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A Note on the Exceptional Set for Goldbach's Problem in Short Intervals

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Abstract. Assume the Generalized Riemann Hypothesis and suppose that $H \log^{-6} x \to \infty$. Then we prove that all even integers in any interval of the form $(x, x+H)$ but $O(H^{1/2} \log^3 x)$ exceptions are a sum of two primes.

1. Introduction

Let $2 \leqslant H \leqslant N$, $L = \log N$, $M = \log H$,

$$R(2n) = \sum_{h+k=2n} \Lambda(h)\Lambda(k)$$

and

$$\mathfrak{S}(2n) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid n \\ p>2}} \left(\frac{p-1}{p-2}\right).$$

In a recent paper [P–P], the two last-named authors proved that

$$\sum_{N \leqslant 2n \leqslant N+H} |R(2n) - 2n\mathfrak{S}(2n)|^2 \ll_{\varepsilon, A} HN^2 L^{-A} \tag{1}$$

provided $0 < \varepsilon < \frac{2}{3}$, $A > 0$ and $H \geqslant N^{1/3+\varepsilon}$.

In this paper we investigate the size of the exceptional set for Goldbach's problem in short intervals under the assumption of the Generalized Riemann Hypothesis (GRH). We prove that an estimate of the type (1) holds, under GRH, in the wider range $HL^{-6} \to \infty$ as $N \to \infty$.

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Theorem. *Assume GRH. Then*

$$\sum_{N \leqslant 2n \leqslant N+H} |R(2n) - 2n\mathfrak{S}(2n) + F(n,N,H)|^2 \ll H^{1/2}N^2L^3$$

where $F(n,N,H)$ *is a certain function satisfying*

$$F(n,N,H) \ll NH^{-1/8}(LM)^{1/2}.$$

Observing that $\mathfrak{S}(2n) \gg 1$ and $F(n,N,H) = o(N)$ uniformly for $N \leqslant 2n \leqslant N+H$ and $H \geqslant L^{4+\varepsilon}$, we easily deduce the following result from the theorem.

Corollary. *Assume GRH and let* $HL^{-6} \to \infty$. *Then all even integers in* $[N, N+H]$ *but* $O(H^{1/2}L^3)$ *exceptions are a sum of two primes.*

In other words, writing

$$E(N,H) = |\{2n \in [N, N+H] : 2n \text{ is not a sum of two primes}\}|$$

we have

$$E(N,H) = O(H^{1/2}L^3).$$

In the case $H = N$, GOLDSTON [Go] obtained, under GRH, the estimate

$$E(N,N) = O(N^{1/2}L^4),$$

which he has subsequently improved (written communication) to

$$E(N,N) = O(N^{1/2}L^3).$$

We remark that our results remain true assuming the Riemann Hypothesis only for the L-functions associated with characters to moduli $\leqslant H^{1/2}$. We also remark that D. WOLKE and G. DUFNER proved independently a result similar to ours for the twin primes and the Goldbach problems respectively, with the condition $H \geqslant L^{27}$ and also S. M. VORONIN [V] announced a similar result with $H \geqslant L^c$, with an unspecified constant $c > 0$.

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2. The Major Arcs

Given $Q = \frac{1}{2}H^{1/2}$ and the Farey dissection of order Q of $I = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right]$, denote by $I_{q,a} = \left\{\alpha = \frac{a}{q} + \eta,\ \eta \in \xi_{q,a}\right\}$, where $\xi_{q,a} \subset$

$\subset \left(-\frac{1}{qQ}, \frac{1}{qQ}\right)$, the Farey arc with center at $\frac{a}{q}$. Let

$$\mathfrak{M} = \bigcup_{q \leqslant P} \bigcup_{a=1}^{q}{}^{*} I_{q,a}, \qquad \mathfrak{m} = I \backslash \mathfrak{M}$$

where * means that $(a, q) = 1$ and P will be chosen later on. Let

$$S(\alpha) = \sum_{n \leqslant 2N} \Lambda(n) e(n\alpha), \qquad e(x) = e^{2\pi i x}.$$

We write

$$S\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} T(\eta) + R(\eta, q, a),$$

with

$$T(\eta) = \sum_{n \leqslant 2N} e(n\eta)$$

so that

$$\begin{aligned}
&\int_{\mathfrak{M}} S(\alpha)^2 e(-2n\alpha) d\alpha = \\
&= \sum_{q \leqslant P} \frac{\mu(q)^2}{\varphi(q)^2} \sum_{a=1}^{q}{}^{*} e\left(-\frac{2na}{q}\right) \int_{\xi_{q,a}} T(\eta)^2 e(-2n\eta) d\eta + \\
&\quad + \sum_{q \leqslant P} \sum_{a=1}^{q}{}^{*} e\left(-\frac{2na}{q}\right) \int_{\xi_{q,a}} R(\eta, q, a)^2 e(-2n\eta) d\eta + \\
&\quad + 2 \sum_{q \leqslant P} \frac{\mu(q)}{\varphi(q)} \sum_{a=1}^{q}{}^{*} e\left(-\frac{2na}{q}\right) \int_{\xi_{q,a}} T(\eta) R(\eta, q, a) e(-2n\eta) d\eta = \\
&= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \quad \text{say.}
\end{aligned} \tag{2}$$

We have

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{q \leqslant P} \frac{\mu(q)^2}{\varphi(q)^2} \sum_{a=1}^{q}{}^{*} e\left(-\frac{2na}{q}\right) \int_0^1 T(\eta)^2 e(-2n\eta) d\eta + O\left(Q \sum_{q \leqslant P} \frac{q}{\varphi(q)}\right) = \\
&= 2n \sum_{q \leqslant P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2n) + O(PQ),
\end{aligned} \tag{3}$$

where $c_q(-m)$ is the Ramanujan sum. By the Cauchy–Schwarz inequality we get

$$\sum\nolimits_2 \ll \sum_{q\leqslant P}\sum_{a=1}^{q}{}^{*}\int_{-1/qQ}^{1/qQ}|R(\eta,q,a)|^2\,d\eta \tag{4}$$

$$\sum\nolimits_3 \ll N^{1/2}\sum_{q\leqslant P}\varphi(q)^{-1/2}\left(\sum_{a=1}^{q}{}^{*}\int_{-1/qQ}^{1/qQ}|R(\eta,q,a)|^2\,d\eta\right)^{1/2}. \tag{5}$$

It is easy to see that

$$R(\eta,q,a)=\frac{1}{\varphi(q)}\sum_{\chi(\operatorname{mod} q)}\chi(a)\tau(\bar\chi)\psi'(2N,\chi,\eta)+O(L^2)$$

where $\tau(\chi)$ is the Gauss sum and

$$\psi'(2N,\chi,\eta)=\sum_{n\leqslant 2N}\Lambda(n)\chi(n)e(n\eta)-\delta_\chi T(\eta),\qquad \delta_\chi=\begin{cases}1 & \text{if}\,\chi=\chi_0\\ 0 & \text{if}\,\chi\neq\chi_0.\end{cases}$$

Hence by the orthogonality of the characters we get

$$\sum_{a=1}^{q}{}^{*}\int_{-1/qQ}^{1/qQ}|R(\eta,q,a)|^2\,d\eta \ll$$

$$\ll \frac{q}{\varphi(q)}\sum_{\chi(\operatorname{mod} q)}\int_{-1/qQ}^{1/qQ}|\psi'(2N,\chi,\eta)|^2\,d\eta+\frac{\varphi(q)L^4}{qQ}. \tag{6}$$

In order to estimate the right hand side of (6) we need the following result.

Lemma 1. *Assume GRH. Then for any* $\chi(\operatorname{mod} q)$

$$\int_{-1/qQ}^{1/qQ}|\psi'(2N,\chi,\eta)|^2\,d\eta \ll \frac{NL^2}{qQ}.$$

Proof. By GALLAGHER's lemma (see [Ga]) we have

$$\int_{-1/qQ}^{1/qQ}|\psi'(2N,\chi,\eta)|^2\,d\eta \ll$$

$$\ll \frac{1}{(qQ)^2}\int_{-2N}^{2N}\left|\sum_{n\in[x,x+2qQ]\cap[1,2N]}(\Lambda(n)\chi(n)-\delta_\chi)\right|^2 dx. \tag{7}$$

Arguing as in Lemma 6 of SAFFARI–VAUGHAN [S–V] we see that the integral on the right hand side of (7) is $\ll NqQL^2$ and the lemma follows.

From (6) and Lemma 1 we get

$$\sideset{}{^*}\sum_{a=1}^{q} \int_{-1/qQ}^{1/qQ} |R(\eta, q, a)|^2 d\eta \ll \frac{NL^2}{Q} \tag{8}$$

and hence from (4), (5) and (8) we obtain that

$$\sum\nolimits_2 + \sum\nolimits_3 \ll \frac{PNL^2}{Q} + \left(\frac{P}{Q}\right)^{1/2} NL \ll \left(\frac{P}{Q}\right)^{1/2} NL \tag{9}$$

provided

$$P \ll QL^{-2}. \tag{10}$$

Since at the end we will choose P as a function of H and N, from (2), (3) and (9) we have that

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-2n\alpha) d\alpha - 2n\mathfrak{S}(2n) = -2n \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2n) - F(n, N, H) \tag{11}$$

where $F(n, N, H)$, defined by (11), satisfies

$$F(n, N, H) \ll \left(\frac{P}{Q}\right)^{1/2} NL + PQ \tag{12}$$

provided (10) holds, the series in (11) being convergent.

From (11) we obtain that

$$\begin{aligned} \sum_{N \leqslant 2n \leqslant N+H} &|R(2n) - 2n\mathfrak{S}(2n) + F(n, N, H)|^2 \leqslant \\ &\leqslant \sum_{N \leqslant 2n \leqslant N+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 e(-2n\alpha) d\alpha \right|^2 + \\ &+ \sum_{N \leqslant 2n \leqslant N+H} \left| 2n \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2n) \right|^2, \end{aligned} \tag{13}$$

with $F(n, N, H)$ satisfying (12). In order to estimate the second sum on the right hand side of (13) we need the following result.

Lemma 2. *If* $P \leqslant H^{1/2}$ *we have*

$$\sum_{N \leqslant m \leqslant N+H} \left| \sum_{q>P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-m) \right|^2 \ll \frac{H \log^3 P}{P^2} + L^2.$$

Proof. It is well known that if $\mu(q)\neq 0$ then $c_q(-m)=\mu(q_1)\dfrac{\varphi(q)}{\varphi(q_1)}$ where $q_1=\dfrac{q}{(q,m)}$. In the sequel all numbers coming from the q's will be square-free. Hence for any m we have

$$\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-m)\leqslant\sum_{d|m}\frac{1}{\varphi(d)}\sum_{l>P/d}\frac{1}{\varphi(l)^2}\ll\sum_{\substack{d|m\\ d>P}}\frac{1}{\varphi(d)}+\frac{1}{P}\sum_{\substack{d|m\\ d\leqslant P}}\frac{d}{\varphi(d)}. \tag{14}$$

By a splitting up of the interval $(P,m]$, writing $d\sim D$ instead of $D<d\leqslant 2D$ and $[d,d']=\dfrac{dd'}{(d,d')}$ we obtain

$$\begin{aligned}
&\sum_{N\leqslant m\leqslant N+H}\Bigg(\sum_{\substack{d|m\\ d>P}}\frac{1}{\varphi(d)}\Bigg)^2=\\
&=\sum_{D,D'}{}^{*}\sum_{N\leqslant m\leqslant N+H}\sum_{\substack{d|m\\ d\sim D}}\frac{1}{\varphi(d)}\sum_{\substack{d'|m\\ d'\sim D'}}\frac{1}{\varphi(d')}\ll\\
&\ll\sum_{D,D'}{}^{*}\sum_{d\sim D}\sum_{d'\sim D'}\frac{1}{\varphi(d)\varphi(d')}\left(\frac{H}{[d,d']}+1\right)\ll\\
&\ll H\sum_{D,D'}{}^{*}\sum_{r\leqslant\min\{D,D'\}}\sum_{l\sim D/r}\sum_{l'\sim D'/r}\frac{1}{r\varphi(r)^2\varphi(l)\varphi(l')ll'}+L^2\ll\\
&\ll H\sum_{D,D'}{}^{*}\sum_{r\leqslant\min\{D,D'\}}\frac{r}{\varphi(r)^2DD'}+L^2\ll\\
&\ll H\sum_{D,D'}{}^{*}\frac{\log\min\{D,D'\}}{DD'}+L^2\ll\frac{H\log P}{P^2}+L^2,
\end{aligned} \tag{15}$$

where $\sum^{*}_{D,D'}$ means that D and D' run over the integer powers of 2 such that $P/2<D,D'\leqslant m$.

In an analogous way, since $P\leqslant H^{1/2}$ we have

$$\begin{aligned}
\sum_{N\leqslant m\leqslant N+H}\Bigg(\frac{1}{P}\sum_{\substack{d|m\\ d\leqslant P}}\frac{d}{\varphi(d)}\Bigg)^2&\ll\frac{H}{P^2}\sum_{d\leqslant P}\sum_{d'\leqslant P}\frac{d}{\varphi(d)}\frac{d'}{\varphi(d')}\frac{1}{[d,d']}\ll\\
&\ll\frac{H\log^3 P}{P^2},
\end{aligned} \tag{16}$$

and Lemma 2 follows from (14)–(16).

From (13) and Lemma 2 we finally get

$$\sum_{N\leqslant 2n\leqslant N+H}\left|2n\sum_{q>P}\frac{\mu(q)^2}{\varphi(q)^2}c_q(-2n)\right|^2\ll\frac{N^2H\log^3 P}{P^2}+(NL)^2 \tag{17}$$

since $P<Q=\frac{1}{2}H^{1/2}$.

3. The Minor Arcs

Due to our choice of Q and writing $\|\cdot\|$ = distance from the nearest integer, arguing as in Sect. 5 of [P–P] we obtain that

$$\begin{aligned}\sum_{N\leqslant 2n\leqslant N+H}\left|\int_{\mathfrak{m}}S(\alpha)^2e(-2n\alpha)d\alpha\right|^2&\ll\\ \ll\int_{\mathfrak{m}}|S(\xi)|^2\int_{\mathfrak{m}}|S(\alpha)|^2\min\left\{H,\frac{1}{\|2(\xi-\alpha)\|}\right\}d\alpha d\xi&\ll\\ \ll HNL\max_{\xi\in[0,1]}\int_{(\xi-(1/H),\xi+(1/H))\cap\mathfrak{m}}|S(\alpha)|^2d\alpha&\ll\\ \ll HNL\max_{\substack{P<q\leqslant Q\\(a,q)=1}}\int_{-1/qQ}^{1/qQ}\left|S\left(\frac{a}{q}+\eta\right)\right|^2d\eta.&\end{aligned}\tag{18}$$

By (8) we get for $P<q\leqslant Q$ that

$$\begin{aligned}\int_{-1/qQ}^{1/qQ}\left|S\left(\frac{a}{q}+\eta\right)\right|^2d\eta&\ll\\ \ll\frac{1}{\varphi(q)^2}\int_{-1/qQ}^{1/qQ}|T(\eta)|^2d\eta+\int_{-1/qQ}^{1/qQ}|R(\eta,q,a)|^2d\eta&\ll\\ \ll\frac{N}{\varphi(q)^2}+\frac{NL^2}{Q}\ll\frac{N\log^2 P}{P^2}+\frac{NL^2}{Q},&\end{aligned}$$

so that by (18) we obtain

$$\sum_{N\leqslant 2n\leqslant N+H}\left|\int_{\mathfrak{m}}S(\alpha)^2e(-2n\alpha)d\alpha\right|^2\ll\frac{HN^2L\log^2 P}{P^2}+\frac{HN^2L^3}{Q}. \tag{19}$$

Now from (13), (17) and (19) we have

$$\sum_{N\leqslant 2n\leqslant N+H}|R(2n)-2n\mathfrak{S}(2n)+F(n,N,H)|^2\ll\frac{HN^2L\log^2 P}{P^2}+\frac{HN^2L^3}{Q}. \tag{20}$$

We choose $P = H^{1/4}M/L$. Hence for $H \gg (ML)^4$ we see that (10) is satisfied, (12) becomes

$$F(n, N, H) \ll NH^{-1/8}(ML)^{1/2}$$

and the right hand side of (20) is $\ll N^2 H^{1/2} L^3$. Now the theorem is proved.

References

[Ga] Gallagher, P. X.: A large sieve density estimate near $\sigma = 1$. Invent. Math. **11**, 329–339 (1970).

[Go] Goldston, D. A.: On Hardy and Littlewood's contribution to Goldbach conjecture. Proc. Amalfi Conf. Analytic Number Theory, 115–155 (1992).

[P–P] Perelli, A., Pintz, J.: On the exceptional set for Goldbach's problem in short intervals, J. London Math. Soc. **47**, 41–49 (1993).

[S–V] Saffari, B., Vaughan, R. C.: On the fractional parts of x/n and related sequences II. Ann. Inst. Fourier **27**, 1–30 (1977).

[V] Voronin, S. M.: On the circle method (Russian). Discrete Mathematics **2**, 60–70 (1990).

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