THE RESIDUAL SET OF A COMPLEX ON A MANIFOLD AND RELATED QUESTIONS

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1. The questions treated in this paper found their starting point in my recent investigations on complexes and manifolds.[1] Among the neighborhoods of a C_k on an M_n there are some whose complementary set is itself an M_n with a boundary in the strict sense of *Tr.* 2, p. 437. The duality theorems of that paper (p. 449) become applicable to them and by relatively simple considerations lead to the two basic duality formulas (3), (4) with their extensions in various directions; (3) is the direct generalization of Poincaré's noted duality relation and of the second duality theorem of *Tr.* 2 (p. 450); (4) includes and extends Alexander's important duality theorem for a C_k on a hypersphere,[2] and its recent generalization by Alexandroff.[3] Alexander's work is the first on the sets $M_n - C_k$ and it is quite probable that his method is applicable to our situation as well. Our attack, however, has been from a different angle and seems decidedly interesting for its own sake. We conclude our discussion with extensions of the coincidence formulas of *Tr.* 1, 2, to residual sets on an M_n.

2. *The Neighborhood of C_k on C_n.*—We take as in *Tr.*1 a C_n whose cells are simplicial and at first C_k a subcomplex of C_n.[4] The cells of C_n abutting on C_k constitute a subcomplex C'_n of C_n, whose cells not abutting on C_k constitute a C_{n-1}. Let PQ be a segment whose end-points P, Q, are, respectively, on C_k and C_{n-1}. Through every point R of C'_n not on C_k or C_{n-1} there passes a unique PQ. To show it we need only consider the case where R is on a certain simplex, or which is the same to assume that C_n is a simplex. Then C_k is a simplex on its boundary, and the property in question goes back to this: Given on an S_n, two independent spaces S_k, S_{n-k-1}, through every point R not on one of them there passes one and only one line intersecting both.

P will be called the *projection* of R on C_k. When R tends to a point on C_k, P tends to the same point, as readily shown by the same consideration as above. Therefore, the points of C_k are to be considered as their own projections on the complex.

3. Given PQ, take for R the point that divides the segment in the ratio $\lambda:1$, $0 < \lambda < 1$. When PQ remains on a definite cell E_h of C'_n, R generates a certain E'_{h-1} which is convex and flat and the set of all these cells determines a complex Φ_{n-1}, polyhedral, but with cells that may not be simplicial. In this connection see *Tr.* 1, p. 10.

The set of all segments PR plus their end-points is an n-complex which we shall denote by $N(C_k)$ or simply N. It is a *closed neighborhood* of C_k, the open neighborhood being $N - \Phi$. The *width* δ of N is the upper bound of PR, and can obviously be made as small as we please. We shall also introduce as of great importance in the sequence the n-complex $K_n = C_n - C_k - N + \Phi$. Regarding the complexes just introduced we have several important theorems.

4. I. *The complexes N, K, Φ remain homeomorphic to themselves when λ varies.*

II. *Every cycle Γ_μ of N is $\sim$ mod. N to a cycle Γ'_μ on C_k and when $\Gamma_\mu \sim 0$ mod. N, then $\Gamma'_\mu \sim 0$ mod. C_k.*

Let C_μ be a complex on N whose cells are oriented and may be singular.[5] As a point R ranges over C_μ its projection P on C_k will define cell for cell a complex C'_μ continuous image of C_μ, the incidence relations being the same between corresponding cells. Hence in the Poincaré congruence notation, from $C_\mu \equiv 0$, follows $C'_\mu \equiv 0$, or the image of a cycle is a cycle. Furthermore, the locus of PR is a $C_{\mu+1}$ whose cells can be so oriented that $C_{\mu+1} \equiv C_\mu - C'_\mu$, hence $C_\mu \sim C'_\mu$ mod. N. Finally, if C_μ, not a cycle, has the boundary $\Gamma_{\mu-1}$, i.e., if $C_\mu \equiv \Gamma_{\mu-1}$, then $C'_\mu \equiv \Gamma'_{\mu-1}$ image of $\Gamma_{\mu-1}$, that is, from $\Gamma_{\mu-1} \sim 0$ mod. N, follows $\Gamma'_{\mu-1} \sim 0$ mod. C_k.

III. *Let K'_n, K''_n correspond to λ', λ'', and let Γ'_μ be a cycle of the first deformed into Γ''_μ when λ varies from λ' to λ''. Then $\Gamma'_\mu \sim \Gamma''_\mu$ mod. the complex common to K'_n and K''_n, and a fortiori mod. $C_n - C_k$. A* similar statement holds for the neighborhood complexes N.

For the difference of the cycles bounds the $C_{\mu-1}$ locus of Γ'_μ.

IV. *The cycles of K_n and of $C_n - C_k$ are in (1,1) correspondence with preservation of their homologies.*

Compare $C_n - C_k$ with K'_n fixed. Every cycle of the latter is also one of the former, and if ~ 0 mod. K'_n, it is likewise ~ 0 mod. $C_n - C_k$. Conversely, Γ_μ of $C_n - C_k$ is on some K''_n, and in the deformation of the latter into K'_n will be reduced to Γ'_μ on K'_n. Furthermore, let there be on $C_n - C_k$ a $C_{\mu+1} \equiv \Gamma_\mu$. Then there exists a K''_n carrying $C_{\mu+1}$. The above deformation will reduce it to a $C'_{\mu+1}$ of K'_n, $\equiv \Gamma'_\mu$; hence $\Gamma'_\mu \sim 0$ mod. K'_n.

V. *The incidence relations between the cells of Φ_{n-1} are the same as between the cells of C_n that carry them.*

VI. *When C_n defines a manifold so do N, K_n, Φ_{n-1}.*

The manifold conditions are as in *Tr.* 2, p. 438. For Φ the theorem follows from V. For K and N we need only verify the conditions in regard to the cells on Φ. The incidence relations being the same for K and N in regard to these cells we need only take up N. Any cell E_j of Φ is carried by an E_{j+1} of C_n. The cells of $N - \Phi$ incident with E_j, other than E_{j+1} itself are of the same dimensionality and have the same incidence relations as those of C_n incident with E_{j+1}. Hence, by condition (*a*) loc. cit., they have the same incidence relations as the respective cells of $j + 2$ less dimensions of a cellular subdivision of a sphere of S_{n-j-1}. Hence, together with E_{j+1} they constitute a set of cells with the same incidence relations as a subdivision of an E_{n-j-1}, with the cells of Φ incident with E_j corresponding to the boundary cells of the subdivision. Hence, (*b*) loc. cit. is satisfied as to E_j and N.

VII. *The preceding properties can all be carried over to any polyhedral* C_k *on* C_n.

All that is necessary is to replace C_n by a subdivision of which C_k is a subcomplex (*Tr.* 1, p. 10, lemma II).

5. *Connection Indices and Similar Invariants.*—In our previous papers we have only considered cycles on complexes. Actually we may speak of cycles and their homologies in regard to any point set G for which the limit process has a meaning (Fréchet $\mathfrak{L}$ space), and that is more than ample for our purpose. Thus in IV we legitimately considered the cycles of $C_n - C_k$, which is not a complex. See in this connection, Alexander, loc. cit., p. 339.

It will be convenient to denote the μth connection index of G by $R_\mu(G)$. When G is not a complex this index need not necessarily be finite.

Recently Alexandroff[6] and Vietoris[7] have given a definition of the numbers $R_\mu(G)$ and similar invariants, also applicable to sets of a very general type. We leave aside the question of their equivalence to the invariants as defined directly by means of the cycles on the sets.

6. It is advisable to introduce special notations for certain simultaneous topological invariants of a set G and its subset H. They come up naturally in these investigations and their relations to be given below are obtained by considerations such as those of *Tr.* 2, p. 440. These invariants are all maximum numbers of independent μ-cycles, respectively:

(*a*) Of H independent mod. G or $R_\mu\ (H;\ G)$
(*b*) —— dependent ———— $r_\mu\ (H;\ G)$
(*c*) —— independent mod. G of the cycles of $G - H$ or $S_\mu\ (H;\ G)$
(*d*) —— dependent ———————— $s_\mu\ (H;\ G)$
(*e*) Of G independent ———————— and also of those of H, or $T_\mu(H;\ G-H)$.

T is also the maximum number of independent μ cycles of G of which every linear combination meets both H and $G - H$. We have

$$R_\mu(H;G) + r_\mu\ (H;G) = S_\mu(H;G) + s_\mu\ (H;G) = R_\mu(H);$$
$$T_\mu(H;G-H) = T_\mu(G-H;H).$$

Also $s_\mu\ (H;G) - r_\mu\ (H;G) =$ the same number for $G - H$, since both represent the maximum number of μ cycles of one of the two sets, homologous mod. G to one on the other, and independent mod. G. Finally we have the fairly obvious formula:

$$R_\mu(H;G) + R_\mu(G - H;G) - [\, s_\mu(H;G) - r_\mu(H;G)] + T_\mu(H;G - H) = R_\mu(G).$$

7. In terms of the preceding invariants the second and first duality theorems of *Tr.* 2., pp. 449, for an M_n with the boundary F_{n-1} are respectively expressed by

$$S_\mu\,(M_n - F_{n-1};\ M_n) = S_{n-\mu}\,(M_n - F_{n-1};\ M_n). \qquad (1)$$

$$s_\mu\,(M_n - F_{n-1};\ M_n) = r_{n-\mu-1}\,(F_{n-1};\ M_n). \qquad (2)$$

(1) *represents the true generalization of the Poincaré duality law* and reduces to it for an M_n without boundary.

8. *Duality Relations for* $M_n - C_k$.—Returning to our main topic we now assume that C_n defines an orientable manifold, without boundary, M_n. We continue to take for C_k a polyhedral complex. Then K likewise defines a manifold with the boundary Φ_{n-1}. Applying to it formula (1) together with IV of No. 4 we find immediately

$$S_\mu\,(M_n - C_k;\ M_n) = S_{n-\mu}\,(M_n - C_k;\ M_n). \qquad (3)$$

This is the duality relation for the cycles of $M_n - C_k$ that do not depend upon those of C_k.

Less immediate is the following formula derived from (2):

$$s_\mu\,(M_n - C_k;\ M_n) = r_{n-\mu-1}\,(C_k;\ M_n) + T_{n-\mu}\,(C_k;\ M_n - C_k), \quad (4)$$

which we shall now prove. Applying (2) to K we have

$$s_\mu\,(K_n - \Phi_{n-1};\ K_n) = r_{n-\mu-1}\,(\Phi_{n-1};\ M_n). \qquad (5)$$

Let Γ^i_μ $(i = 1,2,\ldots h)$ be independent cycles of $M_n - C_k$, $\sim$ mod. M_n, to cycles Δ^i_μ on C_k. We can so choose K that $K - \Phi$ will carry all the Γ's. Let $C^i_{\mu+1} \equiv \Gamma^i - \Delta^i$. Taking the intersection with Φ,[8] we have a $\mu - 1$ cycle $C^i.\Phi \sim \Gamma^i$ mod. K. Moreover, the cycles $C^i.\Phi$ are independent not only mod. Φ but also mod. K for $\Sigma\lambda_i C^i.\Phi \sim 0$ mod. K leads to $\Sigma\lambda_i\Gamma^i \sim 0$ mod. K or *a fortiori* mod. $M_n - C_k$. Hence to h cycles such as the Γ's correspond as many μ cycles of Φ independent mod. K. Conversely, h cycles of Φ independent mod. K are $\sim$ mod. N, hence mod. M_n to as many cycles on C_k. Therefore,

$$s_\mu\,(K_n - \Phi_{n-1};\ K_n) = s_\mu(\,M_n - C_k;\ M_n). \qquad (6)$$

So much for the left side of (5). Concerning the other side we have to investigate the $(n - \mu - 1)$ cycles of Φ that bound on K. Such a cycle, $\Gamma_{n-\mu-1}$, may be of one or two types:

(*a*) It is homologous to $\Gamma'_{n-\mu-1}$ on C_k (its projection on C_k), that is, not ~ 0 mod. C_k. Then $\Gamma'_{n-\mu-1} \sim 0$ mod. M_n, but not mod. C_k, so that there exists on M_n but not on C_k a $C_{n-\mu} \equiv \Gamma'_{n-\mu-1}$. Then $C_{n-\mu}.\Phi = \Gamma_{n-\mu-1}$ ($\sim \Gamma'_{n-\mu-1}$ mod. N) bounds the part of $C_{n-\mu}$ on K. Furthermore, $\Gamma_{n-\mu-1}$ is not ~ 0 mod. Φ, else by II of No. 4, $\Gamma'_{n-\mu-1} \sim 0$ mod. C_k. Hence the number of independent Γ 's mod. Φ is the same as that of the number of independent Γ' 's mod. C_k, or $r_{n-\mu-1}(C_k; M_n)$.

(*b*) $\Gamma_{n-\mu-1}$ is $\sim$ mod. N to $\Gamma'_{n-\mu-1} \sim 0$ mod. C_k. We then have $C_{n-\mu}$ on N and $C'_{n-\mu}$ on K such that $C_{n-\mu} \equiv \Gamma_{n-\mu-1}$; $C'_{n-\mu} \equiv \Gamma'_{n-\mu-1}$. Hence, $\Gamma_{n-\mu} = C'_{n-\mu} - C_{n-\mu}$ is a cycle that intersects Φ into $\Gamma_{n-\mu-1}$.

Now we can select a set of independent $(n - \mu)$ cycles of M_n, in maximum number $R_{n-\mu}(M_n)$, consisting of $T_{n-\mu}(C_k; M_n - C_k)$ cycles $\Gamma^i_{n-\mu}$ $(i = 1,2,\ldots T)$ independent of the cycles on C_k or on $M_n - C_k$, and of $p = R_{n-\mu}(M_n) - T$ cycles $\Delta^i_{n-\mu}$ $(i = 1,2,\ldots p)$ some of which are on C_k, some on $M_n - C_k$. Then

$$\lambda\Gamma_{n-\mu} \sim \Sigma\lambda_i\Gamma^i_{n-\mu} + \Sigma\mu_i\Delta^i_{n-\mu}, \text{ mod. } M_n;\ \lambda \neq 0. \qquad (7)$$

Since the Δ's not on C_k do not have any point on it, the distance from them to C_k exceeds a certain $\delta > 0$. Construct N such that all its points are no farther than δ from C_k. Then Φ will not meet any Δ whatever. Hence, from (7) taking the intersections with Φ we find $\lambda\Gamma_{n-\mu}.\Phi = \lambda\Gamma_{n-\mu-1} \sim \Sigma\lambda_i\Gamma^i_{n-\mu}.\Phi$, mod. Φ. Hence $\Gamma_{n-\mu-1}$ depends mod. Φ upon the T cycles under the sum. These cycles on the other hand are independent mod. Φ. For let $\Gamma'_{n-\mu}$ be a linear combination of the Γ^i such that $\Gamma'.\Phi \sim 0$ mod. Φ. We then have on Φ a $C_{n-\mu} \equiv \Gamma'.\Phi$. Now Γ' can be assumed polyhedral and without $n - \mu$ cells on Φ. It can then be decomposed into $C'_{n-\mu} + C''_{n-\mu}$, the first on N, the second on K, with $C' \equiv \Gamma'.\Phi$, $C'' \equiv -\Gamma'.\Phi$. Also $\Gamma' = (C' - C) + (C'' + C)$. Each parenthesis represents a cycle, the first on N; hence, $\sim$ mod. N to a cycle on C_k, the second on $M_n - C_k$. But Γ', linear combination of the Γ^i, cannot be decomposed into such a sum, a contradiction which proves that the $\Gamma^i.\Phi$ are independent mod. Φ. The number of $n - \mu - 1$ cycles of the type now considered is then, like that of the Γ^i, equal to T.

Adding up the numbers of cycles of the two types (*a*), (*b*), we find that the right sides of (4) and (5) are equal. Then from (6) follows (4) whose proof is thus complete.

Remark. The proof fails when Φ, K, hence $M_n - C_k$, are vacuous sets. Therefore, it is essential that C_k be a true subset of M_n.

9. *Extension to an Arbitrary C_k on M_n.*—We now assume that C_k is any *non-singular* complex on M_n, not homeomorphic to M_n, so that $M_n - C_k$ is a non-vacuous set, i.e., there are points of M_n not on C_k. This is scarcely a restriction since when C_k coincides with M_n the questions here considered lose all interest. The basis of the extension is the theorem and corollary

to be proved below, natural generalizations of proposition II of No. 4:

Theorem: *Let N_δ be the set of all points of M_n whose distance from C_k is $< \delta$. If C_μ is any complex (singular or not) on N_δ whose boundary is on C_k, then for δ sufficiently small there is a C'_μ on C_k, such that $C_\mu \sim C'_\mu$ mod. N_δ.*

The proof is quite simple. For $k = 0$ the theorem reduces to II, No. 4 itself. Granting it for $k-1$ we shall show that it holds for k. Let C_{k-1} be the subcomplex of C_k where it fails to behave like an M_k. Subdivide C_μ into cells of diameter $< \epsilon$ sufficiently small and apply a certain construction due to Alexander,[9] his B vertices being now on C_k and the cells that they determine not necessarily straight but merely cells on C_k. The construction merely requires that δ, ϵ, be sufficiently small. The effect will be to have on N_δ a $C_{\mu+1} \equiv C^1_\mu - C^2_\mu + C^3_\mu$, where C^1 is on C_k, C^2 is very near C_{k-1} and C^3 includes all cells of C_μ, whose points are at a distance exceeding a certain α sufficiently small from C_{k-1}. Then $C_\mu - (C^1 - C^2 + C^3)$ which is $\sim C_\mu$ mod. N_δ, can be written as $C^2_\mu + C^4_\mu$, where C^4 can be brought as near as we please to C_{k-1} by taking δ, ϵ, α, small enough. The common boundary $\Gamma_{\mu-1}$ of C^2 and $-C^4$ is a cycle on C_k very near C_{k-1}, hence by II, No. 4, we can write a congruence $C^5_\mu \equiv \Gamma_{\mu-1} - \Gamma'_{\mu-1}$ where C^5 is on C_k as near as desired to C_{k-1} and Γ' is on C_{k-1}. Now $C^2 + C^4 = (C^2 - C^5) + (C^5 + C^4)$. The first parenthesis represents a complex on C_k, the second a complex whose boundary is on C_{k-1} and as near to the latter as we please. Since the theorem holds by assumption for $k-1$, $C^5 + C^4$ will be $\sim$ mod. a neighborhood of C_{k-1}, which we may manage to choose on N_δ, to a C^6_μ on C_{k-1} itself, hence on C_k. It follows finally $C_\mu \sim C^2 - C^5 + C^6$, mod. N_δ where C^2, C^5, C^6 are on C_k. The theorem is, therefore, proved.

Corollary: Under the same circumstances as in the theorem every cycle on N_δ is $\sim$ mod. N_δ to a cycle on C_k. This is II, No. 4 for our type of complex.

In order to apply properly the results of *Tr.* 2, in particular to derive (3), (4), for our C_k, we need in place of N_δ, a neighborhood which together with its boundary constitutes a manifold. Its construction is very simple. Subdivide M_n into cells whose diameter is $< {}^1/_2\,\delta$; take all n-cells carrying a point of C_k plus their boundaries, and construct a neighborhood of the C_n to which they give rise in the same manner as in No. 3, taking care that Φ_{n-1} be within a distance ${}^1/_2\,\delta$ of the C_n. What we naturally denote now as $N(C_k)$ is a manifold which consists of the C_n plus its neighborhood $+ \Phi$, and $K_n = M_n - N + \Phi$. Thanks to the Theorem and Corollary of No. 9 the proofs of (3), (4) go through without a change. Therefore, *these formulas stand proved for the most general (non-singular) C_k on M_n, actual subset of M_n.*

11. *Further Extensions.*—On examining the derivation of our formulas

it will be found that it is valid when M_n is replaced by a C_n, and C_k by a subset G of C_n such that: (*a*) G is a true subset of C_n. (*b*) There is a neighborhood of G whose points are all within a distance δ arbitrarily small of G, such that its complementary set K_n is an M_n. (*c*) The theorem and corollary of No. 10 are valid. We shall consider several important cases.

I. Let C_n satisfy the manifold conditions for all cells except those of a subcomplex C_k, $k < n$. Then (3), (4) will be applicable with M_n replaced by C_n, for (*a*), (*b*) hold, and (*c*) reduces to II, No. 4. Let, in particular, C_k be the boundary F_{n-1}. Then (3) turns directly into (1) except for C_n in place of M_n. However, in place of (2) we have

$$s_\mu(C_n - F_{n-1}; C_n) = r_{n-\mu-1}\,(F_{n-1}; C_n) + T_{n-\mu}\,(F_{n-1}; C_n - F_{n-1}). \quad (8)$$

Thus, when a C_n behaves like an M_n everywhere except at the boundary, a new term appears at the right in (2). In this connection we recall that Veblen has practically defined an M_n with boundary, as a subcomplex of an M_n without boundary.[10] For such an M_n, (1) holds but (2) must be replaced by the more involved (8).

An interesting example of the type of complex here considered is an algebraic surface or variety with its locus of singular points.

II. G is the homeomorph of a so-called *analytical* manifold (defined by a set of analytical expansions of the coördinates) or obtained by piecing a finite number of such manifolds. The boundaries of the pieces are assumed of the same type as G.[11]

III. Our formulas are valid when G is an *arbitrary closed* (true) subset of M_n, provided that cycles, complexes, etc., by means of which the various invariants are defined, are interpreted thus: A sequence of μ-cycles $\{\Gamma_\mu^p\}$ defines a μ-cycle Γ_μ on G if for every δ sufficiently small there is a p_0 such that all cycles Γ^p with $p > p_0$ are on N_δ and homologous to each other mod. N_δ. By definition $\Gamma \sim 0$ mod. G, if $\Gamma^p \sim 0$ mod. N_δ for every $p > p_0$. From this to the comparison of cycles, and the new definitions of the invariants of No. 6, is but a step and it is clear that all the elements for the extension of our formulas are at hand.

The cycles and connection indices so defined are the same as those recently introduced by Alexandroff[6] and Vietoris.[7] Indeed for δ sufficiently small Γ^p, $p > p_0$, can be reduced by Alexander's construction to a complex whose vertices are *on* G. Furthermore, if on N_δ there is a $C_{\mu+1} \equiv \Gamma_p$, $C_{\mu+1}$ and the cycle can simultaneously be reduced to a complex and cycle ($\sim \Gamma^p$ mod. N_δ) whose vertices are on G. Then the identity with the Vietoris process becomes apparent.

IV. The extension to a set G on an M_n with the boundary F_{n-1} offers no difficulty, provided G be replaced by $G + F_{n-1}$ and the formulas interpreted accordingly.

12. An extension of a different nature is obtained by considering with Alexander,[12] Poincaré congruences and homologies whose coefficients are reduced modulo a prime number p. Then C_μ is a cycle if its boundary is of the form $pC_{\mu-1}$; $C_\mu \sim 0$ if there is a C'_μ such that $C_\mu + pC'_\mu \sim 0$ in the usual sense. The various indices are then defined as in No. 6. Our formulas are then preserved.[13] This is likewise the case for congruences and homologies mod. 2 introduced earlier by Veblen and Alexander (see Veblen, loc. cit., Ch. III). They give rise to their so-called *connectivities*, which are $1 + R_\mu (G)$ in our notation (mod. 2, of course). In that case and with $M_n = H_n$, a sphere of S_n (4) turns into Alexander's main result,[2] the first along the lines investigated in this Note. It also includes the recent generalization given by Alexandroff.[3] Indeed, let G be any subset of H_n; $R (G;H) = S_\mu (\text{—}) = 0$, $T_\mu = 0$ unless $\mu = n$, when its value is 1; $r_\mu (\text{—}) = s_\mu (\text{—}) = R_\mu (G)$. Hence, if G is any closed set to which the formulas apply, (3) disappears and (4) becomes

$$R_\mu (H - G) = R_{n-\mu-1} (G) + \begin{cases} 0 \text{ for } \mu < n \\ 1 \text{ for } \mu = n \end{cases}, \qquad (9)$$

where the indices can also be taken mod. p. The reasoning by which Alexander derives the theorem of Jordan-Brouwer from his formula is also applicable to (9).

13. *Continuous Transformations.*—Let C_n be a complex with a closed subset F that includes in all cases the boundary of C_n, and such that for all elements of C_n without points on F the manifold condition is verified. Thus, C_n may be a manifold with boundary in the sense of Veblen. Let the indices $R_\mu (F)$ in the generalized sense of Alexandroff-Vietoris be finite. Then we can find a neighborhood N of F with the corresponding $K_n = C_n - N + \Phi$ as in No. 10, to which the formulas for coincidence and fixed points of *Tr.* 2 can be applied almost directly. V_n and the cycles Γ, Δ, D, G of *Tr.* 2, pp. 439–441 are defined as in *Tr.* 2 with these reservations: (*a*) The G's are independent cycles of $C_n - F$ (not necessarily of F) homologous mod. C_n to cycles on F. (*b*) A skew-symmetric cycle is a Δ if, and only if its intersection with F (base on F) is ≈ 0 mod. N (not the more stringent mod. F). Then there will be a corresponding skew-symmetric cycle associated with V_n of K_n. It will be found that the whole derivation of the coincidence and fixed point formulas given in *Tr.* 2 for an M_n with boundary is applicable to $C_n - F$.

As an interesting special case let C_n be a manifold in the sense of Veblen with $F = F_{n-1}$ as its boundary, and let it be susceptible of a continuous deformation into a set on $C_n - F$ (deformation into the interior of itself). Then if a cycle of F is ~ 0 mod. C_n it is deformed into a homologous cycle on $C_n - F$, which is ~ 0 mod. $C_n - F$, with a similar remark for the base of a skew-symmetric cycle on F. Therefore, the interpretation of the G's and Δ's can be taken strictly as in *Tr.* 2. From this follows that (37.1) of *Tr.* 2 represents again the number of signed fixed points of a *continuous* deformation and *it is then in this case also equal to the Euler-characteristic of the complex.*

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[1] Lefschetz, *Trans. Amer. Soc.*, **28,** 1926 (1–49); **29,** 1927 (429–462); Respectively referred to as *Tr.* 1, *Tr.* 2. Further references are given at the beginning of *Tr.* 1. The notations and terminology of this Note are as given at the beginning of *Tr.* 1. I take advantage of this occasion to point out the following errata for *Tr.* 2: Page 442, lines 5, 6, 8 replace μ by $\mu - 1$. Page 449, line 6, replace $n-\mu$ by $n-1-\mu$.

[2] Alexander, *Trans. Amer. Soc.*, **33,** 1922 (333–349).

[3] Alexandroff, *Comptes Rendus*, **184,** 1927 (425–427).

[4] As a matter of fact C_n and C_k may be any agglomeration of simplicial cells, no two of which have any common point, that constitutes a closed set. This is what Alexander (loc. cit.) calls a *chain*. The indices are then the maximum dimensions of cells of the set; a similar statement applies to C'_n, and repeatedly in the sequence.

[5] See *Tr.* 2, p. 5; also Veblen, *The Cambridge Colloquium*, New York, **1922,** p. 74.

[6] Alexandroff, *Comptes Rendus*, **184,** 1927 (317–319).

[7] Vietoris, *Math. Ann.*, **97,** 1927 (454–472).

[8] The rigorous justification for our seemingly reckless use of intersections of various cycles and complexes throughout this paper has been given at length in *Tr.* 1, 2.

[9] Alexander, *Trans. Amer. Math. Soc.*, **26,** 1915 (148–154).

[10] Loc. cit., p. 88.

[11] However, I do not think that it would be difficult to prove that these analytical manifolds are complexes.

[12] Alexander, *Trans. Amer. Math. Soc.*, **28,** 1926 (301–329).

[13] In the chain of arguments leading to formulas (3), (4) (*Tr.* 2 down to the duality theorems and present Note), the only step whose extension is not automatic is the proof of Veblen's theorem (*Trans. Amer. Math. Soc.*, **25,** 1923 (540–550); also *Tr.* 2, p. 433): There can be found for an orientable M_n without boundary two associated sets of cycles Γ^i_μ, $\Gamma^j_{n-\mu}$ $(i, j = 1, 2, \ldots, R_\mu)$, whose Kronecker-index matrix $\|(\Gamma^i_\mu \cdot \Gamma^j_{n-\mu})\|$ is $\pm$ the identity matrix of order R_μ. Now, in Veblen's notation, an independent set mod. p is obtained by adding to his circuits Γ_μ: (*a*) The complexes Λ^i_μ whose coefficient of torsion t is a multiple of p; (*b*) the complexes Ψ^i_μ whose boundary is a $\Lambda_{\mu-1}$ of the preceding type. But from Veblen's formulas, loc. cit., p. 546, coupled with his remarks, p. 547, follows that his theorem is still valid for the associated sets mod. p: Γ^i_μ, Λ^i_μ, Ψ^i_μ; $\Gamma^i_{n-\mu}$, $\Psi^i_{n-\mu}$, $\Lambda^i_{n-\mu}$, where now R_μ is the connection index mod. p. The Kronecker indices may, of course, be reduced mod. p. This justifies the assertion of the text. Incidentally Veblen's theorem implies $R_\mu \leqq R_{n-\mu}$, $R_\mu \geqq R_{n-\mu}$, hence $R_\mu = R_{n-\mu}$ which is Poincaré's duality relation for indices mod. p of an M_n without boundary.

The above holds for $p = 2$ as well. Furthermore, from Veblen's remarks at the end of his paper, it appears that the results of *Tr.* 1, 2, and of the present Notes, are applicable to non-orientable manifolds, when congruences, homologies and Kronecker-indices are reduced mod. 2.