ON THE LARGE SIEVE

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1. The purpose of this paper is to give a new and improved version of Linnik's large sieve, with some applications. The large sieve has its roots in the Hardy–Littlewood method, and in its most general form it may be considered as an inequality which relates a singular series arising from an integral $\int_0^1 |S(\alpha)|^2 d\alpha$, where $S(\alpha)$ is any exponential sum, to the integral itself.

Very recently Roth [1] made important progress on this problem by proving the following results:

THEOREM A. (Roth [1].) *Let n_j $(1 \leqslant j \leqslant Z)$ be distinct natural numbers not exceeding N, and let $Z(N; q, a)$ denote the number of those n_j that are congruent to $a \pmod q$. Let $X \geqslant 2$, and let $\mathscr{P}$ be a set of distinct primes $p \leqslant X$. Then, for any $R \geqslant 2$,*

$$\sum_{p \in \mathscr{P}} p \sum_{a=1}^{p} \left(Z(N; p, a) - Z/p\right)^2 \ll ZN + ZX^2 \log R + Z^2 |\mathscr{P}| R^{-2},$$

where $|\mathscr{P}|$ denotes the number of elements of $\mathscr{P}$.

In particular, if $X \geqslant N^{\frac{1}{2}} (\log N)^{-\frac{1}{2}}$, then

$$\sum_{p \leqslant X} p \sum_{a=1}^{p} \left(Z(N; p, a) - Z/p\right)^2 \ll ZX^2 \log X.$$

We shall improve a little upon this result by proving the following

THEOREM 1. *With the notation of Theorem A, we have*

$$\sum_{p \leqslant X} p \sum_{a=1}^{p} \left(Z(N; p, a) - Z/p\right)^2 \leqslant 7 \max(N, X^2) Z.$$

The general version of the large sieve which we shall consider here, and which contains Theorem 1 as a special case, will take the form of the following theorem, where now any reference to prime numbers and sequences of integers has disappeared:

THEOREM 2. *Let the a_n be any complex numbers, and put*

$$S(\alpha) = \sum_{Y < n \leqslant Z} a_n e(n\alpha), \tag{1.1}$$

where as usual $e(t) = e^{2\pi i t}$. Then we have

$$\sum_{q \leqslant X} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |S(a/q)|^2 \leqslant 7 \max(Z - Y, X^2) \sum_{Y < n \leqslant Z} |a_n|^2. \tag{1.2}$$

[MATHEMATIKA 12 (1965), 201–225]

In order to deduce Theorem 1, take $Y=0$ and $Z=N$ in Theorem 2, and take $a_n=1$ if $n=n_j$ and $a_n=0$ otherwise. Then $\Sigma|a_n|^2=Z$ (in the notation of Theorem A), and (1.2) gives

$$\sum_{p\leqslant X}\sum_{a=1}^{p-1}|S(a/p)|^2\leqslant 7\ \max(N, X^2)\,Z.$$

A simple computation shows that

$$\sum_{a=1}^{p-1}|S(a/p)|^2=p\sum_{a=1}^{p}\Big(Z(N;p,a)-Z/p\Big)^2,$$

and Theorem 1 follows immediately.

It may be of interest to remark that Theorem 2 is not far from best possible. Take first $Z-Y\geqslant X^2$ and $a_n=1$. Then $|S(1)|^2=(Z-Y)^2$, and (1.2) gives the upper bound $7(Z-Y)^2$; this shows that for $Z-Y\geqslant X^2$ we cannot replace the factor $7\max(Z-Y, X^2)$ by $\max(Z-Y, X^2)$. Now take $Y=0$, $Z=1$, $a_1=1$. Then $|S(a/q)|=1$, so that the left-hand side of (1.2) is

$$\Big(\sum_{q\leqslant X}\phi(q)\Big)\Big(\sum_{Y<n\leqslant Z}|a_n|^2\Big)\sim\frac{3X^2}{\pi^2}\sum_{Y<n\leqslant Z}|a_n|^2.$$

This shows that for $Z-Y<X^2$ we cannot replace the factor 7 by any number less than $3/\pi^2$.

We may look upon (1.2) as an inequality involving additive characters, and we may ask whether there exists a similar inequality but with multiplicative characters instead. This is in fact the case, though the final result takes a different shape.

Let Q denote a finite set of positive integers, and put

$$M=M(Q)=\max_{q\in Q}q, \tag{1.3}$$

$$D=D(Q)=\max_{q\in Q}d(q), \tag{1.4}$$

where $d(q)$ denotes the number of divisors of q. Also, for any character χ to the modulus q, let $\tau(\chi)$ denote the Gaussian sum

$$\tau(\chi)=\sum_{a=1}^{q}\chi(a)\,e(a/q). \tag{1.5}$$

We have

$$|\tau(\chi)|^2=\begin{cases}\mu^2(q/q^*)\,q^* & \text{if } (q^*, q/q^*)=1,\\ 0 & \text{otherwise,}\end{cases} \tag{1.6}$$

where q^* is the conductor of the character χ (so that χ is the extension to the modulus q of a primitive character to the modulus q^*). Note that $|\tau(\chi)|^2=q$ if χ is a primitive character (mod q), and $|\tau(\chi_0)|^2=\mu^2(q)$ for the principal character χ_0, and $|\tau(\chi)|^2\leqslant q$ always.

The multiplicative analogue of Theorem 2 is:

THEOREM 3. *Let the a_n be any complex numbers, and let Q be any finite set of positive integers. Then*

$$\sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi} |\tau(\chi)|^2 \Big| \sum_{Y < n \leqslant Z} \chi(n) a_n \Big|^2 \leqslant 7D \max(Z - Y, M^2) \sum_{Y < n \leqslant Z} d(n) |a_n|^2, \tag{1.7}$$

where $\sum_{\chi}$ denotes a summation over all characters χ to the modulus q.

In special cases it is possible to prove a slightly better result. In particular, if $a_n = 1$ if n is a prime and 0 otherwise, there is a result of the same general type but without D and $d(n)$.

Theorem 3 has important consequences for the theory of the zeros of Dirichlet's L-functions and the distribution of prime numbers; in fact the large sieve was originally created by Linnik with a view to applications to classical problems in the theory of primes.

The principal result we shall prove on the distribution of the primes runs as follows. Let

$$\psi(z; q, a) = \sum_{\substack{n \leqslant z \\ n \equiv a \pmod q}} \Lambda(n),$$

where $(a, q) = 1$, and consider the error term in the prime number theorem for arithmetical progressions:

$$E(z; q, a) = \psi(z; q, a) - z/\phi(q). \tag{1.8}$$

Define $E(z, q)$ and $E^*(z, q)$ by

$$E(z, q) = \max_{(a, q)=1} |E(z; q, a)|, \tag{1.9}$$

$$E^*(z, q) = \max_{y \leqslant z} E(y, q). \tag{1.10}$$

THEOREM 4. *For any positive constant A there exists a positive constant B such that if $X \leqslant z^{\frac{1}{2}} (\log z)^{-B}$ then*

$$\sum_{q \leqslant X} E^*(z, q) \ll z (\log z)^{-A}. \tag{1.11}$$

We shall show that a possible value for B is $3A + 23$.

It seems likely that this result could also be proved by appealing to Roth's Theorem A; however, the use of Theorem 3 seems to be more appropriate.

We remark that nothing more precise than (1.11) can be proved even on the assumption of the generalized Riemann hypothesis, if we apply this in the form $E^*(z, q) \ll z^{\frac{1}{2}} (\log z)^2$ for $q \leqslant z$. One can say that Theorem 4 may serve as a good substitute for the generalized Riemann hypothesis in many additive problems involving primes. There are several instances

of this general principle, and Professor Davenport and the author have worked out in detail an application to the study of small differences between prime numbers in a paper submitted to *Proc. Royal Soc. A*.

Results like (1.11), such as

$$\sum_{q \leqslant z^{\eta-\varepsilon}} \mu^2(q) E(z, q) \ll z(\log z)^{-A} \tag{1.12}$$

for some positive constant η have been claimed by various authors. The work of Linnik and Rényi [2, 3] on the large sieve led to an inequality slightly weaker than (1.12), for some η. More recently, work on this subject has been published by Barban†‡ and Pan Cheng-Dong§||. However, Barban's work has been subjected to criticism by Pan Cheng-Dong§, and the present writer is unable to understand Pan Cheng-Dong's paper||. (It appears that the exceptional set of primes in Lemma 1.2 of this paper depends on s and a, while the choices of a and $s(=\rho)$ in (2.7) depend on D, with many possible choices for ρ, so that Lemma 1.2 is not applicable.)

Theorem 4 will be deduced from a new type of density theorem (Theorem 5 below) for the zeros of L-functions; most of the known theorems in the so-called statistical theory of L-functions are contained in this density result.

Let $N(\alpha, T; \chi)$ denote the number of zeros of $L(s, \chi)$ in the rectangle

$$\alpha \leqslant \sigma \leqslant 1, \quad |t| \leqslant T, \tag{1.13}$$

where $\frac{1}{2} \leqslant \alpha \leqslant 1$. Our principal density theorem is:

THEOREM 5. *Let Q be a finite set of positive integers and let M and D be defined by* (1.3) *and* (1.4). *Then*

$$\sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi} |\tau(\chi)|^2 N(\alpha, T; \chi) \ll DT(M^2+MT)^{4(1-\alpha)/(3-2\alpha)} \log^{10}(M+T) \tag{1.14}$$

uniformly with respect to Q, for $\frac{1}{2} \leqslant \alpha \leqslant 1$, $T \geqslant 2$.

The proofs of Theorems 4 and 5 are self-contained, except for references to classical work of Landau, Littlewood and Titchmarsh, and we hope that they are given with an adequate amount of detail.

It may be useful if we add some remarks on the significance of (1.14). It was remarked by Littlewood that many results in the theory of Dirichlet's L-functions $L(s, \chi)$, valid for fixed χ and variable s, have

† M. B. Barban, *Trudy Mat. Inst. Akad. Nauk Uz. S.S.R.*, 22 (1961), 1–20.

‡ M. B. Barban, *Mat. Sbornik* (N.S.), 61 (103) (1963), 418–425.

§ Pan Cheng-Dong, *Acta Math. Sinica*, 14 (1964), 597–606 = *Chinese Math.*, 5 (1964), 642–652.

|| Pan Cheng-Dong, *Acta Math. Sinica*, 13 (1963), 262–268 = *Chinese Math.*, 4 (1963), 283–290.

analogues ("q-analogues") for fixed s and variable χ (to variable modulus). A good example of this is

$$\overline{\lim_{t\to\infty}} |L(1+it, \chi)|/\log\log t > 0,$$

where the q-analogue is

$$\overline{\lim_{q\to\infty}} L(1, \chi)/\log\log q > 0$$

for the quadratic character $\chi \pmod q$.

It is easily seen that our inequality (1.14) is related to the q-analogue of the density hypothesis for the zeros of L-functions. This hypothesis asserts that

$$\sum_{\chi} N(\alpha, T; \chi) \ll q^{1+\epsilon} T^{2(1-\alpha)+\epsilon}, \tag{1.15}$$

and its q-analogue is

$$\sum_{\chi} N(\alpha, T; \chi) \ll q^{2(1-\alpha)+\epsilon} T^{1+\epsilon}. \tag{1.16}$$

Here $\sum_{\chi}$ denotes as usual a sum over all characters $\chi \pmod q$. The last two inequalities are unproved and probably very difficult. However, it is possible to prove something very nearly as good as (1.16). In fact, as we shall see later, it is possible to deduce from Theorem 5 the

COROLLARY. *We have, uniformly for* $\frac{1}{2} \leqslant \alpha \leqslant 1$ *and* $2 \leqslant T \leqslant \sqrt{X}$, *that*

$$\sum_{q\leqslant X} \sum_{\chi} N(\alpha, T; \chi) \ll X^{1+2(1-\alpha)+\epsilon} T^{1+\epsilon}. \tag{1.17}$$

Further,

$$\sum_{q\leqslant X} \sum_{\chi} N(\alpha, T; \chi) \ll X^{1+\epsilon} T^{2+\epsilon} \tag{1.18}$$

uniformly for $\frac{5}{6} \leqslant \alpha \leqslant 1$, $2 \leqslant T \leqslant X^2$.

(1.17) shows that the density hypothesis (1.16) is true on the average with respect to q if $2 \leqslant T \leqslant (\max q)^{\frac{1}{2}}$, and (1.18) shows that even more is true on the average with respect to q if $\frac{5}{6} \leqslant \alpha \leqslant 1$ and $T \leqslant (\max q)^2$. The latter result is in a sense surprising, and the explanation is that Theorem 5 is really a new type of density result. We conjecture the following

Density Hypothesis. If $\frac{1}{2} \leqslant \alpha \leqslant 1$ *and* $T \geqslant 2$, *then*

$$\sum_{q\leqslant X} \sum_{\chi}^{*} N(\alpha, T; \chi) \ll X^{4(1-\alpha)+\epsilon} T^{1+\epsilon} \tag{1.19}$$

uniformly in α, *where* $\sum_{\chi}^{*}$ *denotes a sum over the primitive characters* $\pmod q$.

Finally, the author wishes to express his deep gratitude to Professor Davenport for a discussion which originated this work and for his help in revising this paper.

2. In this section we shall prove Theorems 2 and 3. A rectangle in the (m, n) plane of the type

$$y < m \leqslant z, \quad y' < n \leqslant z'$$

will be denoted by $R(y, z; y', z')$, or briefly by R. Let $c_{m,n}$ be a double sequence of complex numbers, defined when (m, n) is in the square $Y < m \leqslant Z$, $Y < n \leqslant Z$; to every such sequence we can associate a sub-rectangle

$$R_0 = R_0(Y, Z_0; Y, Z_0')$$

of the square, depending on the sequence $c_{m,n}$ and on Y, Z, with the property that:

for every rectangle $R = R(Y, z; Y, z')$ contained in $R(Y, Z; Y, Z)$ we have

$$\left|\sum_R c_{m,n}\right| \leqslant \left|\sum_{R_0} c_{m,n}\right|. \tag{2.1}$$

Plainly such a rectangle R_0 always exists, though it need not be unique.

LEMMA 1. (Abel's inequality.) *Let $b_{m,n}$ be real numbers defined for $Y < m \leqslant Z$, $Y < n \leqslant Z$ and satisfying the conditions*

$$\text{(i)} \quad \begin{cases} b_{m,n} \geqslant 0, \quad b_{m,n} - b_{m+1,n} \geqslant 0, \quad b_{m,n} - b_{m,n+1} \geqslant 0, \\ b_{m,n} - b_{m+1,n} - b_{m,n+1} + b_{m+1,n+1} \geqslant 0. \end{cases}$$

Let $B = \max b_{m,n}$. Then

$$\left|\sum_{R_0} c_{m,n} b_{m,n}\right| \leqslant B \left|\sum_{R_0} c_{m,n}\right|. \tag{2.2}$$

Proof. Put $b^*_{m,n} = b_{m,n}$ if $(m, n) \in R_0$ and $b^*_{m,n} = 0$ otherwise. Partial summation gives

$$\sum_{R_0} c_{m,n} b_{m,n} = \sum_{R_0} \left(\sum_{R(Y,m;\ Y,n)} c_{h,k} \right) (b^*_{m,n} - b^*_{m+1,n} - b^*_{m,n+1} + b^*_{m+1,n+1}).$$

Hence

$$\left|\sum_{R_0} c_{m,n} b_{m,n}\right| \leqslant \left(\max_R \left|\sum_R c_{m,n}\right| \right) \left(\sum_{R_0} |b^*_{m,n} - b^*_{m+1,n} - b^*_{m,n+1} + b^*_{m+1,n+1}| \right).$$

By (2.1) we have

$$\max_R \left|\sum_R c_{m,n}\right| \leqslant \left|\sum_{R_0} c_{m,n}\right|,$$

and from the conditions (i) we have

$$\begin{aligned} &\sum_{R_0} |b^*_{m,n} - b^*_{m+1,n} - b^*_{m,n+1} + b^*_{m+1,n+1}| \\ &\quad = \sum_{R_0} (b^*_{m,n} - b^*_{m+1,n+1} - b^*_{m,n+1} + b^*_{m+1,n+1}) \\ &\quad = b_{Y+1,Y+1} \leqslant B. \end{aligned}$$

This proves (2.2).

LEMMA 2. *Let $c_{m,n}$ and R_0 be as before, and suppose that $\eta > 0$. Then*

$$\left|\sum_{R_0} c_{m,n} - (2\eta)^{-1}\sum_{R_0} c_{m,n}\int_{-\eta}^{\eta} e\Big((m-n)\beta\Big)\,d\beta\right| \leqslant \left(\frac{\sinh x}{x} - 1\right)\left|\sum_{R_0} c_{m,n}\right|, \tag{2.3}$$

where $x = 4\pi\eta(Z-Y)$.

Proof. We have

$$(2\eta)^{-1}\sum_{R_0} c_{m,n}\int_{-\eta}^{\eta} e\Big((m-n)\beta\Big)\,d\beta = \sum_{k=0}^{\infty}(-1)^k\frac{(2\pi\eta)^{2k}}{(2k+1)!}\sum_{R_0} c_{m,n}(m-n)^{2k}$$

$$= \sum_{R_0} c_{m,n} \quad + \quad T,$$

where

$$T = \sum_{k=1}^{\infty}\frac{(-1)^k(2\pi\eta)^{2k}}{(2k+1)!}\sum_{r=0}^{2k}(-1)^r\binom{2k}{r}\sum_{R_0} c_{m,n}(Z-m)^r(Z-n)^{2k-r}.$$

The sequence $b_{m,n} = (Z-m)^r(Z-n)^{2k-r}$ satisfies the conditions (i) of Lemma 1, and $B \leqslant (Z-Y)^{2k}$. Hence, by (2.2),

$$\left|\sum_{R_0} c_{m,n}(Z-m)^r(Z-n)^{2k-r}\right| \leqslant (Z-Y)^{2k}\left|\sum_{R_0} c_{m,n}\right|.$$

Thus

$$|T| \leqslant \sum_{k=1}^{\infty}\frac{(2\pi\eta)^{2k}}{(2k+1)!}\sum_{r=0}^{2k}\binom{2k}{r}(Z-Y)^{2k}\left|\sum_{R_0} c_{m,n}\right|$$

$$= \left(\frac{\sinh x}{x} - 1\right)\left|\sum_{R_0} c_{m,n}\right|,$$

where $x = 4\pi\eta(Z-Y)$. This proves Lemma 2.

Proof of Theorem 2. Let

$$S_{m,q} = \sum_{\substack{a=1\\(a,q)=1}}^{q} e(am/q) \tag{2.4}$$

be the well-known Ramanujan sum. Take

$$c_{m,n} = a_m\bar{a}_n\sum_{q\leqslant X} S_{m-n,q} \tag{2.5}$$

for $(m,n)\in R(Y,Z;Y,Z)$. Choose η so that

$$x = 4\pi\eta(Z-Y) = \min\Big(1{\cdot}3168,\ 2\pi(Z-Y)X^{-2}\Big). \tag{2.6}$$

Then $\sinh x < 2x$, and (2.3) implies that

$$\left|\sum_{R_0} c_{m,n}\right| \leqslant \frac{2\pi(Z-Y)}{2x-\sinh x}\left|\sum_{R_0} c_{m,n}\int_{-\eta}^{\eta} e\Big((m-n)\beta\Big)\,d\beta\right|.$$

Since

$$2x - \sinh x > \frac{x}{1\cdot 4632}$$

for $0 < x \leqslant 1\cdot 3168$, we obtain

$$\left|\sum_{R_0} c_{m,n}\right| \leqslant \max\left(7(Z-Y),\ 1\cdot 47X^2\right)\left|\sum_{R_0} c_{m,n}\int_{-\eta}^{\eta} e\Big((m-n)\beta\Big)\,d\beta\right|.$$

Let $\mathfrak{M}_{a,q}$ denote the interval $|\alpha - a/q| < \eta$, and put

$$S(\alpha;\ Y,\ Z) = \sum_{Y<n\leqslant Z} a_n\, e(n\alpha),$$

this being the same as $S(\alpha)$ in (1.1). We have

$$\sum_{R_0} c_{m,n}\int_{-\eta}^{\eta} e\Big((m-n)\beta\Big)\,d\beta$$

$$= \sum_{Y<m\leqslant Z_0}\ \sum_{Y<n\leqslant Z_0'} a_m\bar{a}_n \sum_{q\leqslant X}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}_{a,q}} e\Big((m-n)\alpha\Big)\,d\alpha$$

$$= \sum_{q\leqslant X}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}_{a,q}} S(\alpha;\ Y,\ Z_0)\,\overline{S(\alpha;\ Y,\ Z_0')}\,d\alpha.$$

This has absolute value

$$\leqslant \tfrac{1}{2}\sum_{q\leqslant X}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}_{a,q}} \left\{|S(\alpha;\ Y,\ Z_0)|^2 + |S(\alpha;\ Y,\ Z_0')|^2\right\}\,d\alpha$$

$$\leqslant \max_z \sum_{q\leqslant X}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}_{a,q}} |S(\alpha;\ Y,\ z)|^2\,d\alpha$$

$$\leqslant \max_z \int_0^1 |S(\alpha;\ Y,\ z)|^2\,d\alpha$$

$$= \max_z \sum_{Y<n\leqslant z} |a_n|^2 = \sum_{Y<n\leqslant Z} |a_n|^2.$$

Here we have used the fact that the intervals $\mathfrak{M}_{a,q}$ do not overlap, and this follows from the fact that $q \leqslant X$, $(a,q) = 1$, and that the length 2η of $\mathfrak{M}_{a,q}$ satisfies $2\eta \leqslant X^{-2}$ by our choice of x in (2.6).

We have now proved that

$$\left|\sum_{R_0} c_{m,n}\right| \leqslant \max\left(7(Z-Y),\ 1\cdot 47\,X^2\right)\sum_{Y<n\leqslant Z}|a_n|^2.$$

Since

$$\left|\sum_{R} c_{m,n}\right| \leqslant \left|\sum_{R_0} c_{m,n}\right|$$

by (2.1), and

$$\sum_{R(Y,Z;\,Y,Z)} c_{m,n} = \sum_{q\leqslant X} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{Y<m\leqslant Z} \sum_{Y<n\leqslant Z} a_m \bar{a}_n e\big(a(m-n)/q\big)$$

$$= \sum_{q\leqslant X} \sum_{\substack{a=1\\(a,q)=1}}^{q} |S(a/q;\,Y,Z)|^2,$$

we obtain (1.2).

The proof of Theorem 3 is on similar lines, but we need also the following lemma relating multiplicative characters to Ramanujan sums.

LEMMA 3. *We have*

$$\sum_{\chi} |\tau(\chi)|^2 \chi(m)\bar{\chi}(n) = \begin{cases} \phi(q)\, S_{m-n,q} & \text{if } (mn,q)=1,\\ 0 & \text{if } (mn,q)>1, \end{cases} \tag{2.7}$$

where the summation is over all characters χ to the modulus q.

Proof. If $(mn,q)>1$ the result is trivial, since then $\chi(m)\bar{\chi}(n)=0$ for every χ. Now suppose that $(mn,q)=1$. Write Σ' for summation over all residues (mod q) that are relatively prime to q. We have

$$\sum_{\chi} |\tau(\chi)|^2 \chi(m)\bar{\chi}(n) = \sum_{a}{}' \sum_{b}{}' \sum_{\chi} \chi(am)\bar{\chi}(bn)\, e\big((a-b)/q\big)$$

$$= \phi(q) \sum_{\substack{a\\ am\equiv bn \,(\mathrm{mod}\, q)}}{}' \sum_{b}{}' e\big((a-b)/q\big)$$

$$= \phi(q) \sum_{h}{}' e\big(h(n-m)/q\big)$$

$$= \phi(q)\, S_{n-m,q} = \phi(q)\, S_{m-n,q},$$

since all solutions of $am\equiv bn \pmod q$ are given by $a\equiv hn$, $b\equiv hm$. This proves Lemma 3.

Proof of Theorem 3. We take

$$c_{m,n} = a_m \bar{a}_n \sum_{q\in Q}{}^{*}\, S_{m-n,q}, \tag{2.8}$$

where Σ^* means that we restrict ourselves to those q which satisfy $(q,mn)=1$. Let

$$S_q(\alpha;\,Y,Z) = \sum_{\substack{Y<n\leqslant Z\\(n,q)=1}} a_n e(n\alpha), \tag{2.9}$$

$$S^{(d)}(\alpha;\,Y,Z) = \sum_{\substack{Y<n\leqslant Z\\ d\mid n}} a_n e(n\alpha). \tag{2.10}$$

Choosing η so that

$$x = 4\pi\eta(Z-Y) = \min\big(1{\cdot}3168,\, 2\pi(Z-Y)M^{-2}\big),$$

we obtain as before

$$\left|\sum_{R_0} c_{m,n}\right| \leqslant 7\max(Z-Y, M^2)\left|\sum_{R_0} c_{m,n}\int_{-\eta}^{\eta} e\big((m-n)\beta\big)\,d\beta\right|$$

$$= 7\max(Z-Y, M^2)\left|\sum_{q\in Q}\sum_{a}{}'\int_{\mathfrak{M}_{a,q}} S_q(\alpha;\ Y, Z_0)\,\overline{S_q(\alpha;\ Y, Z_0')}\,d\alpha\right|$$

$$\leqslant 7\max(Z-Y, M^2)\max_z\left(\sum_{q\in Q}\sum_{a}{}'\int_{\mathfrak{M}_{a,q}} |S_q(\alpha;\ Y, z)|^2\,d\alpha\right).$$

By a well-known identity, followed by the use of Cauchy's inequality, we have

$$|S_q(\alpha;\ Y, Z)|^2 = \left|\sum_{d\mid q}\mu(d)\,S^{(d)}(\alpha;\ Y, Z)\right|^2$$

$$\leqslant d(q)\sum_{d\mid q}|S^{(d)}(\alpha;\ Y, Z)|^2$$

$$\leqslant D\sum_{d=1}^{\infty}|S^{(d)}(\alpha;\ Y, Z)|^2. \tag{2.11}$$

Hence

$$\max_z\left(\sum_{q\in Q}\sum_{a}{}'\int_{\mathfrak{M}_{a,q}}|S_q(\alpha;\ Y, z)|^2\,d\alpha\right) \leqslant D\sum_{d=1}^{\infty}\max_z\int_0^1|S^{(d)}(\alpha;\ Y, z)|^2\,d\alpha$$

$$= D\sum_{d=1}^{\infty}\max_z\sum_{\substack{Y<n\leqslant z\\ d\mid n}}|a_n|^2$$

$$= D\sum_{Y<n\leqslant Z} d(n)|a_n|^2.$$

Theorem 3 follows easily, because by Lemma 3

$$\sum_{R(Y,Z;\,Y,Z)} c_{m,n} = \sum_{q\in Q}\sum_{\substack{Y<m\leqslant Z\\ (m,q)=1}}\sum_{\substack{Y<n\leqslant Z\\ (n,q)=1}} a_m\bar{a}_n S_{m-n,q}$$

$$= \sum_{q\in Q}\sum_{Y<m\leqslant Z}\sum_{Y<n\leqslant Z} a_m\bar{a}_n\frac{1}{\phi(q)}\sum_{\chi}|\tau(\chi)|^2\chi(m)\bar{\chi}(n)$$

$$= \sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi}|\tau(\chi)|^2\left|\sum_{Y<n\leqslant Z}\chi(n)\,a_n\right|^2.$$

3. In this section we give the proof of Theorem 4. We shall adhere to the following notation. For any character $\chi \pmod q$ we shall denote by χ^* the (unique) primitive character associated with χ and by q^* its modulus, that is, the conductor of χ. By $\sum_{\chi}^{*}$ we shall denote a sum over the primitive

characters to the modulus q. The principal character (mod q) will be denoted by χ_0. For any character χ we define

$$\psi(z, \chi) = \sum_{n \leqslant z} \chi(n) \Lambda(n). \tag{3.1}$$

LEMMA 4. *Let N be arbitrarily large but fixed, and let $X_0 = (\log z)^N$. Suppose that $X \leqslant z^{\frac{1}{2}}$. For any $D \geqslant 2$ and any positive integer M, let Q_M denote the set of integers q satisfying*

$$1 < q \leqslant M, \quad d(q) \leqslant D. \tag{3.2}$$

Then we have

$$\sum_{q \leqslant X} E^*(z, q) \ll z(\log z)^{-A} + zD^{-1}(\log z)^3 + (\log z)^3 \max_{X_0 < M \leqslant X} M^{-1} \sum_{Q_M} \sum_{\chi}^{*} \max_{y \leqslant z} |\psi(y, \chi)|, \tag{3.3}$$

for every (arbitrarily large) fixed A.

Proof. We have

$$\sum_{q \leqslant X} E^*(z, q) = \sum_{q \in Q_X} + \sum_{q \notin Q_X} = \Sigma_1 + \Sigma_2, \text{ say.} \tag{3.4}$$

It is easy to estimate Σ_2. Plainly

$$\psi(z;\ q, a) \ \ll \ (\log z) \sum_{\substack{n \leqslant z \\ n \equiv a (\mathrm{mod}\ q)}} 1 \ \ll \ (\log z)\Big(1 + z/\phi(q)\Big),$$

whence, by the definition of $E^*(z, q)$ in (1.10),

$$E^*(z, q) \ll z(\log z)/\phi(q) \text{ for } q \leqslant z.$$

Thus

$$\begin{aligned} \Sigma_2 &\ll E^*(z, 1) + \sum_{\substack{q \leqslant X \\ d(q) > D}} z(\log z)/\phi(q) \\ &\ll z(\log z)^{-A} + z(\log z)\, D^{-1} \sum_{q \leqslant X} d(q)/\phi(q) \\ &\ll z(\log z)^{-A} + zD^{-1}(\log z)^3. \end{aligned}$$

For the sum Σ_1, we express $E^*(z, q)$ in terms of the values of $\psi(z, \chi)$. We have

$$\psi(z;\ q, a) = \frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a) \psi(z, \chi),$$

and this gives

$$\phi(q)\, E(z, q) \leqslant |\psi(z, \chi_0) - z| + \sum_{\chi \neq \chi_0} |\psi(z, \chi)|.$$

Now

$$|\psi(z, \chi_0) - z| \ll z \exp\Big(-C(\log z)^{\frac{1}{2}}\Big) \ll z(\log z)^{-A-1}$$

by the prime number theorem in its classical form; and for $\chi \neq \chi_0$

$$\begin{aligned}\psi(z,\chi) &= \sum_{\substack{m\leqslant z\\(m,q)=1}} \chi^*(m)\,\Lambda(m)\\ &= \psi(z,\chi^*) - \sum_{\substack{m\leqslant z\\(m,q)>1}} \chi^*(m)\Lambda(m)\\ &= \psi(z,\chi^*) + O\Big(\sum_{\substack{p^\nu\leqslant z\\ p\mid q}} \log p\Big)\\ &= \psi(z,\chi^*) + O\big((\log z)(\log q)\big).\end{aligned}$$

Hence

$$\phi(q)\,E^*(z,q) \ll z(\log z)^{-A-1} + \phi(q)(\log z)^2 + \sum_{\chi\neq\chi_0} \max_{y\leqslant z} |\psi(y,\chi^*)|.$$

It follows that

$$\Sigma_1 \ll z(\log z)^{-A-1} \sum_{q\leqslant X} 1/\phi(q) + X(\log z)^2 + \sum_{q\in Q_X} \frac{1}{\phi(q)} \sum_{\chi\neq\chi_0} \max_{y\leqslant z} |\psi(y,\chi^*)|$$

$$\ll z(\log z)^{-A} + \sum_{q\in Q_X} \frac{1}{\phi(q)} \sum_{\chi\neq\chi_0} \max_{y\leqslant z} |\psi(y,\chi^*)|.$$

Since $q^*\mid q$ we have $d(q^*)\leqslant d(q)\leqslant D$, which implies that $q^*\in Q_X$, since $q^*>1$ for $\chi\neq\chi_0$. Hence, collecting together all the terms containing the primitive characters χ^* belonging to the same modulus q^*, we get

$$\sum_{q\in Q_X} \frac{1}{\phi(q)} \sum_{\chi\neq\chi_0} \max_{y\leqslant z} |\psi(y,\chi^*)| = \sum_{q^*\in Q_X} \sum_{\chi}^{*} \max_{y\leqslant z} |\psi(y,\chi)| \sum_{\substack{q\in Q_X\\ q\equiv 0 \pmod{q^*}}} 1/\phi(q).$$

Now $\phi(q^*r)\geqslant \phi(q^*)\,\phi(r) \gg q^*\phi(r)(\log X)^{-1}$, and $X<z$. Hence the last expression is

$$\ll (\log z)^2 \sum_{q^*\in Q_X} (q^*)^{-1} \sum_{\chi}^{*} \max_{y\leqslant z} |\psi(y,\chi)|.$$

By the Siegel–Walfisz theorem (Prachar [4; ch. IV, Satz 7.2 and Satz 8.2]) we have

$$|\psi(z,\chi)| \ll z\exp\big(-c(\log z)^{\frac{1}{2}}\big)$$

for $\chi\neq\chi_0$, and uniformly in q for $q\leqslant (\log z)^N = X_0$, where $c=c(N)$. It is easily deduced that the same estimate holds for $\max_{y\leqslant z}|\psi(y,\chi)|$, with possibly a different value of c. Hence

$$\sum_{q^*\in Q_{X_0}} (q^*)^{-1} \sum_{\chi}^{*} \max_{y\leqslant z} |\psi(y,\chi)| \ll z(\log z)^{-A-2}.$$

The remaining sum, that is, the sum over $X_0<q\leqslant X$, can be divided into $\ll \log X$ sums over intervals of the type $2^{m-1}<q\leqslant 2^m$. This gives

$$\sum_{\substack{q^*\in Q_X\\ q^*>X_0}} (q^*)^{-1} \sum_{\chi}^{*} \max_{y\leqslant z} |\psi(y,\chi)| \ll (\log z) \max_{X_0<M\leqslant X} M^{-1} \sum_{Q_M} \sum_{\chi}^{*} \max_{y\leqslant z} |\psi(y,\chi)|.$$

It follows that

$$\Sigma_1 \ll z(\log z)^{-A} + (\log z)^3 \max_{X_0<M\leqslant X} M^{-1} \sum_{Q_M} \sum_{\chi}^{*} \max_{y\leqslant z} |\psi(y,\chi)|,$$

and on substitution in (3.4) we obtain (3.3). This proves Lemma 4.

Proof of Theorem 4. By a well-known explicit formula in the theory of primes (see *e.g.*, Prachar [4; ch. VII, Satz 4.6]), for $\chi \neq \chi_0$,

$$|\psi(z,\chi)| \ll \sum_{|\gamma|\leqslant T} \frac{z^\beta}{|\rho|} + \frac{z(\log z)^2}{T} + z^{\frac{1}{2}}$$

uniformly for $q \leqslant z$, $2 \leqslant T \leqslant z$, where $\rho = \beta + i\gamma$ runs through the zeros of $L(s,\chi)$ with $0<\beta<1$, multiple zeros being counted multiply. It follows that

$$\max_{y\leqslant z} |\psi(y,\chi)| \ll \sum_{|\gamma|\leqslant T} \frac{z^\beta}{|\rho|} + \frac{z(\log z)^2}{T} + z^{\frac{1}{2}} \tag{3.5}$$

uniformly for $q \leqslant z^{\frac{1}{2}}$, $2 \leqslant T \leqslant z^{\frac{1}{2}}$, because $|\psi(y,\chi)| \ll z^{\frac{1}{2}}$ if $y \leqslant z^{\frac{1}{2}}$.

Consider first the contribution of any zeros with $|\rho| < \frac{1}{4}$. The number of these is $\ll \log z$ (see Prachar [4; ch. VII, Satz 3.3]), and from the consideration of the corresponding zeros $1-\rho$ of $L(s,\bar{\chi})$ we deduce that

$$|\rho| > z^{-\epsilon}$$

for any fixed positive ϵ, provided q is sufficiently large (see Prachar [4; ch. VII, Satz 6.9 and Satz 8.1]). Hence the contribution of these zeros to the sum is

$$\ll \sum_{\rho} z^{\beta+\epsilon} \ll (\log z)\, z^{\frac{1}{2}+\epsilon} \ll z^{\frac{1}{2}}.$$

As regards the zeros with $|\rho| \geqslant \frac{1}{4}$, it suffices to take only those with $\beta \geqslant \frac{1}{2}$. We divide the range $|\gamma| \leqslant T$ into $|\gamma| < 1$ and $2^{m-1} \leqslant |\gamma| < 2^m$ for $m = 1, 2, \ldots$. Then

$$\sum_{\substack{|\gamma|\leqslant T \\ |\rho|\geqslant\frac{1}{4}}} \frac{z^\beta}{|\rho|} \ll \sum_{2^{m-1}\leqslant T} 2^{-m} \sum_{\substack{|\gamma|\leqslant 2^m \\ \beta\geqslant\frac{1}{2}}} z^\beta.$$

Further,

$$\sum_{\substack{|\gamma|\leqslant 2^m \\ \beta\geqslant\frac{1}{2}}} z^\beta = \sum_{\substack{|\gamma|\leqslant 2^m \\ \beta\geqslant\frac{1}{2}}} \left(z^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\beta} z^\sigma \log z \, d\sigma \right)$$

$$= z^{\frac{1}{2}} N\left(\tfrac{1}{2}, 2^m; \chi\right) + (\log z) \int_{\frac{1}{2}}^{1} N(\alpha, 2^m; \chi)\, z^\alpha \, d\alpha.$$

Using these results in (3.5), we obtain

$$\max_{y\leqslant z} |\psi(y,\chi)| \ll z^{\frac{1}{2}} + z(\log z)^2 T^{-1} + (\log z) \sum_{2^{m-1}\leqslant T} 2^{-m} \left(z^{\frac{1}{2}} N\left(\tfrac{1}{2}, 2^m; \chi\right) + \int_{\frac{1}{2}}^{1} N(\alpha, 2^m; \chi)\, z^\alpha \, d\alpha \right).$$

Hence

$$M^{-1}\sum_{Q_M}\sum_{\chi}^{*}\max_{y\leqslant z}|\psi(y,\chi)|$$

$$\ll M(z^{\frac12}+z(\log z)^2T^{-1})+M^{-1}(\log z)\sum_{2^{m-1}\leqslant T}2^{-m}\Big\{z^{\frac12}\sum_{Q_M}\sum_{\chi}^{*}N(\tfrac12,2^m;\chi)$$

$$+\int_{\frac12}^{1}\sum_{Q_M}\sum_{\chi}^{*}N(\alpha,2^m;\chi)z^{\alpha}d\alpha\Big\}$$

$$\ll M\Big(z^{\frac12}+z(\log z)^2T^{-1}\Big)+M^{-1}(\log z)\sum_{2^{m-1}\leqslant T}2^{-m}\max_{\alpha}\Big\{\sum_{Q_M}\sum_{\chi}^{*}N(\alpha,2^m;\chi)z^{\alpha}\Big\}$$

$$\ll M\Big(z^{\frac12}+z(\log z)^2T^{-1}\Big)$$

$$+M^{-1}(\log z)^2\max_{2\leqslant T'\leqslant T}(T')^{-1}\max_{\alpha}\Big\{\sum_{Q_M}\sum_{\chi}^{*}N(\alpha,T';\chi)z^{\alpha}\Big\}. \quad (3.6)$$

We now appeal to Theorem 5. Noting that for a primitive character $\chi \pmod q$ we have $|\tau(\chi)|^2=q>\phi(q)$, we deduce from Theorem 5 that

$$\sum_{Q_M}\sum_{\chi}^{*}N(\alpha,T';\chi)\leqslant\sum_{Q_M}\frac{1}{\phi(q)}\sum_{\chi}|\tau(\chi)|^2N(\alpha,T';\chi)$$

$$\ll DT'(M^2+MT')^{4(1-\alpha)/(3-2\alpha)}(\log z)^{10}.$$

Hence

$$\max_{2\leqslant T'\leqslant T}(T')^{-1}\max_{\alpha}\Big\{\sum_{Q_M}\sum_{\chi}^{*}N(\alpha,T';\chi)z^{\alpha}\Big\}$$

$$\ll D(\log z)^{10}\max_{\alpha}(M^2+MT)^{4(1-\alpha)/(3-2\alpha)}z^{\alpha}.$$

The conditions we have imposed on M and T are $M\leqslant X\leqslant z^{\frac12}$ and $T\leqslant z^{\frac12}$. The parameter D is at our disposal, and is independent of M.

Now take $D=(\log z)^{A+3}$, $T=M(\log z)^{A+5}$, $X\leqslant z^{\frac12}(\log z)^{-A-5}$; the condition $T\leqslant z^{\frac12}$ is satisfied since we retain the requirement that $M\leqslant X$. Substituting from the last inequality in (3.6), we obtain

$$M^{-1}\sum_{Q_M}\sum_{\chi}^{*}\max_{y\leqslant z}|\psi(y,\chi)|$$

$$\ll Mz^{\frac12}+z(\log z)^{-A-3}+M^{-1}(\log z)^{2A+20}\max_{\alpha}M^{8(1-\alpha)/(3-2\alpha)}z^{\alpha}. \quad (3.7)$$

Now

$$\frac{8(1-\alpha)}{3-2\alpha}-1=2(1-\alpha)-\frac{(2\alpha-1)^2}{3-2\alpha}\leqslant 2(1-\alpha)-\tfrac12(2\alpha-1)^2=\tfrac32-2\alpha^2.$$

The function $z^\alpha M^{\frac{3}{2}-2\alpha^2}$ increases for $\alpha<\alpha_0$ and decreases for $\alpha>\alpha_0$, where $\alpha_0=(\log z)/(4\log M)$. If $M<z^{\frac{1}{4}}$ then $\alpha_0>1$; the function increases with α for $\frac{1}{2}\leqslant\alpha\leqslant 1$, and its maximum is

$$zM^{-\frac{1}{2}}\leqslant zX_0^{-\frac{1}{2}}=z(\log z)^{-\frac{1}{4}N}.$$

If $z^{\frac{1}{4}}\leqslant M\leqslant X(<z^{\frac{1}{2}})$, the maximum of the function is

$$\exp\Big(\tfrac{3}{2}\log M+\tfrac{1}{8}(\log z)^2/(\log M)\Big).$$

Considered as a function of $\log M$, the expression in brackets is convex, and its maximum for

$$\tfrac{1}{4}\log z\leqslant\log M\leqslant\log X$$

is

$$\max\Big(\tfrac{7}{8}\log z,\quad \tfrac{1}{8}\frac{(\log z)^2}{\log X}+\tfrac{3}{2}\log X\Big).$$

We take $X=z^{\frac{1}{4}}(\log z)^{-B}$, in accordance with the hypothesis of Theorem 4. Then the last expression is $\leqslant\log z-B\log\log z+O(1)$, and it follows that

$$z^\alpha M^{\frac{3}{2}-2\alpha^2}\ll z(\log z)^{-\frac{1}{4}N}+z(\log z)^{-B}.$$

We take $N=2B$, and suppose that $B\geqslant A+5$ so that the earlier condition $X\leqslant z^{\frac{1}{4}}(\log z)^{-A-5}$ shall be satisfied.

Applying the results just proved in (3.7), we obtain

$$M^{-1}\sum_{Q_M}\Sigma^*_\chi\max_{y\leqslant z}|\psi(y,\chi)|\ll Mz^{\frac{1}{4}}+z(\log z)^{-A-3}+z(\log z)^{2A+20-B}$$

for $X_0\leqslant M\leqslant X$. Hence, by Lemma 4,

$$\sum_{q\leqslant X}E^*(z,q)\ll z(\log z)^{-A}+z(\log z)^{2A+23-B},$$

since $D=(\log z)^{A+3}$. Taking $B=3A+23$ $(>A+5)$, we obtain (1.11), and this proves Theorem 4.

4. In this section we prove Theorem 5, the proof being based on Theorem 3.

Let Q be a finite set of positive integers and let M and D be defined by (1.3) and (1.4); also let

$$z=M^2, \tag{4.1}$$

$$Q(s,\chi)=\sum_{n\leqslant z}\chi(n)\mu(n)n^{-s}, \tag{4.2}$$

$$f(s,\chi)=L(s,\chi)\,Q(s,\chi)-1. \tag{4.3}$$

We define

$$F(s)=\prod_{q\in Q}\prod_{\chi\neq\chi_0}\Big(1-f^2(s,\chi)\Big)^{c(\chi)}, \tag{4.4}$$

where

$$e(\chi)=\frac{M!}{\phi(q)}|\tau(\chi)|^2. \tag{4.5}$$

Plainly $F(s)$ is an integral function of s, since $L(s,\chi)$ is an integral function for $\chi\neq\chi_0$, and $e(\chi)$ is a positive integer. Also $F(s)$ is real for real s, since then

$$\overline{f(s,\chi)}=f(s,\bar{\chi}),\ |\tau(\bar{\chi})|^2=|\tau(\chi)|^2.$$

As we shall prove later, we have $F(s)\neq 0$ on $\sigma=2$, provided M is sufficiently large. We define $\arg F(\sigma+it)$ in the usual way (*cf.* Prachar [4; p. 398]) by continuous variation from $\arg F(2)=0$ along the line segments $(2, 2+it, \sigma+it)$, with the proviso that $\arg F(s)$ is to be increased by $-\pi m \operatorname{sgn} t$ whenever the second segment passes through a zero of $F(s)$ of multiplicity m.

We write

$$\log^+ x=\begin{cases}\log x \text{ if } x>1,\\ 0 \quad \text{otherwise.}\end{cases}$$

LEMMA 5. *We have*

$$2\pi M!\int_\alpha^\beta \sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2 N(\sigma,T;\chi)\,d\sigma$$

$$\leqslant \int_{-T}^{T}\{\log|F(\alpha+it)|-\log|F(\beta+it)|\}\,dt$$

$$+\int_\alpha^\beta\{\arg F(\sigma+iT)-\arg F(\sigma-iT)\}\,d\sigma. \tag{4.6}$$

Proof. If ρ is a zero of $L(s,\chi)$ of multiplicity m, then ρ is a zero of $1-f^2(s,\chi)$ and hence a zero of $F(s)$ of multiplicity at least $me(\chi)$. Hence Lemma 5 follows from a familiar theorem of Littlewood (see Prachar [4], Anhang, Satz 8.1).

LEMMA 6. *We have, for* $\frac{1}{2}\leqslant\alpha\leqslant 1$,

$$\int_\alpha^2\{\arg F(\sigma+it)-\arg F(\sigma-it)\}\,d\sigma\ll M!+\int_0^{2\pi}\log^+\left|F\left(2+it+(2-\alpha)e^{i\theta}\right)\right|d\theta.$$

Proof. If $\sigma>1$ it follows from (4.2) and (4.3) that

$$f(s,\chi)=\sum_{n>z}\chi(n)A_z(n)n^{-s},$$

where

$$A_z(n)=\sum_{\substack{d\mid n\\ d\leqslant z}}\mu(d),\quad |A_z(n)|\leqslant d(n).$$

Hence, if M is large enough,

$$|f(2+it,\chi)|^2 \leqslant \Big(\sum_{n>z} d(n)\,n^{-2}\Big)^2 \ll (z^{-1}\log z)^2.$$

Thus $|f(2+it,\chi)|$ is small, and this justifies the earlier remark that $F(2+it)\neq 0$.

We have

$$\Big|\log\Big(1-f^2(2+it,\ \chi)\Big)\Big| \ll z^{-2}(\log z)^2;$$

also

$$\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)^2| \leqslant \sum_{q\leqslant M} q < M^2 = z.$$

It follows from (4.4) that

$$\begin{aligned}|\log F(2+it)| &\leqslant \sum_{q\in Q}\frac{M!}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2\Big|\log\Big(1-f^2(2+it,\chi)\Big)\Big| \\ &\ll M!\,z^{-1}(\log z)^2 \ll M!. \end{aligned} \tag{4.7}$$

We follow the general line of argument of Titchmarsh [5; p. 180]. For fixed t, let

$$g_t(s) = \tfrac{1}{2}\{F(s+it)+F(s-it)\}; \tag{4.8}$$

then $g_t(s)$ is an integral function of s, and

$$g_t(\sigma) = \mathscr{R}F(\sigma+it)$$

by the reflection principle. Let $n(r)$ denote the number of zeros of $g_t(s)$, counted with multiplicity, in the circle $|s-2|\leqslant r$. For $\sigma\leqslant 2$, we have

$$|\arg F(\sigma+it)| \leqslant |\arg F(2+it)| + (N+1)\pi,$$

where N is the number of zeros of $g_t(s)$ on the segment $\sigma\leqslant s\leqslant 2$. Thus, by (4.7),

$$|\arg F(\sigma+it)| \ll M! + n(2-\sigma).$$

By this inequality and Jensen's formula, if $\tfrac{1}{2}\leqslant\alpha\leqslant 2$ and $g_t(2)\neq 0$, we have

$$\begin{aligned}\int_\alpha^2 |\arg F(\sigma+it)|\,d\sigma &\ll M! + \int_0^{2-\alpha} n(r)\,dr \\ &\ll M! + \int_0^{2-\alpha} r^{-1}n(r)\,dr \\ &= M! + \frac{1}{2\pi}\int_0^{2\pi}\log\Big|g_t\Big(2+(2-\alpha)e^{i\theta}\Big)\Big|\,d\theta - \log|g_t(2)|. \end{aligned}$$

Also, using the inequality

$$\log^+(a+b) \leqslant 2+\log^+ a+\log^+ b,$$

we have

$$\int_0^{2\pi} \log^+ \left| g_t\left(2+(2-\alpha)e^{i\theta}\right)\right| d\theta \leqslant 4\pi + \int_0^{2\pi} \log^+ \left| F\left(2+it+(2-\alpha)e^{i\theta}\right)\right| d\theta$$

$$+ \int_0^{2\pi} \log^+ \left| F\left(2-it+(2-\alpha)e^{-i\theta}\right)\right| d\theta$$

$$\ll 1 + \int_0^{2\pi} \log^+ \left| F\left(2+it+(2-\alpha)e^{i\theta}\right)\right| d\theta,$$

since $F(\sigma-it)$ is conjugate complex to $F(\sigma+it)$. Substituting in the previous inequality, we obtain the result of Lemma 6, provided $-\log|g_t(2)| \ll M!$. However, we can dispense with this hypothesis by considering also the function

$$h_t(s) = \frac{1}{2i}\{F(s+it) - F(s-it)\}.$$

Then $|F(2+it)|^2 = |g_t(2)|^2 + |h_t(2)|^2$, and by (4.7) we have either $-\log|g_t(2)| \ll M!$ or $-\log|h_t(2)| \ll M!$. The first alternative we have dealt with, and for the second alternative there is the same argument with $h_t(s)$ in place of $g_t(s)$. This completes the proof of Lemma 6.

LEMMA 7. *If χ is a non-principal character (mod q), then*

$$L(s,\chi) = \sum_{n \leqslant x} \chi(n) n^{-s} + O(qx^{-\sigma}) \tag{4.9}$$

uniformly for $\sigma \geqslant \frac{1}{2}$, $x \geqslant 2q$, $|t| \leqslant x/q$.

Proof. Write, as usual,

$$\zeta(s,w) = \sum_{n=0}^{\infty} (n+w)^{-s}$$

for $0 < w \leqslant 1$ and $\sigma > 1$. By well-known methods (Titchmarsh [5; §4.11 or §4.14]) we derive the approximation

$$\zeta(s,w) = \sum_{0 \leqslant n \leqslant y} (n+w)^{-s} - \frac{y^{1-s}}{1-s} + O(y^{-\sigma}),$$

valid uniformly for $\sigma \geqslant \frac{1}{2}$, any positive integer y, and $|t| \leqslant \pi y$.

We have

$$L(s,\chi) = q^{-s} \sum_{a=1}^{q} \chi(a)\zeta(s, a/q),$$

and on substituting the above approximation we obtain

$$L(s,\chi) = \sum_{0 < n \leqslant qy+q} \chi(n) n^{-s} + O(q^{1-\sigma} y^{-\sigma}),$$

since $\sum_{a=1}^{q} \chi(a) = 0$ for $\chi \neq \chi_0$. Taking $x = qy+q$, we have $|t| \leqslant \pi y$ and we obtain (4.9) when x is an integral multiple of q. But we can omit the latter condition, since the sum over less than q terms can be absorbed in $O(qx^{-\sigma})$.

LEMMA 8. *We have*

$$\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2|f(\tfrac{1}{2}+it,\chi)|\ll D(M^2+M|t|)\log^2(M+|t|). \quad (4.10)$$

Proof. By (4.3)

$$|f(s,\chi)|\leqslant 1+|L(s,\chi)Q(s,\chi)|\leqslant 1+\tfrac{1}{2}|L(s,\chi)|^2+\tfrac{1}{2}|Q(s,\chi)|^2.$$

Hence the sum on the left of (4.10) is $\leqslant\Sigma_1+\Sigma_2+\Sigma_3$, where

$$\Sigma_1=\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2,$$

$$\Sigma_2=\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2|L(\tfrac{1}{2}+it,\chi)|^2,$$

$$\Sigma_3=\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2|Q(\tfrac{1}{2}+it,\chi)|^2.$$

Since $|\tau(\chi)|^2\leqslant q$ we have $\Sigma_1\leqslant M^2$.

Now consider Σ_3. We have

$$Q(\tfrac{1}{2}+it,\chi)=\sum_{n\leqslant z}\chi(n)\mu(n)n^{-\frac{1}{2}-it},$$

and on appealing to Theorem 3 with $Y=0$, $Z=z$, $a_n=\mu(n)n^{-\frac{1}{2}-it}$ we obtain

$$\Sigma_3\ll D\max(z,M^2)\sum_{n\leqslant z}d(n)n^{-1}\ll DM^2(\log M)^2.$$

There remains Σ_2. Here we first approximate to $L(\frac{1}{2}+it,\chi)$ by a finite sum, using Lemma 7 with $x=M^2+M|t|$, and $q\leqslant M$, which is permissible since then $x\geqslant 2q$ and $|t|\leqslant x/q$. We obtain

$$|L(\tfrac{1}{2}+it,\chi)|^2\ll\Big|\sum_{n\leqslant x}\chi(n)n^{-\frac{1}{2}-it}\Big|^2+1.$$

Appealing to Theorem 3 again, with $Y=0$, $Z=x$, $a_n=n^{-\frac{1}{2}-it}$, we get

$$\begin{aligned}\Sigma_2&\ll\Sigma_1+\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2\Big|\sum_{n\leqslant x}\chi(n)n^{-\frac{1}{2}-it}\Big|^2\\&\ll M^2+D\max(x,M^2)\sum_{n\leqslant x}d(n)n^{-1}\ll D(M^2+M|t|)\log^2(M+|t|).\end{aligned}$$

Adding the estimates for Σ_1, Σ_2, Σ_3 we obtain (4.10).

LEMMA 9. *We have, uniformly for* $\sigma\geqslant 1$,

$$\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2|f(\sigma+it,\chi)|^2\ll D\log^9(M+|t|).$$

Proof. Let $x=M^2+M|t|$ as in the preceding proof. Then Lemma 7 gives

$$\begin{aligned}f(\sigma+it,\chi)&=\Big(\sum_{n\leqslant x}\chi(n)n^{-\sigma-it}\Big)\Big(\sum_{n\leqslant z}\chi(n)\mu(n)n^{-\sigma-it}\Big)-1\\&\qquad+O\Big(Mx^{-\sigma}|Q(\sigma+it,\chi)|\Big)\\&=\sum_{z<n\leqslant zx}\chi(n)a_n(x,z)n^{-\sigma-it}+O\Big(M^{1-2\sigma}|Q(\sigma+it,\chi)|\Big),\end{aligned}$$

where $a_n(x, z) = \Sigma\mu(d)$ over $d \mid n$, $nx^{-1} \leqslant d \leqslant z$.

Denoting by S the sum of the present lemma, we have

$$S \ll S_1 + M^{2-4\sigma} S_2,$$

where

$$S_1 = \sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\tau(\chi)|^2 \Big| \sum_{z < n \leqslant zx} \chi(n) a_n(x, z) n^{-\sigma - it} \Big|^2,$$

$$S_2 = \sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\tau(\chi)|^2 |Q(\sigma + it, \chi)|^2.$$

By Theorem 3 applied to S_2, with $Y = 0$, $Z = z$, $a_n = \mu(n) n^{-\sigma - it}$, we have

$$S_2 \ll D \max(z, M^2) \sum_{n \leqslant z} d(n) n^{-2\sigma} \ll DM^2.$$

Now consider S_1. We divide the range $z < n \leqslant zx$ into $\ll \log x$ intervals $(2^{h-1}z, 2^h z)$, $h = 1, 2, \ldots, h_0$, together with a partial interval $(2^{h_0}z, xz)$. By Cauchy's inequality,

$$\Big| \sum_{z < n \leqslant zx} \chi(n) a_n(x, z) n^{-\sigma - it} \Big|^2 \ll (\log x) \sum_{h=1}^{h_0+1} \Big| \sum_{2^{h-1} z < n \leqslant 2^h z} \chi(n) a_n(x, z) n^{-\sigma - it} \Big|^2,$$

with the convention that the upper bound for n in the inner sum is xz when $h = h_0 + 1$. This gives an inequality for S_1 in terms of $h_0 + 1$ sums, the typical one being

$$\sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi} |\tau(\chi)|^2 \Big| \sum_{2^{h-1} z < n \leqslant 2^h z} \chi(n) a_n(x, z) n^{-\sigma - it} \Big|^2.$$

To each such sum we apply Theorem 3, with $Y = 2^{h-1}z$, $Z = 2^h z$ (or xz), $a_n = a_n(x, z) n^{-\sigma - it}$. Noting that $|a_n(x, z)| \leqslant d(n)$, we obtain for the last sum the estimate

$$\begin{aligned} D \max(2^{h-1}z, M^2) \sum_{2^{h-1} z < n \leqslant 2^h z} d(n) a_n^2(x, z) n^{-2\sigma} &\ll D 2^h z \sum_{2^{h-1} z < n \leqslant 2^h z} d^3(n) n^{-2\sigma} \\ &\ll D(2^h z)^{1-2\sigma} \sum_{n=1}^{2^h z} d^3(n) \\ &\ll D(2^h z)^{2-2\sigma} (\log x)^7 \\ &\ll D z^{2-2\sigma} (\log x)^7, \end{aligned}$$

since

$$\sum_{n=1}^{N} d^3(n) \ll N (\log N)^7$$

and $\log(2^h z) \ll \log x$. On substituting in S_1 we get

$$S_1 \ll (\log x) \sum_{h=1}^{h_0+1} D z^{2-2\sigma} (\log x)^7 \ll D (\log x)^9.$$

Collecting the results, we obtain Lemma 9.

LEMMA 10. *Let $f_1(s), \dots, f_K(s)$ be regular functions of s in the strip $\alpha<\sigma<\beta$ and continuous onto the boundary, and suppose that each of them $\to 0$ as $|t|\to\infty$, uniformly in σ. Let $c_1, \dots, c_K$ be positive numbers, and define*

$$J(\sigma;\ \lambda)=\left\{\int_{-\infty}^{\infty}\sum_{k=1}^{K} c_k|f_k(\sigma+it)|^{1/\lambda}\,dt\right\}^{\lambda}. \tag{4.11}$$

Then

$$J(\sigma;\ \lambda u+\mu v)\leqslant J(\alpha;\ \lambda)^u\, J(\beta;\ \mu)^v, \tag{4.12}$$

where

$$u=(\beta-\sigma)/(\beta-\alpha),\quad v=(\sigma-\alpha)/(\beta-\alpha).$$

Proof. When $K=1$ this is a theorem of R. M. Gabriel [**6**], and his proof extends without difficulty to the more general case. In the proof of Gabriel's Theorem 1, one considers

$$\int_{AB}\sum_{k=1}^{K} c_k\phi_k(z)\bar{\phi}_k(z)\,dz$$

instead of $\int\phi(z)\bar{\phi}(z)\,dz$, and follows the same line of argument but with an additional application of Hölder's inequality. This is the only modification of any substance that is needed.

Let us write for brevity

$$\Phi(\alpha,\ T)=\int_{-T}^{T}\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2\log^+|1-f^2(\alpha+it,\ \chi)|\,dt. \tag{4.13}$$

LEMMA 11. *We have*

$$\Phi(\alpha,\ T)\ll DT(M^2+MT)^{4(1-\alpha)/(3-2\alpha)}\log^9(M+T), \tag{4.14}$$

uniformly for $\frac{1}{2}\leqslant\alpha\leqslant 1$, $T\geqslant 2$. Also

$$\Phi(\alpha,\ T)\ll DT\log^9(M+T) \tag{4.15}$$

uniformly for $\alpha\geqslant 1$, $T\geqslant 2$.

Proof. For $T\geqslant 4$, put

$$f_T(s,\ \chi)=f(s,\ \chi)/\cos(s/T),$$

$$J_T(\sigma;\ \lambda)=\left\{\int_{-\infty}^{\infty}\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\tau(\chi)|^2|f_T(\sigma+it,\ \lambda)|^{1/\lambda}\,dt\right\}^{\lambda}.$$

For $T\geqslant 4$, $\frac{1}{2}\leqslant\sigma\leqslant 1$, we have

$$\tfrac{1}{2}\exp(|t|/T)\leqslant|\cos(s/T)|\leqslant\exp(|t|/T), \tag{4.16}$$

so that $f_T(s,\ \chi)$ is a regular function of s for $\frac{1}{2}\leqslant\sigma\leqslant 1$ and $\to 0$ as $|t|\to\infty$, uniformly in the strip $\frac{1}{2}\leqslant\sigma\leqslant 1$.

From (4.16) and Lemma 8, we obtain

$$J_T(\tfrac{1}{2}; 1) \ll \int_{-\infty}^{\infty} e^{-|t|/T} \sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\tau(\chi)|^2 |f(\tfrac{1}{2}+it, \chi)|\, dt$$

$$\ll \int_{-\infty}^{\infty} e^{-|t|/T} D(M^2 + M|t|) \log^2(M+|t|)\, dt$$

$$\ll DT(M^2+MT)\log^2(M+T).$$

In the same way, Lemma 9 gives

$$J_T(1; \tfrac{1}{2}) \ll \left\{ \int_{-\infty}^{\infty} e^{-2|t|/T} \sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\tau(\chi)|^2 |f(1+it, \chi)|^2\, dt \right\}^{\frac{1}{2}}$$

$$\ll \{DT \log^9(M+T)\}^{\frac{1}{2}}.$$

On applying the two-variable convexity result of Lemma 10 with

$$\alpha = \tfrac{1}{2}, \quad \beta = 1, \quad \lambda = 1, \quad \mu = \tfrac{1}{2}, \quad u = 2(1-\sigma), \quad v = 2\sigma - 1,$$

$$c_k = \frac{1}{\phi(q)} |\tau(\chi)|^2, \quad f_k(s) = f_T(s, \chi),$$

we obtain

$$J_T(\sigma; \tfrac{3}{2}-\sigma) \ll \{DT(M^2+MT)\log^2(M+T)\}^{2-2\sigma} \{DT \log^9(M+T)\}^{\sigma-\frac{1}{2}}$$

$$\ll \{DT \log^9(M+T)\}^{\frac{3}{2}-\sigma} (M^2+MT)^{2-2\sigma}, \tag{4.17}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T \geqslant 4$.

For every complex w and every λ in $\frac{1}{2} \leqslant \lambda \leqslant 1$ we have

$$\log^+|1-w^2| \ll |w|^{1/\lambda}. \tag{4.18}$$

Hence it follows from the definition of $\Phi(\alpha, T)$ in (4.13) that, for $\frac{1}{2} \leqslant \sigma \leqslant 1$,

$$\Phi(\sigma, T) \ll \int_{-T}^{T} \sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\tau(\chi)|^2 |f(\sigma+it, \chi)|^{1/(\frac{3}{2}-\sigma)} dt.$$

By (4.16) we can introduce a factor

$$\left| \cos\Big((\sigma+it)/T\Big) \right|^{-1/(\frac{3}{2}-\sigma)}$$

into the integral, and this has the effect of replacing $f(\sigma+it, \chi)$ by $f_T(\sigma+it, \chi)$. Hence

$$\Phi(\sigma, T) \ll \{J_T(\sigma; \tfrac{3}{2}-\sigma)\}^{1/(\frac{3}{2}-\sigma)}.$$

Now the first conclusion of Lemma 11, namely (4.14), follows from (4.17).

In order to obtain the second conclusion, namely (4.15), we use again the inequality (4.18), which implies that

$$\Phi(\alpha, T) \ll \int_{-T}^{T} \sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\tau(\chi)|^2 |f(\alpha+it, \chi)|^2\, dt.$$

Now (4.15) for $\alpha \geqslant 1$ follows at once from Lemma 9. This completes the proof of Lemma 11.

Proof of Theorem 5. We write for brevity

$$\mathscr{N}_Q(\sigma, T) = \sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\tau(\chi)|^2 N(\sigma, T;\ \chi).$$

We start from Lemma 5 with $\beta = 2$; using Lemma 6 and (4.7) we obtain

$$\begin{aligned} 2\pi M! \int_\alpha^2 \mathscr{N}_Q(\sigma, T)\, d\sigma &\leqslant \int_{-T}^{T} \{\log|F(\alpha+it)| - \log|F(2+it)|\}\, dt \\ &\quad + \int_\alpha^2 \{\arg F(\sigma+it) - \arg F(\sigma-it)\}\, d\sigma \\ &\ll \int_{-T}^{T} \log^+|F(\alpha+it)|\, dt \\ &\quad + \int_0^{2\pi} \log^+\left|F\left(2+iT+(2-\alpha)e^{i\theta}\right)\right| d\theta + M!\, T. \end{aligned} \tag{4.19}$$

The function $\mathscr{N}_Q(\sigma, T)$ is a non-decreasing function of T for fixed σ, so on integrating (4.19) with respect to T from 0 to $2T$ we get

$$\begin{aligned} 2\pi M!\, T \int_\alpha^2 \mathscr{N}_Q(\sigma, T)\, d\sigma &\leqslant 2\pi M! \int_0^{2T} \int_\alpha^2 \mathscr{N}_Q(\sigma, U)\, d\sigma\, dU \\ &\ll \int_0^{2T} \int_{-U}^{U} \log^+|F(\alpha+it)|\, dt\, dU \\ &\quad + \int_0^{2T} \int_0^{2\pi} \log^+\left|F\left(2+iU+(2-\alpha)e^{i\theta}\right)\right| d\theta\, dU \\ &\quad + M!\, T^2. \end{aligned} \tag{4.20}$$

Obviously

$$\int_0^{2T} \int_{-U}^{U} \log^+|F(\alpha+it)|\, dt\, dU \leqslant 2T \int_{-2T}^{2T} \log^+|F(\alpha+it)|\, dt.$$

Also

$$\begin{aligned} &\int_0^{2T} \int_0^{2\pi} \log^+\left|F\left(2+iU+(2-\alpha)e^{i\theta}\right)\right| d\theta\, dU \\ &\leqslant 2\pi \max_\theta \int_0^{2T} \log^+\left|F\left(2+iU+(2-\alpha)e^{i\theta}\right)\right| dU \\ &\ll \max_{\alpha \leqslant \sigma \leqslant 4} \int_{-2}^{2T+2} \log^+|F(\sigma+it)|\, dt. \end{aligned}$$

Using these results in (4.20), we find that

$$M! \int_\alpha^2 \mathscr{N}_Q(\sigma, T)\, d\sigma \ll M!\, T + \max_{\alpha \leqslant \sigma \leqslant 4} \int_{-2T-2}^{2T+2} \log^+|F(\sigma+it)|\, dt. \tag{4.21}$$

We have

$$\log^+|F(s)| = \log^+ \prod_{q\in Q} \prod_{\chi\neq\chi_0} |1-f^2(s,\chi)|^{e(\chi)}$$

$$\leqslant M! \sum_{q\in Q} \frac{1}{\phi(q)} \sum_{\chi\neq\chi_0} |\tau(\chi)|^2 \log^+|1-f^2(s,\chi)|,$$

so that

$$\int_{-2T-2}^{2T+2} \log^+|F(\sigma+it)|\,dt \leqslant M!\,\Phi(\sigma, 2T+2)$$

by the definition (4.13) of $\Phi(\alpha, T)$. Hence, by (4.21) and Lemma 11, we have

$$\int_\alpha^2 \mathcal{N}_Q(\sigma, T)\,d\sigma \ll DT(M^2+MT)^{4(1-\alpha)/(3-2\alpha)} \log^9(M+T), \qquad (4.22)$$

provided $T \geqslant 2$ and $\frac{1}{2} \leqslant \alpha \leqslant 1$.

The function $\mathcal{N}_Q(\sigma, T)$ is a non-increasing function of σ for fixed T. Hence if $0<\delta<1$ we have

$$\mathcal{N}_Q(\alpha+\delta, T) \leqslant \delta^{-1} \int_\alpha^{\alpha+\delta} \mathcal{N}_Q(\sigma, T)\,d\sigma$$

$$\ll \delta^{-1} DT(M^2+MT)^{4(1-\alpha)/(3-2\alpha)} \log^9(M+T)$$

provided $T \geqslant 2$ and $\frac{1}{2} \leqslant \alpha \leqslant 1$. We take

$$\delta = 1/\log(M+T),$$

and note that

$$4(1-\alpha)/(3-2\alpha) = 4(1-\alpha-\delta)/(3-2\alpha-2\delta) + O(\delta)$$

and that

$$(M^2+MT)^\delta = O(1).$$

Hence

$$\mathcal{N}_Q(\alpha, T) \ll DT(M^2+MT)^{4(1-\alpha)/(3-2\alpha)} \log^{10}(M+T), \qquad (4.23)$$

uniformly for $\frac{1}{2}+\delta \leqslant \alpha \leqslant 1$.

We have also

$$\sum_{q\in Q} \frac{1}{\phi(q)} |\tau(\chi_0)|^2 N(\alpha, T;\ \chi_0) = N(\alpha, T) \sum_{q\in Q} \frac{\mu^2(q)}{\phi(q)} \ll T(\log T)(\log M), \qquad (4.24)$$

where $N(\alpha, T)$ denotes as usual the number of zeros of $\zeta(s)$ in the rectangle $\frac{1}{2} \leqslant \sigma \leqslant 1$, $|t| \leqslant T$. On adding (4.23) and (4.24) we obtain the conclusion of Theorem 5, namely (1.14), for $\frac{1}{2}+\delta \leqslant \alpha \leqslant 1$.

Finally, suppose that $\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}+\delta$. By a known result (Prachar [**4**; p. 223]), we have

$$\sum_\chi N(\alpha, T;\ \chi) \ll \phi(q)\, T \log(M+T).$$

Hence

$$\sum_{q\in Q}\frac{1}{\phi(q)}\sum_{\chi}|\tau(\chi)|^2 N(\alpha, T;\ \chi)\ll\sum_{q\in Q}qT\log(M+T)\ll M^2T\log(M+T).$$

This inequality is superior to (1.14) when $\frac{1}{2}\leqslant\alpha\leqslant\frac{1}{2}+\delta$, in view of the definition of δ, and the proof of Theorem 5 is now complete.

We now deduce the corollary (1.17) and (1.18). Each character χ mod q arises from a primitive character χ^* mod q^*, where $q^*|q$, and

$$N(\alpha, T;\ \chi)=N(\alpha, T;\ \chi^*).$$

Since the number of values of q for a given q^* is at most X/q^*, we have

$$\sum_{q\leqslant X}\sum_{\chi}N(\alpha, T;\ \chi)\leqslant\sum_{q^*\leqslant X}X/q^*\sum_{\chi^*}N(\alpha, T;\ \chi^*).$$

Dividing the sum over q^* into intervals $(2^h, 2^{h+1})$, we get the estimate

$$(\log X)\max_{M\leqslant X}XM^{-1}\sum_{q^*\leqslant M}\sum_{\chi^*}N(\alpha, T;\ \chi^*)\ll\max_{M\leqslant X}XM^{-1}T(M^2+MT)^{\beta}(XT)^{\epsilon},$$

where $\beta=4(1-\alpha)/(3-2\alpha)$. This is

$$\ll\max(X^{1+\epsilon}T^{1+\beta+\epsilon},\ X^{2\beta+\epsilon}T^{1+\epsilon}).$$

Since $T\leqslant X^{\frac{1}{2}}$ we have

$$T^{\beta}\leqslant X^{2(1-\alpha)/(3-2\alpha)}\leqslant X^{2(1-\alpha)}.$$

Also $2\beta\leqslant 1+2(1-\alpha)$. Hence (1.17). For (1.18) we note that $\beta\leqslant\frac{1}{2}$ when $\frac{5}{6}\leqslant\alpha\leqslant 1$.

We end the paper with two simple remarks about Theorem 5. It is possible to prove many other similar inequalities, for example

$$\sum_{q\in Q}1/\phi(q)\sum_{\chi}|\tau(\chi)|^2 N(\alpha, T;\ \chi)\ll DT^2M^{5(1-\alpha)}\log^{10}(M+T);$$

which is better than our Theorem 5 if $\alpha>7/10$ and T is not too large. The inequality of Theorem 5 is analogous to another result of Ingham (see Titchmarsh [**5**; Theorem 9.19(B)]), and, by a happy circumstance, it gives a useful bound in the whole range $T\ll M^{1+\epsilon}$; this seems to be essential in the proof of Theorem 4.

References

1. K. F. Roth, " On the large sieve of Linnik and Rényi ", *Mathematika*, 12 (1965), 1–9.

2. Yu. V. Linnik, " The large sieve ", *Doklady Akad. Nauk SSSR*, 30 (1941), 292–294 (in Russian).

3. A. Rényi, " On the representation of an even number as the sum of a single prime and an almost prime number ", *Izv. Akad. Nauk SSSR*, Ser. Mat. 12 (1948), 57–78 (in Russian); also *American Math. Soc. Translations* (2), 19 (1961), 299–321.

4. K. Prachar, *Primzahlverteilung* (Springer, 1957).

5. E. C. Titchmarsh, *The theory of the Riemann zeta-function* (Oxford, 1951).

6. R. M. Gabriel, " Some results concerning the integrals of moduli of regular functions along certain curves ", *Journal London Math. Soc.*, 2 (1927), 112–117.

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