Transcription of a video by Alain Connes at the Collège de France, Language and Thought Symposium (October 2018) (Denise Vella-Chemla, 19.1.2019)

So that's it... thank you. So I'm going to try to lighten the mood, because... so I'm going to bounce on the introduction of Jean-Noël Robert, to point out a variant of "I think therefore I am" which I think, if you will, is actually the best graffiti I have ever seen, it was in Jussieu's restroom *(laughs)* and there was a sign that was put... if you want, which said "Please leave these toilets as clean when you go out as they were when you came in" and there was a clever little guy who wrote below : "I think therefore I wipe" [1] *(Laughs)*... Okay !... So I'm going to talk to you about mathematical language and so if you want, my talk will be in fact both an introduction to mathematical language and at the same time, a reflection on mathematical language. So this language, the mathematical language is a coded language, it is a coded language at the outset which means that there are certain habits. For example, in mathematics, if you like, it's traditional to call the unknown x. This is how in a school, a professor had posed the following exercise : "we give a triangle, the triangle is an equilateral triangle and, well, what do we know ?"... We have Pythagoras' theorem : Pythagoras' theorem says that the square of the hypotenuse is equal to the sum of the squares of the 2 sides, so you have to understand square as really a square, so, here, what do we write ? We write that we have 3 squared, it's worth 9, 4 squared it's worth 16, 9 plus 16 is 25, so we guess... the answer. Hence if you want the teacher's embarrassment when he saw the answer given to him by a student *(AC shows a comic drawing, "Find x", Pupil : "it's here", with an arrow on the x letter, bursts of laughter)*. Okay ? So obviously, it was... it was an unstoppable answer, eh, it is unstoppable, the professor cannot say "it's false" so... so, to get back to the serious stuff, because we're going to get back to the serious stuff now, so my presentation will be divided into 4 parts.

In the first part, I will give you elements of language. So I'll talk to you about geometry, about theorems, I will explain to you what a lemma, a demonstration, a counterexample, a conjecture, I will speak to you about algebra and the great difficulty of this part of the presentation, it will be not let myself be overwhelmed by the demonstration, therefore, because what interests us is language, not, if you will, the semantic content, it is the language which will interest us therefore... And yet, I can't explain it all to you in abstract, I have to explain it to you on an example because that, I will say, we will see that it is extremely important in fact to have a support which is a specific example.

In the second part, I will tell you about a book, which is a book by Hans Freudenthal which is a mathematician, this is the book *(AC shows the Lincos book)*, and it was a mathematician who understood that in fact, there was a particularity, a specificity of mathematical language, which I will not return to, which is that it is without doubt one of the only languages that is not entirely self-referential. You see, when you take a dictionary, the dictionary defines words in relation to other words. But in mathematics, it is impossible, and this is what Hans Freudenthal did in the Lincos, it is possible to build mathematics, gradually, starting if you want to... one impulse, two impulses, all that, it will mean integers, etc., etc. And he understood that, in fact, if there was a possibility of communiculating with an extraterrestrial intelligence, well, we would have to build the language entirely, it is because there, it is really a language, starting from mathematics. And that's what he did. So there are some extremely interesting reflections, but what we will see, what we will see, which is also quite extraordinary, is that in fact, the universe communicates with us, it communicates in mathematical language and I will explain to you in what form it communicates with us.

So the third part will be, it will come from the fact that the mathematical language of course evolves and very gradually, the paradigm that was at the center of mathematics during..., until last fifty years, which was the sets paradigm, has been replaced, very gradually for the past fifty years, by a new paradigm which is thecategories paradigm. And this replacement in fact, at the level of language is extremely important and in fact, it is thanks to this replacement that an extraordinary concept was born in the hands of Alexandre Grothendieck. This concept, this is the concept of topos, and it is a concept that perfectly illustrates the fact that precisely mathematics are not at all, if you like, confined to language, confined to calculations or things like that. In fact, mathematics is a factory that manufactures new concepts and for showing you the richness, the variety if you want of these concepts, the concept of topos is so... powerful that in fact it gives nuances to the notion of truth and we will see at the end of my talk that the concept of topos allows for example to define in a perfectly rigorous way what it is to be at three steps from the truth, four steps from the truth, etc. So it's something extraordinary because that it shows that mathematical language if you will, and not just mathematics but the mathematical language touches... in fact... has a philosophical scope which is far beyond what is normally accepted by the general public. The general

1. *French pun : Je suis, I am, j'essuie, I wipe.*

public confines mathematics to calculations or geometric figures but it is very far from being there. In fact, if you will, most of the concepts really important have a mathematical origin and have a precise mathematical formulation.

So, so, this is the plan, this is the plan, so what I chose to do, like I said, is to go of a concrete example. So this concrete example is a theorem, it is a theorem that has been somewhat found by chance, by Franck Morley. Why was it found by chance ? It was found by chance, it was found in 1899, Franck Morley was not concerned with the question that we are going to see, with the statement that is here. Actually, he was looking for a geometric problem that was much more complicated on cardioid, etc., but basically, he fell in his path, he fell on a fact, that we are going to consider as a fact, and that is a very striking fact, quite extraordinary, which says the next thing : it says that when you take a triangle, any triangle, the triangle is absolutely whatever here, and what you do is, you divide each of the angles of the triangle into three equal parts, therefore, these are called the angle bisectors, and you intersect these bisectors 2 to 2. What the fact says, what Morley's theorem says if you like, is that you always have an equilateral triangle here. So this is something extremely simple, extremely striking, and the reaction of a surveyor, it is doubt. That is to say if you are a surveyor and you are given a similar statement, well, the first thing you have to do is to doubt it. So you doubt, you say, "Boah, I say "it can't be true !". I'm going to make you a triangle that won't work. Okay ?". Well So there, what will start in the brain, maybe Stanislas will tell us things much more precise, but it is the visual areas, that is to say that will make the surveyor doubt, and he will try to build other triangles and he will look at what is going on. So in fact, we're just going to look, we're going to look, we're going to build other triangles, you see. We build another triangle, well, it still seems to be true, what *(laughs)*.

Okay ? So we continue, we continue like this, then you see what I'm doing here, I'm trying to do an extremely weird operation on the triangle, I'm going down the top that's there without moving the vertexes which are there, and normally when we do that, it is an affine transformation, so that does not preserve the lengths at all, so it is very surprising that despite this, the triangle between remains equilateral. So we continue like this, we continue, we look for all kinds of examples, we try to flatten the triangle, to the..., we do the same operation, well, it seems to always work. So at the end, after a while, when we have done this enough times, well, the doubt begins to dissipate, and there, we have to set off because it is not enough to have taken examples, of course, examples never demonstrated a theorem, so the doubt started to dissipate and we're going to leave, we're going to set off, we're going to set out for the demonstration. And then, in a demonstration in general, the demonstration is preceded by stages, these stages are called lemmas, okay ? So it's no coincidence that the name of lemma was used when in fact the astronauts went to the moon, they used the term lemma, and it's exactly in the same sense, that is to say that the lemma is not, if you want the result, but it is what allows you to move forward and to go to the result. So in mathematical language, the lemma has an extremely precise meaning : in general, the statement of the lemma in itself is not enough, how to say, convincing, so that it makes a theorem, it's just a small result that allows you to move forward, it's not really a theorem, Okay ? So here, we are going to see a first lemma and we are going to see how, precisely, this lemma is going to allow to better illustrate, better, if you want, to initiate in mathematical language.

So what does the lemma say ? The lemma says the following : I must not get lost in the demonstration so I will go very fast, he says if you look at the rotations, relative to the vertices, so the center of the rotation is the top, but the angles are double, and you make this product, you make the product $R_A.R_B.R_C$. So you will see, there, there is a peculiarity of mathematical language, is that when the product is well made, one starts by applying R_C. So it will seem weird but it comes because of the notation in mathematical language, because when we take a function of x, by example $\sin x$, the x appears afterwards, so that's why when we apply $R_A.R_B.R_C$, we will first apply R_C and R_B and R_A. Well. So, then, what does the lemma, the little lemma say ? It says that if I make the product of these 3 rotations, I get what is called the identical transformation. So now, what is absolutely amazing is that now we will see that, we will act. At the beginning, if you want, there was a theorem, there was a fact, we were exposed to this fact, but now we're going to start acting, to prove this lemma. How do we act ? Well, that's what we calls a group that acts on a whole. But what does that mean here ? What does that want to say here ? It means something incredibly simple, it means that unlike current language, when we will demonstrate that the product of the 3 rotations R_A, R_B, R_C is identity, well, we're just going to have a very small operation to do, which is the operation to play with parentheses. So you see the group that is going to be in action here, it will contain the rotations, but it's going to be in fact the group of transformations of the plane which preserves the lengths. And then, the thing essential is that in fact, if you take the rotation around point B which is here, well actually, we can break it down into two things : symmetry around 1, then symmetry around 3. You see, if you make the symmetry around 1, for example, if I take the point which is here, and the symmetry around 3, that will turn well from the double angle, which is there. And that works in general. So what is the result ?

Why is the demonstration so simple ? Well, because the R_B that is there, I can, as I told you, as I showed it, I can write it as s_3s_1. The others are s_1s_2 and s_2s_3. And now I can play with parentheses. It's not at all the same as in everyday language, of course. So the role of parentheses is not at all the same. In a group, we can manage with the parentheses as we want, we can do a set of parentheses, and with this set of parentheses, we immediately get the result since you see, now, I'm going to replace this product with a product where s_3, s_3 will be contiguous, s_1, s_1 will be contiguous but since s_3 was a symmetry with respect to a straight line, s_3s_3, it is worth 1, s_1s_1 it's worth 1, there are only s_2s_2 and s_2s_2, it's worth 1. So we're done, okay ? So here is the demonstration.
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So you see the role of language, the role of writing, there, in the demonstration. Then another little lemma, okay, but that will show you how the same type of action is absolutely crucial if you want to demonstrate, it is a lemma which will also make us progress a lot, and which is the following, and which says how we will find the 3 vertices of the Morley triangle, for example, the point P which is here which is at the intersection of these 2 trissectrices. Well, what does the lemma say ? We will have these 2 little ones lemmas. What does the lemma say ? It simply says that the vertex P of the Morley triangle is simply the fixed point of the same thing as earlier, but I put powers 1 / 3. The fact of having put powers 1/3, that means that the angle by which I move is going to be a third, a third of double the angle in C. Well, that's exactly the angle that is here. What does point P do ? That's the action now, that's it, now, we're going to be in action. Well, if we do $R_C^{1/3}$, converting P to P'. And then if we do $R_B^{1/3}$, afterwards, well, we bring P' back to P. So we have the fixed point. So now we start to control things. We start by controlling things because we have 3 symbols, R_A, R_B, R_C, and we control the vertices of Morley's triangle thanks to these 3 things, okay ?. But we are far from having finished and when we are in this situation now, what do we see ? Well, with a little thought, we realize that the two lemmas that I have explained to you, the two lemmas that I have explained to you, in reality, these are really geometric lemmas, that is to say they are lemmas that are going to be still true in non-Euclidean geometry.

What is non-Euclidean geometry ? It may seem to you very complicated, but in fact, in general, if a result is true in non-Euclidean geometry, it is going to be true for spherical geometry. What is spherical geometry, well, it's geometry of the Earth, if you will, but made perfect, a perfect sphere, and in which the lines are the large circles. So, the guess we can do at that time, well, we can say "well, maybe that Morley's theorem is true for spherical geometry. "So it's the same, we're going to use our visual areas this time, and it's going to be more difficult, because now we can't take a straight line, I mean we can no longer draw on a sheet of paper, we have to show you a figure which is more complex. So this is what we do, we try, we do like a while ago, we look, "Ah, it still seems to be true ? !". You see ? I took an extremely special triangle, a extremely particular type of triangle, I took a triangle which had a vertex at the North Pole and whose the base was on the equator. It is a very special triangle because it has 2 right angles, this angle is right, that angle is right and the sum of the angles obviously does not equal $\pi = 180°$. So you see, we try, we try, we look, we look, well, it still seems quite true, I mean, it's really extremely amazing, really extremely amazing, so we use our visual areas, etc., we try, and then there, then there, we can ask the question on the computer.

So here we go... we see the question that really arises "is it true ? Is it true ?". We talk to the computer, Gérard will talk about it much more precisely than I do, we communicate with the computer, it's not very difficult for spherical geometry, it's very very simple, to tell the truth, the spherical geometry is even simpler than ordinary geometry, so we talk to the computer, we asks it the question, and after a moment, the computer tells us "no, that's not true !". Unbelievable ! The computer calculates the difference between the length of the side that is there, since the triangle is isosceles by definition, so these two lengths are equal, but this length is not a priori the same. If it was the same, it would be an equilateral triangle, equilateral, that means that there are the 3 sides which have the same length, so in fact, we look and amazing ! That's the curve of all, all the differences. At first, this difference is almost 0, but you see, it's 0 to 4 decimal places. So it means to say that if we trusted our view, we would have believed that the theorem was true, okay ? In fact, it is not true. It is not true, this is called a counterexample, we made a conjecture, we have a counter example to this guess, and now what is happening ? Because we have a counterexample to our conjecture, that forces us to completely change our strategy for the demonstration.

Why ?

Because if the conjecture had been true, if the theorem had been true in non-Euclidean geometry, the evidence would have been entirely geometric. The fact that the theorem is wrong in non-Euclidean geometry tells us that there will now be a total change of scenery and this total change of scenery, that is going to be the transition to algebra. Algebra was born from a... how to say ? was born from an heresy. Algebra was born out of a heresy a very, very long time ago, long before Jesus-Christ, heretics made the next thing : they added lengths with surfaces... and then surfaces with volumes. This is completely crazy ! You see when at the start, I told you about this little student triangle, when we write the rule of Pythagoras, I told you "the square on one side is equal to the sum of the squares...", that, the dimensions are preserved, because we add between them surfaces, we add between them squares, but there is a very old mathematician, very old, who had the idea of posing a problem, I believe it was among the Babylonians, in which he posed a relationship between surface and length, it's amazing, so, at that time, algebra was born. So it was born out of this heresy, which was to add between them quantities which absolutely do not have the same dimension, which a physicist never would do of course, which is to add lengths, areas and volumes. So then, now... So, here is the demonstration. What does the demonstration say ? It says that now Morley's theorem has nothing to do with a geometric statement for non-Euclidean geometry, it has to do with a statement of algebra, a statement of algebra that is so self-explanatory that it can be given to a computer. When, if you will, a fundamental difference, between Morley's theorem at the outset and the statement that is here, no matter what, I said I don't want to get lost in the technical details, etc., what is the fundamental difference ? The fundamental difference is that when you are in front of a geometric problem, we can dry, we can dry indefinitely. When we are faced with a problem of algebra like that, not only we are not allowed to dry because we have to do maths, but we can delegate the problem to the computer, this is called formal calculus, and this formal calculus is so powerful if you will, that in fact it can do infinitely more complicated calculations than that one, I remember once, a long time ago, in the early 2000s, I had delegated a calculation to the computer that took a whole night, I wanted to demonstrate that a certain product was associative, and well, the next morning, the computer had done the calculation, so it's absolutely phenomenal the power that computer have for making extremely complicated calculations. So when we did that, if you want, we can say, "Okay, well, we gave a proof of Morley's theorem, so if you want, we arrived. The lemma arrived on the moon, okay ? The lemma has arrived on the moon, okay ?. It was the third lemma".

But... now what is the difference ? There is a difference which is absolutely crucial, the crucial difference is the generative power of mathematical language. Because now, once we formulate the result in the form of the lemma, the last, the last lemma, which is purely algebraic, well, Morley's theorem has meaning on any field... Of course, I didn't tell you that there was a secret behind the lemma that I gave here, which was that this lemma, it applies to a field, what is called a field which is the field of complex numbers. The field of complex numbers, it's a miracle, it's the fact that you can, we're used to rational numbers, for example, that you can add, multiply, etc. We are used to real numbers, well, there are an extension of real numbers, which is wonderful, and which makes it sufficient to actually add a root of an equation which is the equation $x^2 + 1 = 0$ so that we can solve all the other equations. This is what we call the field of complex numbers, and this field of complex numbers, well, we learned it in... I think we learned it in the second, I learned it in the second, in physics, to do electromagnetism, and in fact, what is extraordinary in the field of complexes is that a triangle equilateral is characterized by the property that 3 points form the vertices of an equilateral triangle if and only if $a + jb + j^2c = 0$ where j is a cubic root of unity. So now, what's wonderful, is that we have a Morley theorem for any field that contains a cubic root of unity. So these fields abound, for example, you can take the integers modulo 7, the integers modulo 13 or the integers modulo 31, these are fields, this is called Galois fields, and they all have a cubic root of unity, so there is, for all of them, a Morley theorem that applies ; we would not have been able to imagine, if we had not had this path, through geometry and algebra, which led us here. Okay ? So, this is where we are, this is where we are. So now, yes, of course, you have to let me tell you, yes, anyway, that there is, in mathematical language, a constant use of words of ordinary language. We use, in mathematical language, words of ordinary language, and there is a rather provocative sentence, which is made through the words of ordinary language, and which is called "Disintegration of atomic measurements on nuclear spaces". So if you want, this sentence is all the more ironic as the inventor of the nuclear space denomination, first of all, it is ironic because a mathematician who knows the meaning of these words will tell you "it's trivial." So triviality, it's common, it's a kind of mathematical jargon, which is for something which is true, but which is irrelevant, okay ? So... But what is really ironic in this sentence that I have given is that the inventor, the inventor of the concept of nuclear space is a formidable mathematician, well known, Alexandre Grothendieck, but who spent most of his life being an activist anti-nuclear. *(laughs)* So, then, he is... there is always, he had an absolutely extraordinary gift for finding the right terminology. So, so, I'm coming to Lincos now, so, I'm not going to paraphrase what Hans Freudenthal says, his book is extremely precise, etc., he wrote an introduction which I recommend that you read, in which he makes reflections on mathematical language and on language in general and it explains a point, which is a delicate point, which is a point which is not at all obvious, which is that in mathematics, we use variables, well I told you about x at all the time, and it's a bit the same when we talk about everyday language. But, in mathematics, the variables have a specific meaning, and in addition, for example, there is a notion of a generic object, that is to say in mathematics, we can speak for example of a generic triangle, etc., there is what is called a generic point, when we talk about a mathematical set. So what's really amazing is that if.... So... Hans Freudenthal discusses in great detail what is happening in ordinary language, and the fact that in ordinary language, of course, there are variables : for example, we say a door, we said a goat. You will see that I am not taking the example of the goat at random.

Okay, but then of course, if we were to ask the question, it would be extremely difficult to describe, a generic door, or describe a generic goat. Only a great artist got there for the goat, and I guess you know the work in question, it's called, it's Picasso's goat, and it has this property, this goat, this incarnation of the goat has this extraordinary property, how to say, that it concretely abstract the properties of the goat through an artistic work. So, of course, in mathematics, all that, it has a precise meaning, it has a much more precise meaning, and good... to return to communication with an extraterrestrial intelligence, we tried, we tried, we tried to communicate with a possible extraterrestrial intelligence, by sending the probe, for example, by sending the Pioneer probe in space, and on this probe, we gave a certain amount of information, well, for example, we gave the Sun, finally, we gave the position of the planets, etc., etc., but, hey, is it included in what we gave in this probe, that we gave the aliens the possibility of replying us ? I pretend there is another way, it would have been to send them a little triangle from Morley *(bursts of laughter)*, okay ?... And imagine that they respond to us like that... Well here, there, you would know that... not only they did understand us, but also that they are intelligent people, etc., etc. So in fact, well, in fact, this mean of communication I am talking about is not very practical because it means that they should have received the probe as a physical object, and that they should return us a physical object. This is not how we communicate in the Universe. We know well that the Universe is written in mathematical language, but how does the Universe communicate with us in mathematical language ? So there, you are going to be really surprised, because the way the universe communicate with us, we'll see, it's... you know, we've looked a lot... we've looked a lot at... how to say... at how to label, label objects, etc. And while trying to label the objects, we felt gradually on the right idea. And the good idea of labeling objects is called barcodes. What I'm going to explain to you now is that the Universe communicates with us through barcodes. And in addition, these barcodes are mathematical objects. And I will tell you, I don't know how much time I have left, well, but I'm going to... And I need to tell you about the topos as well. So... I'll tell you about it. Why ? Because what is amazing is how it has been discovered. First, it was discovered in physics. In physics, there is a German optician, who was called Fraunhofer and who had the absolutely brilliant idea of looking at the sunlight, the rainbow which comes to us from the sun when we pass the sunlight through a prism, to look at it with a microscope. And he noticed that there were black lines. There were a number of black stripes. So at the start, he must have thought his lens was dirty, etc. Then, well... And finally, during his existence, he found 500 black rays. Then there were physicists, I think it was Bunsen and Kirchhoff who succeeded in heating bodies like sodium to obtain rays which this time were bright lines, not black lines, on a black background, which corresponded to the lines of the spectrum of Fraunhofer, that came from the sun. And they realized, if you will, that each chemical element, had a barcode. So each chemical element, for example hydrogen, etc., had a barcode. Except that... there was a barcode they couldn't find on Earth. There was a barcode missing on Earth. And so, as good physicists, they gave it a name, they called it helium.

They said "there is a chemical body, which is not present on Earth, which is in the heliosphere and which is called helium." Miracle ! There was an eruption of Vesuvius. They did the spectrography of the lavas of Vesuvius and they found helium in it. So this is extraordinary ! What does it mean ? It was the first step. It was the first step in barcodes in physics. The second step which is absolutely staggering, it's at the level of quantum mechanics, it's Schrödinger, when he had the idea of his equation, he had the idea if you like, that he had an operator and this operator, in fact, this is what we call spectra of course. So he went to mathematicians and asked them, "What is it a spectrum in mathematics ?". So his friends told him, "Well, go and see Hermann Weyl, and he will tell you right away what it is.". And Schrödinger said, "Especially not, he would calculate the spectrum before me.". So what Schrödinger did which was extraordinary is that he kicked off the fact that all these spectra, all these barcodes that come to us from the Universe, in fact, they have a reason to be mathematical, in fact, they are mathematical beings. These are the spectra of operators in a Hilbert space.

And when you see them, they seem very complicated, but the operator is a lot simpler. And it is the operator who gives us the key, for example, the key which allows us to understand the elements table of Mendeleïev. So in fact... the Universe speaks to us, it speaks to us, but it speaks to us in a spectral manner, it speaks to us by sending us barcodes, these barcodes are shifted towards the red, it is which allowed us to understand the expansion of the Universe, etc. But you see the phenomenal role, there, of writing. There, I'm not talking about mathematical language, I'm talking about writing. So the Universe sends information in writing things to us, and its writing is spectral. So now I come to the evolution of mathematical language, okay ? So the mathematical language has evolved of course. He has evolved over the past 50 years, and it has evolved in ways that will be quite difficult to get across teaching. The reason why I think it will be quite difficult to pass in teaching, this is what happened when Lichnerowicz promulgated the teaching of set theory, in secondary classes, even in middle school. And I remember an exercise I saw, I mean I had witnessed an exercise. Exercise is the following : we take three sets A, B, C. We trace Venn diagrams, okay, *(laughs)*, and the subject of the exercise was : suppose $A \cup B = C$ and the subject of the exercise was "Hatch the empty set." *(He shows the exercise with the 3 sets.)*. So there was only one student who found it, and he put "I couldn't, it's empty.". *(laughs)*. Okay, so it is clear that we will have problems for categories, well, categories is already the level above set theory. But as I said in my introduction, the categories allowed, they allowed precisely, thanks to their flexibility compared to set theory of inventing a language, if you will, of developing a language, that was, how to say, that was formulated, initiated etc. by Alexandre Grothendieck and that is the concept of topos.

So the concept of topos, it's good that I simply give you an idea. And this idea is that instead of focusing on a space, like the space of earlier, the sphere, or something like that, instead of the space I'm showing you, which is front and center, the space will play a completely different role. The space will disappear, will be behind the stage, but it will play the role of a Deus ex machina, that is to say that in fact, we are going to do set theory as we usually done, but the space in question will serve as a parameter. That is, the space in question will never be at the front of the stage, it will be behind, and then, what we realize when we do this, is that all the properties of set theory, which are ordinary properties, for example the demonstration that I gave you of Morley's theorem, will continue to take place, provided that you never used the absurd reasoning, provided you never used the excluded middle rule. All these reasonings will continue to work, it's extraordinary because it gives you reasonings that will work with parameters, that will work with this hazard, if you want.

The topos introduces a hazard, okay ? And so now, what's amazing is that, from this fact alone, the notion of truth will become more subtle. And instead of only having the true or the false, which allowed us to reason by the absurd, we are going to have a notion of truth which is going to be a lot more nuanced, much more subtle, than the ordinary notion, we will continue to work as if we was working in set theory, and what seems likely to me is that gradually this notion will allow us to formalize situations in which to say that one is right, that one is wrong is completely naive, you see, when for instance you see a chat on TV or a stuff like that and where the notion of truth will become much more interesting and much more subtle, and adapted to a given situation. So that was done by Grothendieck, so there is the notion of truth in a topos, and there is as I told you the possibility of having nuances. So I will finish showing you a sentence, sorry, a text, a page of math language text, but to show you the richness, the variety of mathematical language, in all its splendor, this page of text, this is probably the last letter that Grothendieck wrote to a mathematician and it is about of... it is a question concerning original paradise, topological algebra, perennial semi-simplicial category, the eye of the surveyor, bundles of sets, from having felt, etc., categorical topos, etc. So you see the immense richness of mathematical language, and to what extent, precisely, the philosophical significance of this language is something which is often overlooked, but which is of an incomparable power. Here, so, I will finish on that, thanks.