Grain d'sel, Denise Vella-Chemla, juin 2026.

L'un des verbes anglais traduisant l'expression française “mettre son grain d'sel” est “to chip in”. C'est adéquat lorsqu'une informaticienne met son grain d'sel. Est fourni ci-dessous un programme sympathique, basé sur le fait qu'on a choisi l'expression

$$f(x) = \frac{2\pi x}{W\left(\dfrac{x}{e}\right)}$$

avec $W(x)$ la fonction de Lambert pour tout de même assez bien approximer les parties imaginaires des zéros de zeta, et qu'on a trouvé cette image dans un article de Knuth et al. pour la fonction de Lambert qu'on calcule en utilisant à peine cinq logarithmes.

6 A Final Pair of Expansions

The iterations (76–77) may be used to show that $W(z)$ can be written as

$$W(z) = \cfrac{z}{\exp \cfrac{z}{\exp \cfrac{z}{\exp \cfrac{z}{\ddots}}}} \qquad (98)$$

or

$$W(z) = \ln \cfrac{z}{\ln \cfrac{z}{\ln \cfrac{z}{\ddots}}} \qquad (99)$$

according as $|W(z)| < 1$ or $|W(z)| > 1$. These curious formulae are just the iterated exponential in disguise, and indeed are naturally discovered from rewriting $W(z) = z/\exp W(z)$ and $W(z) = \ln(z/W(z))$ as iterations.

Cela nous permet, par le programme `Python` ci-dessous de calculer, très lentement, les parties imaginaires des 100000 premiers zéros de la fonction ζ de Riemann assez bien.

Le programme :

```
import math
from math import pi,e,log,exp
import mpmath
from mpmath import zetazero

tic = time.time()
for x in range(4,101):
    numerateur = 2*pi*x
    denominateur = log((x/e)/log((x/e)/log((x/e)/log((x/e)/log(x/e)))))
    res1 = numerateur/denominateur
    res2 = zetazero(x).imag
    print(x, ' --> ', res1,'    ',res2,'   ',res1/res2)
tac = time.time()
print(tac-tic,' s.')
tic = time.time()
for x in [999,1000,1001,9999,10000,10001,99999,100000,100001,
          999999,1000000,1000001,9999990,10000000,10000001,
          99999999,100000000,100000001,999999999,1000000000,1000000001,
          9999999999,10000000000,10000000001]:
    numerateur = 2*pi*x
    denominateur = log((x/e)/log((x/e)/log((x/e)/log((x/e)/log(x/e)))))
    res1 = numerateur/denominateur
    res2 = zetazero(x).imag
    print(x, ' --> ', res1,'    ',res2,'   ',res1/res2)
tac = time.time()
print(tac-tic,' s.')
```

Son résultat : la fonction démarre très mal, elle nous satisfait à partir de $x = 9$. La colonne ratio fournit le rapport $\dfrac{f(x)}{\mathfrak{Im}(x^{\text{ieme}} \text{ zero de } \zeta)}$

x	$f(x)$	$\mathfrak{Im}(\text{zetazero}(x))$	ratio
4	-40.55657415398964	30.4248761258595	-1.33300704286249
5	87.20362822471078	32.9350615877392	2.64774450147603
6	51.21664818022682	37.5861781588257	1.36264580995183
7	46.651091201129944	40.9187190121475	1.14009168242243
8	46.81591627756783	43.327073280915	1.08052339409202
9	48.4925035226753	48.0051508811672	1.01015209061033
10	50.78139695584081	49.7738324776723	1.02024285509099
11	53.340578460437115	52.9703214777145	1.00698989495237
12	56.01773580759968	56.4462476970634	0.99240849645554
13	58.738616479508586	59.3470440026024	0.989747972568489
14	61.4651044027874	60.8317785246098	1.01041110244576
15	64.17717830867335	65.1125440480816	0.985634630729256
16	66.86438214275536	67.0798105294942	0.996788476517177
17	69.52150829219724	69.546401711174	0.999642060288322
18	72.14629862429595	72.0671576744819	1.0010981555589
19	74.73817166878499	75.7046906990839	0.987233036402715
20	77.29749679760648	77.1448400688748	1.0019788326555

x	$f(x)$	$\mathfrak{Im}(\text{zetazero}(x))$	ratio
21	79.82517118846009	79.3373750202494	1.00614837796292
22	82.32236957641726	82.910380854086	0.992907869055582
23	84.79039506872199	84.7354929805171	1.00064792315798
24	87.2305902440869	87.4252746131252	0.997773134029034
25	89.64428475997335	88.8091112076345	1.00940414267165
26	92.03276530742295	92.4918992705585	0.99503595485922
27	94.39725932917929	94.6513440405199	0.997315572072259
28	96.73892722091314	95.8706342282453	1.00905692342246
29	99.05885973005148	98.8311942181937	1.00230357948883
30	101.35807849045516	101.317851005731	1.00039704241972
31	103.63753839234552	103.725538040478	0.999151610588913
32	105.89813096579495	105.446623052326	1.00428186223891
33	108.1406882602511	107.168611184276	1.00907053908073
34	110.36598689710472	111.02953554317	0.994023674486083
35	112.57475209732677	111.874659176993	1.0062578328773
36	114.76766156669478	114.320220915453	1.00391392395553
37	116.94534917288847	116.226680320858	1.00618333802572
38	119.10840838183584	118.790782865976	1.00267382290272
39	121.25739544162117	121.370125002421	0.999071191853867
40	123.3928323152422	122.946829293553	1.00362760897741
41	125.51520937128033	124.256818554346	1.01012733813384
42	127.62498784593873	127.516683879596	1.00084933173485
43	129.72260209210143	129.578704199956	1.00111050571955
44	131.80846163186996	131.087688530933	1.00549840422861
45	133.88295302895853	133.497737202998	1.0028855607146
46	135.94644159672978	134.756509753374	1.008830236443
47	137.99927295675053	138.116042054533	0.999154558036518
48	140.04177446169496	139.736208952121	1.00218673106895
49	142.07425649531461	141.123707404021	1.00673557341129
50	144.09701366109095	143.111845807621	1.00688390152409
51	146.11032587012352	146.000982486766	1.00074892224351
52	148.1144593378068	147.42276534256	1.0046919076144
53	150.10966749792024	150.053520420785	1.00037418033897
54	152.09619184190518	150.925257612241	1.00775837158199
55	154.07426269032695	153.024693811199	1.00685882031839
56	156.04409990281758	156.112909294238	0.999559233174685
57	158.0059135321637	157.597591817594	1.00259091341346
58	159.9599044276318	158.849988171421	1.00698719760063
59	161.90626479211548	161.188964137596	1.00445006057553
60	163.84517869722936	163.030709687182	1.00499580116906
61	165.77682256006588	165.5370691879	1.00144833645625
62	167.70136558496358	167.184439978175	1.00309194807158
63	169.61897017330887	169.094515415569	1.00310154800971
64	171.5297923040979	169.911976479412	1.00952149376523
65	173.43398188772443	173.411536519592	1.00012943411138
66	175.3316830952232	174.754191523366	1.00330459353692
67	177.22303466498786	176.44143429771	1.00442980057597
68	179.1081701887941	178.3774077761	1.0040967206655
69	180.9872183787883	179.916484020257	1.0059512854776
70	182.86030331695116	182.207078484366	1.00358506836298

x	$f(x)$	$\mathfrak{Im}(\text{zetazero}(x))$	ratio
71	184.7275446884076	184.874467848387	0.999205281498901
72	186.5890579998319	185.598783677707	1.00533556471924
73	188.44495478408666	187.228922583502	1.00649489503975
74	190.2953427921334	189.416158656017	1.00464154770298
75	192.14032617316357	192.026656360714	1.00059194808994
76	193.9800056438164	193.079726603846	1.00466273210454
77	195.81447864727704	195.265396679529	1.00281197783675
78	197.64383950298202	196.876481840958	1.00389766037491
79	199.46817954759842	198.015309676252	1.00733715930209
80	201.28758726788834	201.264751943704	1.00011345913263
81	203.10214842602113	202.493594514141	1.00300529956684
82	204.91194617785047	204.189671803105	1.00353727183343
83	206.71706118463194	205.394697202163	1.00643816028594
84	208.51757171862008	207.906258887806	1.0029403291372
85	210.31355376294977	209.576509716856	1.00351682565517
86	212.10508110617496	211.690862595365	1.00195671417146
87	213.89222543181026	213.347919359713	1.00255126027819
88	215.6750564031944	214.547044783491	1.00525764230797
89	217.45364174397147	216.169538508264	1.00594025987458
90	219.22804731446323	219.067596349021	1.00073242674004
91	220.99833718418697	220.714918839314	1.00128409237746
92	222.76457370075357	221.430705554693	1.00602386260171
93	224.5268175553655	224.007000254604	1.0023205404303
94	226.28512784511761	224.983324669582	1.00578622072301
95	228.03956213228975	227.421444279679	1.00271794005428
96	229.790176500808	229.337413305525	1.0019742229964
97	231.5370256100378	231.250188700499	1.00557326997527
98	233.28016274606233	231.98723525318	1.00557326997527
99	235.0196398705888	233.693404178908	1.00567510964351
100	236.75550766761566	236.524229665816	1.00097781949074

Exécution en 8.548126459121704 s.

x	$f(x)$	$\mathfrak{Im}(\text{zetazero}(x))$	ratio
999	1418.845086069349	1418.69696385245	1.00010440722767
1000	1420.0041288365396	1419.422480946	1.00040977784863
1001	1421.1629971287484	1420.41652632375	1.00052552951276
9999	9876.381175553825	9876.47901706378	0.999990093482729
10000	9877.234801767912	9877.7826540055	0.999944536921212
10001	9878.088417961295	9878.65477238569	0.999942668871679
99999	74917.67699298714	74920.2597932589	0.999965526010202
100000	74918.34637684899	74920.8274989942	0.999966883412423
100001	74919.01576007373	74921.9297939584	0.99996110572842
999999	600259.6318192366	600269.005560249	0.999984384099586
1000000	600260.1797379042	600269.677012445	0.999984178320338
1000001	600260.7276565284	600270.301090712	0.999984051461207
9999990	4992340.390705031	4992376.23342942	0.999992820508168
10000000	4992345.015587815	4992381.01400318	0.999992789329327
10000001	4992345.478076076	4992381.26271121	0.999992832150982

x	$f(x)$	$\mathfrak{Im}(\text{zetazero}(x))$	ratio
99999999	42653397.17720029	42653549.3447753	0.999996432475671
100000000	42653397.576620035	42653549.7609516	0.999996432082854
100000001	42653397.97603978	42653550.0467585	0.999996434746497
999999999	371869495.18624645	371870203.802552	0.999998094452584
1000000000	371869495.5373369	371870203.837028	0.999998095303996
1000000001	371869495.88842726	371870204.366313	0.999998094824814
9999999999	3293528046.039799	3293531632.14024	0.999998911168666
10000000000	3293528046.352748	3293531632.39714	0.999998911185685
10000000001	3293528046.6656966	3293531632.68696	0.999998911192708

Exécution en 22.373121976852417 s.