What is New Today in a Mathematician's Work ?
Conversation between Alain Connes and Jean-Christophe Yoccoz

Jean-Christophe Yoccoz : Mathematicians'work has changed and we are increasingly working collectively. Fifty years ago, people travelled less. They maintained postal correspondence, but collaborations were not as common. Most articles were single author ones. This was still the case when I started out. Today, there are far more conferences and collective work. This trend took place over the course of our generation and intensified in the last 10 to 15 years. Articles co-signed by two or three authors have become the norm. Speaking for myself, I have different collaborators in France, Brazil, and Italy. We see one another quite regularly, which involves travelling.

Alain Connes : As in other domains, this trend has perverse effects : there is now a plethora of information that needs to be filtered. First, there is less time to devote to each article and second, considerable redundancy develops, as for example in the field of theoretical physics. This is not as true of mathematics ; but still, communication takes place on a more superficial level. On the other hand, Internet and search engines have profoundly altered the picture. Previously, a lot of importance was placed on a certain form of mathematical erudition. Now, Google instantaneously provides all the references one might need on virtually any subject. This wonderful tool is of immense relief to our memory. It is a sort of collective memory - metaphorically speaking of course, for it is no one's memory, but everyone can access it and draw from it what is needed. This is a great step forward.

Jean-Christophe Yoccoz : We should specify that although the number of actors in our discipline has increased, it is still very small in comparison to other fields like biology. What's more, mathematics as a discipline has at the same time witnessed tremendous expansion. As a result, there are few mathematicians in the world working on the same problem. The situation is very different when it comes to finding, for example, a vaccine against Aids : given the urgency of the issue, one can understand that a large number of teams are mobilized to try and solve such a problem.

In mathematics, except for a few particularly exciting problems, the rule is rather the absence of competition. It is therefore an unusual domain, particularly in view of its demographics and the virtual absence of economic pressure.

Joint work corresponds to quite a general trend in science. The production of knowledge has become a collective endeavour in many domains. In medicine, in physics, great experiments, like those of the LHC at CERN, require the collaboration of hundreds, if not thousands of researchers, engineers, technicians, etc. from different disciplines. They can no longer be handled entirely by one

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individual.

JEAN-CHRISTOPHE YOCCOZ : In mathematics, collaborations are on a smaller scale. But the classification of simple groups, for example, mobilized about 200 authors and represents about 10,000 pages, spread across a large number of articles of 50 to 100 pages each. On the scale of our discipline, this work is of enormous proportions, and no one person is entirely responsible for the full proof, as it is simply too long. Still, in mathematics the authors of an article know and understand everything it contains, which is not the case in articles presenting wide-scale interdisciplinary research findings. A mathematician never uses a theorem whose demonstration he or she has not understood.

The question of mathematical proof has long been of interest to philosophers. Wittgenstein, for example, says that there must be a unified perception of proof. Is this possible for such long proofs ?

ALAIN CONNES : There can be seemingly very long proofs that are still manageable. A mathematician, if he or she knows the subject well, will identify strategic points. A proof is not homogeneous ; there are fundamental articulations where things happen. Mathematicians must be able to identify those crucial points and then to introduce hierarchy levels in the proof, so as to extract the essence, in such a way that it may then exist as an entity of its own. One of the crucial aspects of mathematical work is introducing levels of conceptualization. Furthermore, certain complications are disappearing with time. When one examines, for example, Descartes' writings and the way in which he used "Cartesian" coordinates, one gets a sense of great complication, especially because negative numbers were not commonly used. Once real numbers, followed by complex numbers, started to be manipulated, once the hierarchy of structures was understood, and the right notations were adopted, all this complication which was, so to speak spurious, disappeared to give way to great simplicity.

There is therefore a significant striving for simplification, a constant effort that will surely be made for instance for the classification of simple groups. As long as this process is still ongoing, we will not really have finished understanding, we will not really have isolated all important concepts, and things will continue to seem complicated.

There is therefore collective historical prioritization and simplification, on the one hand, and the work of mathematicians who have knowledge about a domain and a sort of virtuosity of proof and understanding, on the other.

JEAN-CHRISTOPHE YOCCOZ : Or a technique, which requires constant practice to maintain its fluidity. While this technical dimension of the activity is necessary, it is not enough.

ALAIN CONNES : Mathematics requires daily practice. If it is interrupted for too long, the know-how wanes. One loses the reflexes. Fortunately, that is temporary. One can compare that with the experience of musicians : Arthur Rubinstein said, “When I stop playing for a day, I hear it, when I stop for two days, the audience hears it”.

Moreover, there is another equally important part of mathematical activity, which is stable : it consists in manipulating mental pictures. Should a layperson try to read an article, they would only see a collection of formulas with no meaning. But a mathematician who has sufficiently worked in a specific domain “inevitably limited” has constructed mental pictures that come to mind as soon as questions relating to that field of activity are addressed : the meaning immediately becomes visible, it jumps to mind.

Jean-Christophe Yoccoz : Even in a familiar domain, with certain difficult articles, due to their length or technicality, one can sometimes struggle to understand until a mental image suddenly forms and allows one to move past the abstract literal rendering involved in deciphering formulas.

What does this mental image correspond to ? Is it the representation of something or on the contrary a pure construction ? For many people, mathematics is like a foreign language, but everyone shares a basic knowledge made up of numbers and elementary geometric objects, which seems to be a sort of product of evolution, an innate mathematical baggage of our species. Do your mental images relate to objects of the same nature, only more complex ?

Jean-Christophe Yoccoz : Whole numbers are concrete images of this kind, which everyone shares. I will draw an analogy with physics. Certain postulates of classical physics allow for the association of images with theory, like in ballistics, where objects’trajectory can be seen. The relation to observation remains quite simple, and the properties grasped are basic, as are the properties of whole numbers or geometry. But there are domains of physics where observation is far less immediate, like electromagnetism or quantum mechanics.

As for the nature of mathematical objects, the Platonic position is prevalent among mathematicians. Even those who challenge it, in practice behave as though they were Platonist : they discover and manipulate mathematical objects as though they were real.

Alain Connes : To begin with, and until the nineteenth century, a large part of mathematics was very close to physics. But then things evolved within mathematics itself. Without seeking to maintain a direct relationship with physics and the outside world, mathematicians discovered an extraordinary universe. Take the example of what is called the p-adic world, in number theory : it is a world that exists in as many versions as there are prime numbers. The real world corresponds to only one of these versions. There are therefore as many of these worlds as there are prime numbers, and these worlds are as beautifully, as brightly coherent as the “real” world of physics. The domain of mathematics is absolutely not limited to geometry or numbers : it is an extraordinary source of creation of concepts. In reality they encompass everything, so I believe that most of the qualities found in the real world, if truly understood, have a mathematical formulation.

One could think that mathematics was created by humans through an adaptive process, as an adaptation of humans to reality, which would explain the astonishing fact that intrinsically mathematical relations, produced from purely mathematical objects, are able to account for physical phenomena as though they were the rules governing them. Yet in fact mathematicians opened doors, discovered horizons that do not simply relate to the real world, but open to other worlds which, while incredibly coherent, have no relation to the "real" world, in the classical physics sense of the term. Or rather worlds that have no materialization in the real world...

Jean-Christophe Yoccoz : Yes, because computer science, for example, gives them new applications, and I'm not just thinking about applications like cryptography. Computer science relies on the same validation and control principles as mathematics. I would readily compare their relations with those that exist between chemistry and physics : chemistry is more oriented towards industry and applications ; computer science is likewise governed by applications. But they function by the same principle, respectively, as physics and mathematics. And just as chemists create products that do not necessarily exist in nature, so computer scientists create things that have a mathematical structure and that do not exist in the world.

Alain Connes : I must admit that I have undue faith in the explanatory power of mathematics for our understanding of the world, and have a profound aversion to the all too widespread tendency to want to found our understanding of reality on the classical model which is valid up to a certain scale, but has no validity for microscopic objects, which are governed by quantum rules. Quantum physics has an explanatory power that is far from being fully integrated into the culture of the society in which we live. Think about electrons or quantum physics equations concerning electrons : this is a wonder. Starting from something extremely simple - Pauli's exclusion principle, which posits that electrons cannot coexist in the same quantum state - the periodic table of elements is reconstituted. That is breath-taking ! Such an explanation does not fit in with the evolutionary framework. Without being mystical in any way, I think that nature is far more subtle and complex than we think, but that it has extremely simple ingredients and that these ingredients are of a mathematical nature.

Our sensory perception affords us only a very partial image of reality. Colour, for instance, captures only three of the infinity of parameters governing the distribution of the intensity of light frequencies. Mathematics has made it possible to simplify, to model parts of outer reality, to the extent that one can end up doubting the idea that mathematics was created to explain the outer world, starting from the material world that surrounds us. I have come to imagine a radically opposite point of view, according to which the mathematical world is in fact the one that pre-exists and it is from this world that a certain image emerges, that which we perceive in the physical world. But we are very far from understanding the fundamental explanation, which is, I think, far simpler and more mathematical than we believe it to be.

While these are crucial aspects of scientific culture, very few people know and understand them. The first step is quantum physics : our world is filled with quantum objects - laser, microchips, etc. - but we have not integrated the quantum dimension into our culture. We live in a quantum world and we continue to think as though we lived in a classical world.

Jean-Christophe Yoccoz : Regarding the question of building, I would add another analogy with physics : in the same way that physicists create instruments, like the telescope, to explore the physical world, so mathematicians create instruments to analyse mathematical realities. There is an element of discovery and an element of invention. Mathematical techniques are human creations, just as physics instruments are.

Mathematics has moreover allowed for logical and mathematical operations to be physically materialized in the form of the computer. What role does this instrument play in the mathematics of today ? Are computers capable of demonstration in the same way as mathematicians are ?

Jean-Christophe Yoccoz : No. First of all, we should specify that the question does not boil down to the sole problem of proofs. The computer is an unparalleled tool for exploration. To demonstrate something, one must have reasonable certainty that it is true. With a computer, counter-examples can, for instance, be discovered, propositions tested, etc. It allows for the observation of interesting phenomena, if only through the mass reiteration of complex calculations, which would be impossible without it.

Alain Connes : The computer dramatically enhances the computational power. In Witt's work on rings, for example, we find very complicated polynomials : whereas these would be extremely long to calculate and manipulate by hand, the result is very easily obtained using a small computer program, thanks to which one can quickly become familiar with these objects that initially seem rather exotic. Yet this exploratory use of computers should not hide another very important aspect, regarding formal proofs. The computer can do much more than check specific cases : it is capable of doing general proofs. It does so very effectively, not as a proof, in the sense of logical deduction, but through formal computation.

In practice, I very often use it in the following way. In a given context, I wish to know whether a formula is true : alone, I can verify it on a very small number of cases only and I'm still at the mercy of a computing error. The machine, however, is capable of demonstrating it (through formal computation) for values such that any mistake can be ruled out, so convincing is the verification. After experimenting in this way, then, I'm sure that it is true. Of course, the direct proof remains to be found, but that is not the most difficult part. It is not a test of specific cases ; it is a formal computation which shows me that my formula is true. This power of formal calculation offers exceptional resources.

Over the course of the history of science, the question of the reliability of instruments has been raised. Can we rely on what we see of celestial bodies with the astronomical telescope ? Is what we see in the microscope a reliable image of reality or an artefact ? Can this question be transposed to the computer ? Do we know what the computer does ? What are its limits ?

JEAN-CHRISTOPHE YOCCOZ : There is a difference between numeric simulation, where the margins of error are not entirely controlled, and computer assisted demonstration, where errors are in principle controlled. In the first case, the role of the computer is to suggest answers and directions for research, their validation still being the mathematician's responsibility. In the second case, the mathematician entrusts the computer with this validation process. But there can be errors in the writing of the programme...

Often, proofs by computers are founded on the exploration of a huge number of cases, as in the case of the four colour theorem (which posits that any map divided into connected regions can be coloured in such a way that two adjacent regions will always receive two distinct colours). Therefore, even if we can be sure that it is true, that is not satisfactory proof for a mathematician.

ALAIN CONNES : There are limits to the use of computers to formally produce statements - including proofs. One cannot overlook the question of meaning, which in my opinion is certainly just as important as that of the nature of mathematical objects. The crucial question is understanding why certain statements have meaning, while others, even if they are true and proven almost mechanically by the computer, are totally uninteresting and devoid of meaning.

There are two human activities which for the moment entirely defy computers and which form a truly fundamental part of mathematics : firstly, the human mind is capable, often in highly complex situations, after having carried out a lot of experiments, calculations, etc., of forming a concept. This is a crucial point.

The second activity, which is out of the reach of computers, is analogical reasoning. When confronted with a given difficulty, mathematicians are able to recognize that the situation is not very different from another they encountered in a different, sometimes far removed context, and to use this analogy to resolve the new problem. This also relates to meaning. It is difficult to objectify : one knows that there is something analogous, but this stems from an intuition and not an explicit, well-formulated perception. A lot of time would be needed to crystallize it, just as it is difficult, in the phase of creation of a concept, to give a set definition. A whole system of maturation and distillation performed by the human mind is at play. This is an extraordinary power, from which the computer, it seems to me, is very far removed.

The computer therefore introduces a new way of working, but not a qualitative break. As a result, the history of mathematics has a cumulative aspect and builds a coherent whole, while other sciences are subject to upheavals, to paradigm overthrows that can lead to part or the whole of the construction being left to decay, having become practically unusable.

Jean-Christophe Yoccoz : The history of mathematics presents a singularity. In physics or biology, centuries were needed before arriving at fundamental concepts like the electron or DNA. The objects that are now considered as fundamental only emerged at a late stage. In mathematics, it is the other way around. At the start, there are whole numbers or basic geometric figures. These are the fundamental mathematical concepts from which increasingly sophisticated concepts are built, like a knowledge pyramid resting on its tip. The concepts that form the basis of the edifice are also the first from a historical point of view. That is why it is so crucial that there be no mistakes : demonstration establishes things, it constitutes a validation and affords a solid grounding upon which to build. Things are different in physics, for example, where theory always has its limits - as seen for instance in the case of classical mechanics - and where one can go back on these theories.

Alain Connes : As a result, in theoretical physics, the “cultural” rules are not the same at all. In a theoretical physics paper, rigorous justification does not weigh in as significantly as in mathematics. In both cases, one must be convincing, but the modalities differ. Unlike physicists, mathematicians cannot do without rigorous proof. This is a general feature of mathematics.