference is distributed regularly over all the fifths, so as to give the best possible fifth and octave, as some calendars closely match Moon and Sun. The interval between two new moons is approximately 29 and a half days. The solar year lasts 12 lunar months, or 354 days. plus a remainder of approximately 11 days. To build a lunisolar calendar, where each month begins with a new moon, while keeping the cycle of the seasons, we must take into account these 11 days lag. How ? In adding a thirteenth moon, therefore a thirteenth month, every two or three years. It was in the fifth century before Jesus-Christ that was discovered in Greece the cycle of 19 years which brings back the new moons almost exactly on the same dates of the solar calendar. It is a question, over nineteen years, of distributing the years of twelve lunar months and the years of 13 moons.

Pierre Cartier : In the minds of many philosophers, the order that governs the movement of the stars is not of a different nature from the harmony that presides over intervals in music. This desire for musical harmony which, for the Pythagoreans, had its foundation in the notion of numbers, we find it in the XVII$^{\text{th}}$ century with the great astronomer Kepler. Kepler is the inventor of a number of formulas, such as the harmony of worlds, the music of the spheres. Having discovered in his astronomical work that the planets do not move at constant speed in their movement of revolution around the Sun, he had imagined to translate this variation by a musical theme which would be specific to each planet and thus the planets would play around the Sun a symphony.

Transcription of a small video viewable here : http://webusers.imj-prg.fr/ jean-michel.kantor/MOSAIQUE/Mosaique-musique.html.

Edwige Kertès : Kepler put all his energy into finding the secret of the world to show its mathematical beauty, sign of divine perfection. God, for Kepler, is an architect who leaves nothing to chance. So Kepler will discover according to which law the planets revolve around the Sun by describing an ellipse. His approach, at the same time metaphysical and scientific, joins that of astrophysicists who, nowadays, build the model of Big-Bang to try to unravel the mysteries of the beginning.

Pierre Cartier : Mathematics claim to establish a world order and are driven by a desire of perfection, but they cannot help but be confronted with natural phenomena. The function of mathematics, it is most often to track the order under the disorder, to seek the exact relation where one had first seen an approximation. When it comes to relating different cycles or lengths immeasurable, it can be done thanks to continuous fractions. These provide a simple representation, but which reveals to the laymen their perpetually unfinished character.

Edwige Kertès : Because whatever the precision of the calculations, there is always a small residue.