INTERSECTIONS AND TRANSFORMATIONS OF COMPLEXES AND MANIFOLDS*

BY

SOLOMON LEFSCHETZ

INTRODUCTION

In writing this paper my first objective has been to prove certain formulas on fixed points and coincidences of continuous transformations of manifolds. To this proof for orientable manifolds without boundary is devoted most of the second part, the remainder of which is taken up by a study of product complexes in the sense of E. Steinitz, as they are the foundation on which the proof rests. With suitable restrictions the formulas derived are susceptible of extension to a wider range of manifolds, but this will be reserved for a later occasion. It may be stated that our formulas include and completely generalize the early results due to Brouwer and whatever has been obtained since along the same line.† No such generality would have been possible without that powerful instrument, the product complex.

The principle of the method is best explained by means of a very simple example. Let $f(x)$ and $\varphi(x)$ be continuous and uni-valued functions over the interval 0, 1, and let their values on the interval also lie between 0 and 1. It is required to find the number of solutions of $f(x)=\varphi(x)$, $0 \leqq x \leqq 1$.

Graphically the problem is solved by plotting the curvilinear arcs

$$y=f(x), \quad y=\varphi(x), \quad 0 \leqq x \leqq 1,$$

and taking their intersections. A slight modification of the functions may change the number of solutions, even make them become infinite in number. However, the difference between the numbers of *positive* and *negative* crossings of sufficiently close polygonal approximations to the arcs is a fixed number, their Kronecker index. Its determination is then a partial answer to the question, and indeed seemingly the only possible general answer.

---

* Presented to the Society under somewhat different title at the Chicago Meeting of April 13, 1923, and the Southwestern Section Meeting of December 1, 1923; received by the editors in November, 1924.

† A good bibliography is found in Kérékjárto's recently published volume *Topologie*, Berlin, J. Springer. For a list of the most recent titles see a paper by J. W. Alexander, these Transactions, vol. 25 (1923), p. 173, to which must be added my notes in the Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 90, vol. 11 (1925), pp. 287, 290, summarizing the results of the present paper.

The two complexes whose product is taken in this case are the unit segments on the x and y axes, their product being the square whose sides they are. Replace the unit segments by two identical manifolds of n dimensions, M_n and M_n', the square by the M_{2n} image of their pairs of points (product of the two), the arcs by manifolds on M_{2n} and the exact situation of Part II is obtained.*

In all questions of the above type, the Kronecker index plays then an essential part. In order to put everything on a solid basis, it seemed vital to discuss thoroughly this index. There is in existence an excellent treatment of it by Hadamard,† leaving little to be desired for euclidean spaces, but distinctly insufficient for general manifolds. Then the Kronecker index is only a special topic in the more interesting and far reaching theory of the intersection of complexes on a manifold,‡ needed in any case, to some slight extent, for a good treatment of the index itself. To this theory is devoted most of Part I, of which the chief result is as follows: given several complexes on an orientable manifold M_n, which do not intersect on each other's boundaries nor on that of M_n, there exists a well defined cycle of M_n, their *intersection*. It is well defined in this sense: no matter how the complexes are approximated by means of straight complexes, the cycle intersection of the latter remains homologous to itself. If the approximating complexes intersect in isolated points there is a definite Kronecker index independent of the mode of approximation.

The independence from covering complexes and related modes of defining straightness has presented some of our most serious difficulties. It is a little surprising that the necessity of freeing the Kronecker index from this vitiating circumstance has never been considered in the literature. That the wider problem has not been attacked is natural enough since intersections of general complexes have been studied but very little if at all.§

* This concept appeared first, applied to the special case of algebraic correspondences, in Severi's paper in the Torino Memorie, vol. 54 (1904). Needless to say, Severi did not suspect the analysis situs aspect of the problem, hence did not and could not derive the Hurwitz coincidence formulas. See in this connection Enriques and Chisini, *Lezioni sulla Teoria Geometrica delle Equazioni*, vol. 3, p. 427, also Chisini, Istituto Lombardo Rendiconti, ser. 2, vol. 7 (1924), p. 481. Their work is anticipated by my first Note, which seems to have escaped their notice.

† Note to Tannery's *Introduction à la Théorie des Fonctions*. See also the first chapter of my recent Borel Series Monograph, *L'Analysis Situs et la Géométrie Algébrique*, and my paper in the 1921 Transactions which both contain important applications of the index to algebraic geometry, and finally a very interesting paper by Veblen that has just appeared in these Transactions, vol. 25 (1923); results are recalled in Part I, §7 of this paper and derived anew in Part II.

‡ Considered and actually applied, I believe, for the first time in my Monograph.

§ Some very important results along that line have been obtained of late by J. W. Alexander. See Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 99, 101, 493.

PART I. QUESTIONS OF INTERSECTION

§1. PRELIMINARIES

1. In notation and terminology, we shall follow essentially Veblen's *Colloquium Lectures on Analysis Situs*. We shall assume the reader fairly familiar with this fundamental work and briefly refer to it as "Coll. Lect." The following designations will recur with special frequency in our paper: S_n = euclidean n-space; $E_n = n$-cell; $C_n = n$-dimensional complex; $M_n =$ manifold of n dimensions; Γ_n (also γ_n, δ_n in Part II) $= n$-cycle.* The various numerical invariants, the signs $\sim$, $\equiv$, for congruence or homology, and also the definition of orientation are as in Chapter I of my Borel Series Monograph, *L'Analysis Situs et la Géométrie Algébrique*. In Part II we shall introduce the sign $\approx$ for homologies with division allowed, that is with zero-divisors neglected.

2. Our ordinary complexes shall be restricted to Veblen's *regular* type. Such a C_n is the homeomorph of an n-dimensional polyhedron Π_n whose faces are all *simplicial* cells (interiors of simplexes) no two intersecting. The cells of Π_n define those of C_n. We shall apply the term rectilinear segment, polygonal or polyhedral configuration, etc., to C_n, as if it were Π_n itself, meaning thereby the images of the Π_n configurations. Distances on C_n shall also be measured by reference to Π_n, which for the purpose is assumed immersed in some S_n', $n' \leqq n$. By a *subcomplex* of C_n we shall mean one made up with cells of C_n.

C_n may be subdivided into new complexes, and this can be carried out so that the cells of the new complex be of diameter $<\epsilon$, assigned. Of importance in this connection is the method of *regular* subdivision (Coll. Lect., p. 89).

3. We shall define manifolds in accordance with a suggestion due to Veblen (Coll. Lect., p. 92). It amounts essentially to demanding of a C_n defining an M_n that its cells be grouped about any particular one much as if they were all immersed in an S_n. It will be found worth while to examine the matter at closer range.

Let E_k, $k<n$, be any simplicial cell on C_n, E_{k+1} another incident with it (i. e. with E_k on its boundary). We define a set $\{e\}$ of elements such that (*a*) to every E_{k+1} corresponds one and only one e; (*b*) to e corresponds together with a given E_{k+1} all others having such a cell in common with it; (*c*) two elements of the set are said to tend towards one another if and

* Or set of oriented n-circuits in Veblen's terminology.

only if there are corresponding cells whose vertices opposite to E_k tend towards one another. This last condition gives a definition of continuity for $\{e\}$.

Now C_n is said to define an M_n *without boundary* if for every E_k, $0 \leqq k \leqq n-1$, the set $\{e\}$ is homeomorphic to the boundary of an $(n-k)$-cell. The M_n *with boundary* is then defined as in Coll. Lect., p. 88.

To verify the manifold condition we proceed much as in Coll. Lect., pp. 88, 90. We shall show in §3 (No. 14, Lemma II) that there exists a subdivision C_n' of C_n of which $E_k = A_0A_1 \cdots A_k$ is a cell, the A's being its vertices. (Incidentally this method, now quite customary, of naming a cell by its vertices will prove very convenient.) Let $E_h = A_0A_1 \cdots A_h$ be any cell of C_n' incident with E_k. Then $A_{k+1} \cdots A_h$ is also a cell of C_n', and the totality of such cells gives rise to a C_{n-k-1} homeomorphic to $\{e\}$. Hence the manifold condition is equivalent to demanding that all these complexes be homeomorphic to cell boundaries. As the complexes for a given E_k are all homeomorphic the verification is really independent of the particular C_n' chosen.

It is as yet unknown whether for a given M_n the manifold condition is verified simultaneously for all defining complexes. We shall therefore agree to consider only defining complexes such that the cells of M_n for which the condition is verified have for logical sum one and the same point set.

4. The *orientation* of a simplicial cell is best defined by the order of naming the vertices (Monograph, p. 13). The oriented C_n is a complex as previously understood plus an assigned orientation for each n-cell. With the orientation of E_n there is attached one for its boundary $(n-1)$-cells. M_n is called *orientable* if its defining C_n can be so oriented that every non-bounding E_{n-1} receives opposite orientations from its two adjacent E_n's. This property is independent of the particular defining C_n. For match the complex with a copy of itself so that corresponding boundary points coincide. There will result a set of n-circuits, orientable or not at the same time as C_n itself. From the known independence for the circuit (Coll. Lect., pp. 100-102) follows that for M_n.

5. It is frequently convenient to orient M_n by means of a special n-cell used as indicatrix, thus: on an $E_n = A_0A_1 \cdots A_n$ of the defining complex we choose another $E_n' = B_0B_1 \cdots B_n$ reducible to the first by an affine transformation of the common S_n with coincidence of vertices in the order named, and instead of assigning the order of the A's, we do it for the B's.

Let $x_1, x_2, \cdots, x_n$ be cartesian coördinates on S_n, the origin being B_0. Then E_n' is completely defined if we give ourselves the matrix H' of the coördinates of the B's, it being understood that the ith row corresponds to B_i. Let E_n'' be another indicatrix with the same first vertex B_0, and let H'' be the corresponding matrix. Then E_n'' is the indicatrix of $+M_n$ or $-M_n$ according as the signs of the determinants of H', H'' (certainly $\neq 0$) are or are not the same.

Remark. Whenever we derive from C_n a new complex C_n', subdivision of the first, we shall agree to orient each n-cell of C_n' so that it constitutes an indicatrix of the n-cell of C_n that carries it.

6. With Veblen we shall call *singular* k-cell on C_n a point set $\bar{E}_k$, of C_n, uniform and continuous image of an ordinary cell E_k which we may as well assume simplicial. Any statement concerning $\bar{E}_k$, in particular regarding its orientation or boundary cells, is to be interpreted by reference to E_k.

Let $E_k^1, \cdots, E_k^p$ be sensed cells on C_n, and let $E_{k-1}^1, \cdots, E_{k-1}^q$ be their bounding $(k-1)$-cells, all cells being possibly (but not necessarily) singular. We shall extend the term "k-complex on C_n" to cover a symbol

$$C_k = \sum x_i E_k^i$$

where the x's are arbitrary integers. The points of C_k are those of the E_k's whose x coefficient is not zero plus their limit points. To the cells correspond Poincaré congruences

$$E_k^i \equiv \sum y_{ij} E_{k-1}^j ,$$

and for C_k by definition

$$C_k \equiv \sum x_i y_{ij} E_{k-1}^j .$$

The C_{k-1} at the right is the *boundary* of C_k. If it reduces to zero, C_k is a *k-cycle*. The fact that C_{k-1} is a boundary is also expressed by the homology

$$C_{k-1} \sim 0 \qquad \pmod{C_n} .$$

Remark. The boundary (sensed) of an n-cell is a cycle; the verification is immediate. Hence, by summation the boundary of a C_n is also a cycle.

§2. Intersection of cells

7. Let E_h, E_k, E_n be simplicial cells, the first two on the third. E_h and E_k may intersect in various ways. Assume that in the S_n of E_n the spaces S_h and S_k which carry the other cells are linearly independent, so that their intersection is an S_l, $l = h + k - n$. Grant furthermore that $l \geqq 0$, so that S_l is an actual space (possibly a point) and also that the cells themselves intersect. This intersection will consist of an element of S_l, bounded by a convex polyhedron, and therefore constitutes an l-cell E_l. If we assume the linear spaces of the boundaries of E_h and E_k also as independent as possible

from those of the cells themselves, no bounding $(l-1)$-cell of E_l (necessarily polyhedral like E_l itself) will be an $(l-1)$-cell of the boundary of E_h or E_k, but will be merely on one of their bounding $(h-1)$-cells or $(k-1)$-cells respectively, and this we shall assume for the present. E_l with its bounding polyhedron constitutes an M_l decomposable into simplexes, for example by the regular subdivision process. To this M_l we now propose to assign an orientation corresponding to given orientations of the other cells as follows. Let $E_n' = A_0A_1 \cdots A_n$ be a small simplicial cell such that $E_l' = A_0A_1 \cdots A_l$ lies on E_l, $E_h' = A_0A_1 \cdots A_h$ on E_h, and $E_k' = A_0A_1 \cdots A_l A_{h+1} \cdots A_n$ on E_k. Let $\alpha_s E_s'$, $\alpha_s = \pm 1$, be the indicatrix of E_s $(s = h, k, l, n)$. Then α_l is to be determined by the relation

$$\alpha_h \cdot \alpha_k \cdot \alpha_l \cdot \alpha_n = +1 \ .$$

The cell so sensed shall be designated by $E_h \cdot E_k$.*

The relation between the α's shows that if one of the cells E_h, E_k, E_n is inverted so is E_l. Furthermore if E_h and E_k are permuted the only indicatrix changed is E_n', whose vertices undergo $(n-h)\,(h-l) = (n-h)\,(n-k)$ transpositions. Hence

$$E_h \cdot E_k = (-1)^{(n-h)\cdot(n-k)} E_k \cdot E_h = -(-E_h) \cdot E_k = -E_h \cdot (-E_k) \ .$$

8. We have tacitly assumed throughout that $l > 0$. With suitable conventions we may let it take any value whatever. The case $l < 0$ may be dismissed at once; we simply write, then,

$$E_h \cdot E_k = 0.$$

Let now $l = 0$. Then there is a unique point of intersection constituting a zero-cell E_0. It is this point with the value of α_0 attached which we designate by $E_h \cdot E_{n-h}$. The value of α_0 is called the *Kronecker index* or simply index of E_h and E_{n-h} and denoted by $(E_h \cdot E_{n-h})$. The above symbolic relation still holds, and we derive from it and our discussion

$$(E_h \cdot E_{n-h}) = (-1)^{(n+1)h}\,(E_{n-h} \cdot E_h) = -(-E_h \cdot E_{n-h}) = -(E_h \cdot -E_{n-h}) \ ,$$

which may also be obtained directly by means of the indicatrices.

To make our conventions complete, when the cells do not intersect, we shall write

$$E_h \cdot E_k = 0 \ , \qquad (E_h \cdot E_{n-h}) = 0 \ .$$

The case $h = 0$, $k = n$, is not exceptional. We then have a point $E_0 = A$ and an attached unit α_0 in place of α_h. The point is on the "intersecting"

* In previous papers the same notation was used, sometimes with, sometimes without the "dot." In this paper it has been essential to use the dot throughout, for in Part II another "product" symbol comes in, whose meaning is wholly different and which will be written as a "cross" product.

cell $\bar{E}_n$ (in place of E_k) and a_k is replaced by $\bar{a}_n$ whose value is $+1$ if E_n is sensed like E_n, -1 otherwise. The index is now

$$(E_0 \cdot E_n) = a_0 = a_n \cdot \bar{a}_n \cdot \bar{a}_0 .$$

Its sign is that of $\bar{a}_0$ if E_n and $\bar{E}_n$ are sensed alike, its opposite otherwise.

9. Our next task is to determine the boundary congruences. We first assume that $E_h \cdot E_k$ *has no boundary* $(l-1)$*-cell on the boundary of* E_n. This is indeed the general case, but the exception here referred to is of importance later. Let the boundary congruences for E_h and E_k be

$$E_h \equiv \sum E_{h-1}^i , \qquad E_k \equiv \sum E_{k-1}^j .$$

The boundary of $E_h \cdot E_k$ is then the sum of the cells $E_h \cdot E_{k-1}^j$, $E_{h-1}^i \cdot E_k$, affected with signs that are to be determined.

Let for example E_h actually intersect E_{k-1}^j and choose E_n' with the vertices of

$$E_{k-1}' = A_0 A_1 \cdot \cdot \cdot A_{l-1} \, A_{h+1} \cdot \cdot \cdot A_n \text{ on } E_{k-1}^j .$$

As A_l must be transposed l times to come to first place, $(-1)^l \cdot a_k \cdot E_{k-1}'$ is an indicatrix of E_{k-1}^j. Hence, a_{k-1} corresponding to E_{k-1}^j as a_k to E_k, we have

$$a_{k-1} = (-1)^l a_k .$$

Therefore

$$a_{l-1} = (-1)^l \cdot a_l$$

corresponds to $E_h \cdot E_{k-1}$ as a_l to $E_h \cdot E_k$ itself, so that $E_h \cdot E_{k-1}$ has for indicatrix $(-1)^l \cdot a_l \cdot A_0 A_1 \cdot \cdot \cdot A_{l-1}$, which is the precise indicatrix that it should have as a boundary cell of $E_h \cdot E_k$, since A_l must be transposed l times to be brought to first place, and since $a_l A_0 A_1 \cdot \cdot \cdot A_l$ is the indicatrix of $E_h \cdot E_k$.

We conclude then that in the boundary congruence for $E_h \cdot E_k$ we must affect $E_h \cdot E_{k-1}$ with the sign $+$. Similarly $E_k \cdot E_{h-1}$ must be affected with the sign $+$ in the congruence for $E_k \cdot E_h$, hence $(-1)^{(n-h+1)(n-k)} E_{h-1}^i \cdot E_k$ with the sign $+$ in the congruence for $(-1)^{(n-h)(n-k)} E_h \cdot E_k$, from which at once

$$E_h \cdot E_k \equiv (-1)^{(n-k)} \sum E_{h-1}^i \cdot E_k + \sum E_h \cdot E_{k-1}^j .$$

In the exceptional case at first excluded, E_l will have some bounding E_{l-1}'s on some bounding E_{n-1} of E_n. This will be due to the fact that for example E_{h-1}^p and E_{k-1}^q will both lie in E_{n-1}. It is found by considering now intersections in E_{n-1} that $E^p_{h-1}\cdot E^q_{k-1}$ is positively related to $E_h\cdot E_k$. The method is the same as above: we merely assume in our indicatrix A_l exterior to E_{n-1} and the rest goes through about as before. We shall then have the general congruence

$$(9.1)\qquad E_h\cdot E_k\equiv(-1)^{n-k}\sum E^i_{h-1}\cdot E_k+\sum E_h\cdot E^j_{k-1}+\sum E^p_{h-1}\cdot E^q_{k-1}\,.$$

We have here a case where it would be distinctly worth while to have a more complicated notation to indicate in which complex intersections are taken. Such instances are comparatively rare and the doubt will always readily be cleared up by reference to the context.

10. **Fundamental theorem on Kronecker indices.** Just as before, the case where $l=1$, and the cells at the right are points (zero-cells) offers no exception. $E_h\cdot E_k$ is then a rectilinear one-cell E_1 and the three sums at the right reduce to two terms corresponding to the initial and terminal points of E_1, the sensed intersection. To each of these terms corresponds a Kronecker index, computed either as to E_n or as to one of its bounding $(n-1)$-cells. I say, and this is our theorem, that *in all cases the sum of these two-indices is zero* so that they are units of opposite signs.

11. Let then

$$E_1=E_h\cdot E_{n-h+1}\equiv(-1)^{h-1}E'_{h-1}\cdot E_{n-h+1}+\cdot\cdot\cdot\;,$$

where we have not written the term that we do not wish to discuss. Assume first that the term written represents the initial point A_0 of E_1 and take it for vertex of same name of the indicatrix previously considered. Here then $\alpha_l=\alpha_1=+1$. The situation is as follows:

$A_0A_2\cdots A_h$	is indicatrix for	$-\alpha_h\cdot E'_{h-1}\,,$
$A_0A_1A_{h+1}\cdots A_n$	" " "	$\alpha_{n-h+1}\cdot E_{n-h+1}\,,$
$A_0A_2\cdots A_hA_1A_{h+1}\cdots A_n$	" " "	$(-1)^{h-1}\cdot\alpha_nE_n\,.$

The Kronecker index for the point $(-1)^{h-1}\cdot E'_{h-1}\cdot E_{n-h+1}$ is then the number β defined by the condition

$$(-1)^h\cdot\alpha_h\cdot\alpha_{n-h+1}\cdot(-1)^{h-1}\cdot\alpha_n\cdot\beta=1\;.$$

We have also

$$\alpha_h\cdot\alpha_{n-h+1}\cdot\alpha_n\cdot\alpha_1=\alpha_h\cdot\alpha_{n-h+1}\cdot\alpha_n=1\;.$$

Hence finally $\beta=-1$. If A_0 were the terminal point of E_1, we would have merely $\alpha_1=-1$, the rest being the same, hence $\beta=+1$. Thus we see that the Kronecker indices for the end points have the same signs that the points receive in the boundary congruence for E_1. The other two cases (where A_0 is $E_h\cdot E_{n-h}$ or $E_{h-1}\cdot E_{n-h}$ and on an E_{n-1}) lead to exactly the same conclusion; the proofs, essentially similar, are omitted here. The sum of the indices is therefore always zero as was to be proved.

12. The extension to the intersection of s cells $E_i, E_h, \cdots, E_k$ on E_n, goes through with ease. It is denoted by $E_i \cdot E_h \cdots E_k$, a symbol which obeys the associative but in general not the commutative law. A similar remark holds for the index $(E_i \cdot E_h \cdots E_k)$, which exists only when $i+h+\cdots+k=n(s-1)$. The boundary congruences can be written down at once.

§3. Intersections of polyhedral complexes and their Kronecker indices

13. The complexes which are to occupy us in the rest of Part I shall all be immersed in a connected, orientable and oriented manifold M_n with an assigned defining complex C_n. We shall assume throughout that intersecting complexes have non-intersecting boundaries and no common points on the boundary C_{n-1} of M_n. Of several intersecting complexes so restricted let one, say C_h, have points on C_{n-1}. We may subdivide C_h into C_h' with cells so small that those h-cells which have points of C_{n-1}, or whose boundary has some, carry no points of the intersecting complexes on themselves or on their boundary. Let $\bar{C}_h$ be the complex sum of these cells plus their boundaries. As far as the intersection with the other complexes is concerned, C_h' may be replaced by $C_h'-\bar{C}_h$ which carries no points of C_{n-1}. A similar remark applies in case there are points common to some, but not all, the boundaries of the intersecting complexes. Henceforth it shall then be understood once for all that

I. *intersecting complexes have no points on the boundary of* M_n;

II. *their boundaries do not meet.*

Our general plan is as follows. We shall first define the intersection of a still narrower class of polyhedral complexes, and then approximate general complexes by means of these. But before defining our special polyhedral complexes, we must prove two lemmas.

14. Lemma I. *Any polyhedral* C_h *is a sum of simplicial cells.*

Each h-cell of C_h is a sum of a finite number of polyhedral regions of a certain S_h. Each region is decomposable into a sum of convex polyhedral h-cells.* Remove these from C_h and let C_{h-1} be the remaining complex. The lemma is true for $h=1$. Grant it for the dimensionality $h-1$; C_{h-1} can then be decomposed into a sum of simplicial cells. Select a point on each convex h-cell and join it by rectilinear segments to the simplicial

* This has been proved by Veblen and others. For detailed references see Coll. Lect., p. 83.

cells of C_{h-1} on the boundary of its h-cell. There will follow the requisite decomposition of C_h.

Remark. A region of the initial decomposition of C_h may lie on several cells of C_n. If so the boundaries of the latter decompose it into regions each of which lies on a unique cell. Then C_h will appear as a sum of simplicial cells, each also on a unique cell of C_n. Whenever we shall consider in the sequel a polyhedral C_h, arising in some manner in the course of the discussion, we shall assume that it *has been decomposed into a sum of simplicial cells each on a unique cell of* C_n. Strictly speaking, the initial complex is thus replaced by a subdivision and should be designated by a new notation, but it will simplify matters a good deal to avoid this.

LEMMA II. *There exists a subdivision C_n' of C_n with C_h as a sub-complex.*

Decompose C_h as just stated into a sum of simplicial cells, any one, say E_k, on an E_n of C_n or on its boundary. The S_k of E_k is the intersection of certain S_{n-1}'s of the S_n of E_n. Extend these S_{n-1}'s as far as possible on the simplex of their E_n. There will result a decomposition of C_n into a new complex $\bar{C}_n$ with C_h as a subcomplex. Apply now Lemma I to $\bar{C}_n$ and C_n' follows.

15. We now seek to define the intersection of two polyhedral complexes C_h, C_k and its boundary congruences. Let

$$C_h = \sum E_h^i\,; \qquad C_k = \sum E_k^j\,;$$
$$E_h^i \equiv \sum E_{h-1}^{ip}\,; \qquad E_k^j \equiv \sum E_{k-1}^{jq}\,.$$

We impose the following restrictive conditions:

(*a*) *Intersecting h- and k-cells are on one and the same n-cell of C_n and in general position as understood in* §2. *Their intersection is then an l-cell, where, as before,* $l = h + k - n$.

(*b*) *Let E_{h-1}^{ip} intersect E_{k-1}^{jq} on an E_{l-1}. Then both are on an E_{n-1} of C_n and E_{l-1} is not on the boundaries of C_h and C_k.*

When these two conditions are satisfied, the intersection, to be denoted by $C_h \cdot C_k$, is a C_l defined by the relation

$$C_h \cdot C_k = \sum E_h^i \cdot E_k^j\,,$$

it being understood that, whenever E_h^i and E_k^j do not intersect, $E_h^i \,.\, E_k^j = 0$.

The symbols $C_h \cdot C_k$ obeys the distributive law, as follows at once from the definition. Thus:

$$(C_h' + C_h'') \cdot C_k = C_h' \cdot C_k + C_h'' \cdot C_k\,,$$

it being granted of course that both C_h' and C_h'' satisfy conditions (*a*) and (*b*) as to C_k. Similarly for C_k while the effect of permuting the C''s or inverting them is as in No. 7 for the cells.

The boundary congruences will present no great difficulty. From (9.1) follows

$$(15.1)\qquad C_h \cdot C_k \equiv (-1)^{n-k} \sum E_{h-1}^{ip} \cdot E_k^j + \sum E_h^i \cdot E_{k-1}^{jq} + \sum E_{h-1}^{ir} \cdot E_{k-1}^{js} ,$$

the meaning of each sum being readily apprehended by reference to (9.1). I say that the terms in the third sum cancel each other. Indeed let E_h^i and E_k^j be on the cell E_n and give rise to the term $E_{h-1}^{ir} \cdot E_{k-1}^{js}$, intersection of E_{h-1}^{ir}, E_{k-1}^{js}, situated in a bounding cell E_{n-1} of E_n. We assume the cells E_{h-1}^{ir}, E_{k-1}^{js}, E_{n-1} positively related to E_h^i, E_k^j, E_n, and E_{h-1}^{ir}, E_{k-1}^{js} is the intersection of the two sensed cells, oriented as indicated in §2.

According to (*b*) there exist E_n', $E_h^{i'}$, $E_k^{j'}$ of C_n, C_h, C_k, with E_{n-1}, E_{h-1}^{ir}, E_{k-1}^{js} on their boundaries and negatively related to them. There is a cell labelled $E_{h-1}^{i'r'} = -E_{h-1}^{ir}$ and one labelled $E_{k-1}^{j's'} = -E_{k-1}^{js}$. Indeed according to (*b*) the cells of C_h adjacent to E_{h-1}^{ir} can be grouped in pairs oppositely related, and we may assume that we have such a pair in E_h^i, $E_h^{i'}$. Similarly for E_k^j and $E_k^{j'}$.

There are now several possibilities. It may be that E_{n-1} separates E_h^i and $E_h^{i'}$ (that is, one of them is on E_n, the other on E_n') but not E_k^j and $E_k^{j'}$. Then the third sum in (15.1) contains these terms pertaining to the couples considered above and no others:

$$E_{h-1}^{ir} \cdot E_{k-1}^{js} + E_{h-1}^{ir} \cdot E_{k-1}^{j's'} .$$

They represent intersections on E_{n-1} and as $E_{k-1}^{j's'} = -E_{k-1}^{js}$ their sum is zero.

A second possibility is that E_{n-1} separates the two pairs of h- and k-cells. Then in the sum in question there correspond the terms

$$E_{h-1}^{ir} \cdot E_{k-1}^{js} + E_{h-1}^{i'r'} \cdot E_{k-1}^{j's'} .$$

The first intersection is taken on E_{n-1}, the second on $-E_{n-1}$ (that is in the scheme of §2, E_{n-1} must now be replaced by $-E_{n-1}$, the reason being that boundaries of cells on E_n' are now involved and E_n' is negatively related to E_{n-1}). When intersections are referred to E_{n-1}, the second term must be written

$$-E_{h-1}^{i'r'} \cdot E_{k-1}^{j's'} = -(-E_{h-1}^{ir}) \cdot (-E_{k-1}^{js}) = -E_{h-1}^{ir} \cdot E_{k-1}^{js}$$

and the sum is again zero.

Finally we must consider the case where E_{n-1} separates none of the two pairs of cells. Then the terms to be considered are now

$$E_{h-1}^{ir} \cdot E_{k-1}^{js} + E_{h-1}^{ir} \cdot E_{k-1}^{j's'} + E_{h-1}^{i'r'} \cdot E_{k-1}^{js} + E_{h-1}^{i'r'} \cdot E_{k-1}^{j's'} ,$$

the intersections being all referred to E_{n-1}. It is immediately verified that the fourth term is the same as the first, and the other two terms its negative. The sum is then again zero. This completes the proof of our assertion.

16. The boundary of C_h is a cycle Γ_{h-1}, and that of C_k is a cycle Γ_{k-1}. We have

$$C_h \equiv \Gamma_{h-1}, \quad C_k \equiv \Gamma_{k-1} .$$

The first two sums in (15.1) are respectively $\Gamma_{h-1} \cdot C_k$ and $C_h \cdot \Gamma_{k-1}$. Hence in the last analysis we have this fundamental congruence:

$$C_h \cdot C_k \equiv (-1)^{n-k} \Gamma_{h-1} \cdot C_k + C_h \cdot \Gamma_{k-1} . \tag{16.1}$$

17. A series of important corollaries follows at once from the preceding discussion.

I. A complex C_h is called a *generalized manifold* if every non-bounding E_{h-1} of it is incident with just two h-cells. Orientability is defined for it as for an ordinary manifold. From our discussion we obtain the following: *if C_h and C_k are orientable generalized manifolds so is $C_h \cdot C_k$.*

II. *If one of the complexes is a cycle the boundary of $C_h \cdot C_k$ is the intersection of this cycle with the other complex or its opposite. If both are cycles so is their intersection.* In symbols,

$$C_h \cdot \Gamma_k = (-1)^{n-k} \Gamma_{h-1} \cdot \Gamma_k ; \quad \Gamma_h \cdot C_k \equiv \Gamma_h \cdot \Gamma_{k-1} ; \quad \Gamma_h \cdot \Gamma_k \equiv 0 . \tag{17.1}$$

III. *When the boundary of each complex does not meet the other, $C_h \cdot C_k$ is a cycle.*

IV. *Let Γ_h bound C_{h+1} satisfying our restrictive conditions as to its intersection with C_k. Then $\Gamma_h \cdot C_k$ is a bounding cycle also.*

This can be read off from (17.1).

18. **Kronecker index.** This time the dimensions are h and $n-h$. We make the same assumptions as previously, and, in addition, agree that for two non-intersecting cells the index is zero. Then

$$(C_h \cdot C_{n-h}) = \sum (E_h^i \cdot E_{n-h}^j) .$$

From No. 8 follows the distributive law for indices. The result of permuting the two complexes or of changing the sign of one is as for cells and need not be written down.

From No. 10 and (16.1) it follows that if

$$C_h \equiv \Gamma_{h-1}\,, \quad C_{n-h+1} \equiv \Gamma_{n-h}\,,$$

then here

(18.1) $$(C_h \cdot \Gamma_{n-h}) = (-1)^h \cdot (\Gamma_{h-1} \cdot C_{n-h+1})\,,$$

a formula of great importance later (§6), as it transforms an index corresponding to dimensionalities h, $n-h$, to one with h replaced by $h-1$.*

19. From the theorem of No. 10, together with (16.1), now follows

Let Γ_h bound a C_{h+1} not intersecting the boundary of C_{n-h}, a condition that disappears if we deal with a Γ_{n-h}. Let furthermore the usual restrictions as to intersecting complexes, C_{h+1} and C_{n-h} or Γ_{n-h}, hold. Then

$$(\Gamma_h \cdot C_{n-h}) = 0\,, \quad (\Gamma_h \cdot \Gamma_{n-h}) = 0\,.$$

Observe that owing to the distributive law, it is not necessary that $\Gamma_h \sim 0$, but merely ≈ 0, for then $t\Gamma_h$ bounds and the multiples of the indices are zero; hence also the indices themselves. This result will have important applications in Part II.

20. The extension to several intersecting complexes offers no particular difficulties. The symbols follow the associative and distributive laws, but not in general the commutative law.

§4. Approximation of complexes

21. A first, but somewhat inelastic, approximation to a general complex C_k by a polyhedral C_k' will be obtained by direct application of processes due to Alexander (these Transactions, vol. 16 (1915), p. 148) and Veblen (Coll. Lect., pp. 95, 118). C_k appears then as a subcomplex of a subdivisior of C_n with cells of suitably small diameter. There are two associated complexes C_{k+1} and C_k^0 such that

(21.1) $$C_{k+1} \equiv C_k - C_k' + C_k^0\,,$$

and therefore

(21.2) $$C_k' \sim C_k + C_k^0\,.$$

C_k^0 appears only when C_k is not a cycle. Our C_{k+1} is the same as Veblen's B_{i+1}. When C_k is not a cycle, the boundary cells of C_{k+1} which join boundary cells of C_k and C_k' are also part of the boundary of C_{k+1}, and their sum is precisely C_k^0.

* An analogous formula for cells was given by Veblen in the Transactions paper already quoted. p. 542.

In the light of this and upon examining the construction, we find by the simplest continuity considerations that it may be carried out so that the following statements hold.

(*a*) *C_k' and C_{k+1} are both as near as we please to C_k.*

(*b*) *The complex C_k^0 and the boundary of C_k' are as near as we please to the boundary of C_k.*

(*c*) *C_k' includes any particular polyhedral boundary subcomplex of C_k.* This is proved by a very transparent application of Lemma II, No. 14.

22. From the preceding method of approximation we may derive this interesting result: Let the approximation be made by means of cells of C_n. *If the points of a cell of C_k are sufficiently near those of E_h of C_n, then its approximation is E_h itself or a cell on its boundary. The cell of C_k need only be within a certain distance δ of E_h in order that this be true.*

From this we have the following

THEOREM. *To every polyhedral complex C_k corresponds a positive number δ such that every cycle Γ_h whose points are all within δ of C_k is homologous to a cycle Γ_h' on C_k.*

For C_k is a subcomplex of a subdivision C_n' of C_n (Lemma II, No. 14) and in this case C_h^0 corresponds to C_k^0 of No. 21, for Γ_h is absent. We can also affirm that $\Gamma_h - \Gamma_h'$ will bound, by (*a*), a C_{h+1} whose points are as near as we please to C_k at the same time as those of Γ_h.

Incidentally, since C_k can have no cycle of more than k dimensions, we have this very interesting result: *A non bounding cycle cannot be homologous to a cycle as near as we please to a complex of smaller dimensionality.*

23. The approximations which we have obtained so far are not flexible enough for our purpose, which demands the approximation of two or more complexes at the same time by others with a well-defined intersection. This will be based upon the all-important

THEOREM. *Let C_k be a subcomplex of C_n. By subtraction of bounding k-cycles, it may be reduced to another complex with the same boundary, whose $(k-i)$-cells not on the boundary of M_n nor on its own are on cells of at least $n-i$ dimensions of C_n.*

Let C_n' be a regular subdivision of C_n and C_k' the corresponding subdivision of C_k. Any vertex of C_n' shall be affected with an upper index, such as A_i^p to indicate the dimensionality p of the cell of C_n which carries it.

Let us attach to any cell $E_r = A_0A_1 \cdots A_r$ of C_n' a symbol $(p_0, p_1, \cdots, p_r)$ to describe its type. Observe that *the p's are all distinct*, for $p_0 = p_1 = p$ would mean that two points on distinct p-cells of C_n are joined by a rectilinear segment wholly on a p-cell.

Another and more significant property is that *the highest p indicates the dimensionality of the cell of C_n which carries E_r.* For then E_r has points on such a cell (in the vicinity of the corresponding A^p); hence, due to the mechanism of regular subdivision, it lies entirely on it.

Our theorem will then be proved if we can show that *C_k is reducible to a complex of which every E_{k-i} not on its boundary nor on that of M_n has in its symbol an integer $\geqq n-i$.*

24. (*a*) For $n=1$, the reduction is immediate. Then $k=0$ alone needs to be considered. C_0 is a sum of points each with an assigned sign. Any such point A, say affected with $+$, may be reduced to any point B of C_1 by adding the end points of a polygonal line AB. These end points constitute the bounding Γ_0 of the theorem.

(*b*) Let now $i=0$ and E_k be as yet not reduced. Its symbol is then of type $A_0^{p_0} \cdots A_k^{p_k}$ with all the p's less than n. The cell lies therefore on the boundary of an n-cell of C_n on which there is a vertex A_{k+1}^n. Let Γ_k be the boundary of the simplex $A_{k+1}^n A_0^{p_0} \cdots A_k^{p_k}$ which is positively related to E_k; $C_k'-\Gamma_k$ is a complex which has the same structure as C_k except that E_k has been replaced by $k+1$ cells of same dimensionality in every one of whose symbols appears A_{k+1}^n, so that they are on n cells of C_n. This carries out the reduction for $i=0$.

(*c*) Assume that the process goes through for any $M_{n'}$, $n'<n$, and also for all cells of more than $k-i=m$ dimensions of C_k'. I say that it goes through for all dimensionalities.

Consider an unreduced $E_m=A_0^{p_0} \cdots A_m^{p_m}$ of C_k', the p's being then all $<n-i=n-k+m$. To any $E_h=A_0^{p_0} \cdots A_m^{p_m}\ A_{m+1}^{p_{m+1}} \cdots A_h^{p_h}$ incident with E_m corresponds $E_{h-m+1}=A_{m+1}^{p_{m+1}} \cdots A_h^{p_h}$ which we sense so that if the first set of A's is, as we shall assume, an indicatrix of E_h, the last is one for E_{h-m-1}. The totality of these cells is a subcomplex of C_k which is an M_{n-m-1} homeomorphic to the boundary of a cell (No. 3). The incidence relations (boundary congruences) between corresponding cells are formally identical.

Let $q_0, q_1, \cdots, q_{n-m-1}$ be the set of integers in increasing order which together with $p_0, \cdots, p_m$ constitute the set $0, 1, \cdots, n$. The manifold M_{n-m-1} carries two defining complexes. The first C_{n-m-1} has the points A^{q_0} for vertices, the second C'_{n-m-1} the remaining points A^q. In fact C'_{n-m-1} is a regular subdivision of C_{n-m-1}, its vertices q_i being on i-cells of it. To show this it will suffice to examine the relation between those which correspond to q_0 and q_1. Let, for example, the sequence of the p's and q's in increasing order read $p_0, p_1, q_0, p_2, p_3, q_1, \cdots$, so that $p_0=0$, $p_1=1$, $q_0=2$, $\cdots, q_1=5, \cdots$. Consider now $E_2=A^0 A^1A^3$ of C_n'. It is on a certain three-cell of C_n on which it is incident with exactly two cells $A^0A^1A_1^2A^3$ and

$A^0A^1A_2^2A^3$ of C_n'. Hence $E_{m+1}=A^0A^1A^3\cdots A^5A^{p_4}A^{p_5}\cdots A^{p_m}$ whose symbol is $(p_0,\cdots,p_3,q_1,p_4,\cdots,p_m)$ is incident with the two $(m+2)$-cells obtained by placing first A_1^2, then A_2^2 between A^1 and A^3. It follows that A^3 is on the one-cell $A_1^2A^3+A^3A_2^2$ of C_{n-m-1}. With the q's the statement is that A^{q_1} is on the one-cell $A_1^{q_1}A^{q_1}+A^{q_1}A_2^{p_1}$. The same reasoning applies to the other q vertices.

Observe that since the p's are all $< n-k+m$ the q's must include all integers from $n-k+m$ to m. Hence $q_{n-m-1}=n$, $q_{n-m-2}=n-1,\cdots, q_{n-k-1}=n-k+m$.

25. Let now $(p_0,\cdots,p_m;\, q_0',\cdots q_{k-m-1}')$ be the symbol for any k-cell of C_k' incident with E_m. The reduction being achieved by assumption for all cells of more than m dimensions, $(p_0,\cdots,p_m;\, q_i')$, the symbol of an $(m+1)$-cell, must possess an integer $\geqq n-k+m+1$ which can only be q_i'. Similarly $(p_0,\cdots,p_m;\, q_i',\, q_j')$ must include at least one integer $\geqq n-k+m+2$, and this can only be q_i' or q_j', etc. Finally, then, among the $k-m$ integers q_i' there must be one at least equal to each integer of the sequence $n-k+m+1,\, n-k+m+2,\cdots, n$. As they are all distinct and $\leqq n$ they constitute that very sequence and our cell has then the symbol $(p_0,\cdots, p_m;\, n-k+m+1,\ \cdots, n)$. The symbol of the corresponding cell of C_{n-m-1}' is $(n-k+m+1,\ \cdots, n)$.

26. Let $\bar{C}_k$ be the subcomplex of C_k' which is the sum of its cells incident with E_m. Since the latter is not on the boundary of C_k', on any E_{k-1} of $\bar{C}_k$, there are as many positively related incident k-cells as negatively related. Hence the complex of C_{n-m-1}' which corresponds to $\bar{C}_k$ is a cycle Γ_{k-m-1}. Since C_{n-m-1}' is homeomorphic to the boundary of a cell, Γ_{k-m-1} bounds on the complex, and in fact bounds a subcomplex C_{k-m} of C_{n-m-1}' (Coll. Lect., pp. 95, 118). Furthermore since the reduction to be proved applies by assumption to an M_{n-m-1}, C_{k-m} may be so reduced without changing its boundary that in the symbol of its s-cells not on the boundary Γ there appears $q_{n-k+s-1}$ or a higher q. But since the cells of Γ already satisfy this condition, it holds for all cells of C_{k-m} without exception. From this we conclude immediately that its $(k-m)$-cells have all the same symbol:

$$(q_{n-k-1},\, q_{n-k},\cdots, q_{n-m-1}) \equiv (n-k+m,\, n-k+m+1,\cdots, n).$$

That it has this last simple form follows from the remark at the end of No. 24.

27. To C_{k-m} corresponds a subcomplex C_{k+1} of C_n' whose boundary is a Γ_k. The cells of this cycle incident with E_m constitute $\bar{C}_k$ so that E_m is not a cell of $C_k''=C_k'-\Gamma_k$. However, among the new cells of C_k'' are found those on k-cells of Γ_k not incident with E_m and we must examine these.

The symbol for any $(k+1)$-cell of C_{k+1} is $(p_0,\cdots, p_m;\, n-k+m,\cdots, n)$. The new k-cells introduced have then a symbol such as $(p_0,\cdots, p_{s-1},$

$p_{s+1}, \cdots, p_m; n-k+m, \cdots, n)$. In the symbol of any cell of $m+j$ dimensions of its boundary will then appear $n-k+m+j$ or a higher integer. The new cells of m or more dimensions have then the desired behavior. Thus E_m and its incident cells have been replaced bv a set of cells which fulfil all requirements. *The proof of our theorem is therefore complete.*

Remark. We have incidentally obtained the following interesting proposition: *Every cycle of a complex defining a manifold without boundary is homologous to a cycle on its dual.* More precisely *every* Γ_k *of* C_n *is homologous to a cycle whose k-cells have all the same symbol* $(n-k, n-k+1, \cdots, n)$.

28. We return to our approximation problem. The reduction of C_k has been obtained by adding the boundaries of $(k+1)$-cells incident with its k-cells and belonging to C_n. Let us replace C_n by a subdivision $\bar{C}_n$ with cells $<\epsilon$ and C_n' by a regular subdivision of $\bar{C}_n$. If we make the same reduction, we shall merely add to C_k the boundaries of $(k+1)$-cells within a distance of ϵ from the complex.

Applying this reduction to the approximating complex C_k' we find that we do not thereby disturb (21.1) or (21.2). The complex C_{k+1} is simply increased by cells as near as we please to C_k', hence to the approximated complex.

29. The essential property of C_k is that *the* S_{k-i} *of any* E_{k-i} *of the complex has the maximum degree of generality relatively to the space of the cell* C_n *that carries it.* We mean thereby that, by modifying the complex without changing its cellular structure, S_{k-i} may be brought into coincidence with an arbitrary neighboring S'_{k-i}. The weakest case is when E_{k-i} lies on an E_{n-i} of C_n with its vertices on $(n-k)$-cells of the boundary of E_{n-i}. Let A be a vertex of E_{k-i} on the cell E_{n-k}; S'_{k-i} will intersect the latter at a point B very near A. Impress upon A the rectilinear displacement AB, and similarly for the other $k-i$ vertices of E_{k-i}, leaving the remaining vertices of C_k' unchanged. There results an obvious deformation of C_k', into say C_k'', whereby S_{k-i} is brought into coincidence with S'_{k-i} thus proving our assertion.

The displacement of C_k' may be so carried out, and in a continuous way, that every point will describe a rectilinear path. Their locus is a $\bar{C}_{k+1} \equiv C_k'' - C_k'$. By adding this to (21.1) we see that C_k'' may take the place of C_k' with $C_{k+1}+\bar{C}_{k+1}$ in place of C_{k+1}. If S'_{k-i} is sufficiently near S_{k-i} conditions (a), (b), (c) of No. 21 may still be fulfilled. The reasoning in case E_{k-i} lies on a cell of more than $n-i$ dimensions is the same.

§5. Intersections of general complexes

30. To arrive at something significant, we must narrow down the problem once more. We replace then the second condition of No. 13 by the somewhat more sweeping

III. *Intersecting complexes must not meet the boundaries of one another.*

To complexes so restricted we shall ascribe definite cycles or Kronecker indices. Indices and cycles are fixed, in the sense that the former remain the same when we vary the approximating complexes or even the C_n by means of which we construct them, while the cycles remain homologous to themselves. We shall at first maintain C_n fixed and merely vary the polyhedral approximations, then examine the effect produced by a change of C_n.

31. Starting first with two complexes C_h, C_k always restricted as in No. 13, we approximate them as closely as we please by C_h', C_k' constructed as in No. 32 with the system of relations

$$C_{h+1} \equiv C_h - C_h' + \overset{0}{C_h}\ ; \qquad C_h' \sim C_h + \overset{0}{C_h}\ ; \tag{31.1}$$

$$C_{k+1} \equiv C_k - C_k' + \overset{0}{C_k}\ ; \qquad C_k' \sim C_k + \overset{0}{C_k}\ ; \tag{31.2}$$

the various complexes have the same meaning as those of similar designation in §4. If the approximation is sufficiently fine, C_h' and C_k' will also fulfil the restrictive conditions I, III of Nos. 13 and 30. This we assume henceforth for all our approximating complexes.

It follows at once from No. 29 that we may so choose C_h' and C_k' that they satisfy the two conditions of No. 14 for a well defined intersection $C_h' \cdot C_k'$. Since the boundary of each complex does not meet the other, the intersection will be an l-cycle. It remains to be shown that this cycle is independent of the approximating complexes.

32. As a preliminary step let $\bar{C}_h'$ be another approximation whose intersection with C_k' is well defined. I say that

$$C_h' \cdot C_k' \sim \bar{C}_h' \cdot C_k'\ . \tag{32.1}$$

We have now congruences such as (21.1)

$$\bar{C}_{h+1} \equiv C_h + \bar{C}_h^0 - \bar{C}_h'\ , \tag{32.2}$$

$$C_{h+1} \equiv C_h + C_h^0 - C_h' \tag{32.3}$$

with C_h^0, $\bar{C}_h^0$ very near the boundary of C_h and C_{h+1}, $\bar{C}_{h+1}$ very near the complex itself. Their approximation is in fact assumed such throughout that none of these complexes meets the boundary of C_k'. We have then from (32.2) and (32.3)

$$\bar{C}_{h+1} - C_{h+1} \equiv C_h' - \bar{C}_h' - (C_h^0 - \bar{C}_h^0)\ .$$

To the complex at the left we may apply everything said previously for C_k with the following result. There exists a polyhedral complex C_{h+1}' very near it, having a well defined intersection with C_k' and whose boundary is $C_h'-\bar{C}_h'$ plus a certain h-complex very near $C_h^0-\bar{C}_h^0$ and therefore not meeting C_k'. Hence by (16.1)

$$C_{h+1}' \cdot C_k' \equiv (C_h'-\bar{C}_h') \cdot \bar{C}_k' \sim 0 \qquad (\mathrm{mod}\ M_n)\ ,$$

from which (32.1) follows.

33. Let now $\bar{C}_h'$, $\bar{C}_k'$ be two approximations with a well defined intersection. In order to show that the intersections are independent of the particular polyhedral approximations provided they have a well defined intersection, we must prove that

$$\bar{C}_h' \cdot \bar{C}_k' \sim C_h' \cdot C_k' \ . \tag{33.1}$$

By a slight displacement such as is used in No. 29, we may replace C_h' by a complex C_h'' with a well defined intersection with both C_k' and $\bar{C}_k'$. All that is necessary is to replace throughout C_k' by $C_k'-\bar{C}_k'$. We have then according to the preceding number

$$C_h' \cdot C_k' \sim C_h'' \cdot C_k' \ ; \quad \bar{C}_h' \cdot \bar{C}_k' \sim C_h'' \cdot \bar{C}_k' \ ;$$

$$C_h'' \cdot C_k' \sim C_h'' \cdot \bar{C}_k' \ ,$$

whence (33.1) follows.

Regarding the Kronecker indices, Corollary IV at the end of §3 yields at once for $h+k=n$

$$((C_h'-\bar{C}_h'-(C_h^0-\bar{C}_h^0)) \cdot C_k') = 0 \ ,$$

and as C_k' does not intersect $C_h^0-\bar{C}_h^0$,

$$(C_h' \cdot C_k') = (\bar{C}_h' \cdot C_k')$$

and the rest is as before.

34. The extension to more than two intersecting complexes is easy. With obvious notations we must show that

$$C_h' \cdot C_k' \cdots C_l' \sim \bar{C}_h' \cdot \bar{C}_k' \cdots \bar{C}_l' \ .$$

Introduce C_h'' in general position as to C_k', $\bar{C}_k'$, $\cdots$, C_l', $\bar{C}_l'$. As above it may be shown that in the homology C_h' and $\bar{C}_h'$ may both be replaced by C_h'', the process continuing in an obvious way. The treatment for indices is the same.

35. As is natural we denote the cycles and indices defined by means of our approximations as $C_h \cdot C_k \cdots C_l$, or $(C_h \cdot C_k \cdots C_l)$. These symbols have the same properties as those for polyhedral complexes them-

selves. Those pertaining to permutation of complexes hold obviously, others less so. We shall examine them in turn, with particular reference to our future needs.

I. *Associative law.* The scheme of the proof is sufficiently illustrated with three complexes and if we show that

$$C_h \cdot (C_k \cdot C_l) \sim C_h \cdot C_k \cdot C_l .$$

By definition $C_k \cdot C_l$ and $C_h \cdot C_k \cdot C_l$ are the polyhedral cycles $C_k' \cdot C_l'$ and $C_h' \cdot C_k' \cdot C_l'$. In the approximation that leads to the determination of $C_h \cdot (C_k' \cdot C_l')$, the cycle in parentheses can be taken as its own approximation. Hence the cycle at the left in the homology is by definition $C_h' \cdot (C_k' \cdot C_l')$ and we are back to the case of polyhedral complexes in general position, for which the law holds.

II. *Distributive law.* We wish to show that, say,

$$C_h \cdot (C_k + \bar{C}_k) \sim C_h \cdot C_k + C_h \cdot \bar{C}_k .$$

On examining the two successive approximations of §4 it will be seen that $C_k' + \bar{C}_k'$ is an approximation for $C_k + \bar{C}_k$. If each of the two primed complexes has a well defined intersection with C_h' so has their sum. Hence the left side is by definition $C_h' \cdot (C_k' + \bar{C}_k')$. As the terms at the right are also defined by means of the primed symbols, we are again back to the case of polyhedral complexes where the law holds.

The two preceding proofs hold without modification for the Kronecker index.

III. *If $C_h, C_k, \cdots, C_l$ do not actually have a common point, then*

$$C_h \cdot C_k \cdots C_l \sim 0 , \textit{ or } (C_h \cdot C_k \cdots C_l) = 0 .$$

For then the primed complexes may be taken without any common point and everything is once more reduced to the known case of polyhedral complexes.

IV. *Let C_h bound $\bar{C}_{h+1}$ such that it is the only one of the set $\bar{C}_{h+1}, C_k, \cdots, C_l$ whose boundary may have points in common with the other complexes. Then*

$$C_h \cdot C_k \cdots C_l \sim 0 \textit{ or } (C_h \cdot C_k \cdots C_l) = 0 .$$

From (31.1) follows $\bar{C}_{h+1} - C_{h+1} \equiv C_h'$, for C_h^0 is now absent since C_h is a cycle. Moreover C_h being a subcomplex of $\bar{C}_{h+1}$ the sequence $C_h, C_k, \cdots, C_l$ behaves like that of the statement. Let us approximate $\bar{C}_{h+1} - C_{h+1}$ by, say, C'_{h+1}, in our usual manner, which may be done without changing C_h since it is on the boundary (property (*c*), No. 21). Since C_{h+1} is very near C_h'

it will be seen that the primed sequences corresponding to the two considered above behave as they do. The generalization of (16.1) gives here

$$C'_{h+1}\cdot C_k'\cdots C_l'\equiv C_h'\cdot C_k'\cdot\cdots\cdot C_l'$$

from which IV follows.

V. *Let* $\Gamma_h\approx 0$ (*i. e., some multiple of* $\Gamma_h\sim 0$). *Then*

$$\Gamma_h\cdot\Gamma_k\cdots\Gamma_l\approx 0 \text{ or } (\Gamma_h\cdot\Gamma_k\cdots\Gamma_l)=0\ .$$

This is an immediate corollary of IV. In both IV and V it is of course not at all necessary that the complex or cycle singled out be the first.

VI. *In* C_h *we may suppress any subcomplex not intersecting* $C_k,\cdots,C_l$, *without affecting intersection cycle or index.*

This is an immediate but important corollary of II and III.

36. Before we proceed with a thorough examination of the effect of passing from C_n to a new defining $\bar{C}_n$, let us observe that *instead of approximating to* C_h *by means of* $\bar{C}_n$ *it is sufficient to do this for* C_h'. Indeed, let $\bar{C}_h'$ be the approximation to C_h' by means of $\bar{C}_n$. We shall have a congruence such as (21.1):

$$\bar{C}_{h+1}\equiv C_h'-\bar{C}_h'+\bar{C}_h^0\ ,$$

with $\bar{C}_{h+1}$ very near C_h' and $\bar{C}_h^0$ very near its boundary. Add this congruence to the first of (31.1) (which is the same as (21.1) with k replaced by h):

$$(C_{h+1}+\bar{C}_{h+1})=C_h-\bar{C}_h'+(C_h^0+\bar{C}_h^0)\ .$$

This is analogous to (21.1) with $\bar{C}_h'$ as the approximation to C_h. As the congruences such as (21.1) plus the structure of the approximating complexes themselves were alone used in defining the intersection cycles and indices and deriving their properties, our assertion is proved.

§6. Proof that intersection cycles and indices are invariant when the defining complex is changed*

37. For indices the invariance is conditioned upon a certain simple sense convention. Let C_n, $\bar{C}_n$ be any two defining complexes. By practically the same reasoning as Veblen's in *Colloquium Lectures*, pp. 101, 102, we may show that one of the two complexes $C_n\pm\bar{C}_n$ is a bounding cycle, but not both (loc. cit., p. 120). We shall assume in the future that *the complexes are so oriented that* $C_n-\bar{C}_n$ *is the bounding cycle.* Once any particular complex has been assigned an orientation, a definite one follows for the rest.

* A first type of proof is outlined in my second Proceedings note. Just as it appeared in print I discovered the much simpler treatment embodied in this section.

We must examine if the present convention agrees with our previous mode of sensing a subdivision C_n' of C_n, where it will be remembered a cell and its subdivisions were always so sensed as to have a common indicatrix. It is clearly sufficient to consider the case where C_n is a simplex with a simplicial subdivision:

$$C_n' = \overset{1}{\Sigma}_n + \cdots + \overset{p}{\Sigma}_n .$$

Let the first q simplexes, but no others, have an $(n-1)$-simplex on a given simplex Σ_{n-1} of Σ_n, namely $\overset{i}{\Sigma}_{n-1}$ for $\overset{i}{\Sigma}_n$, with A and A^i as the vertices of Σ_n and $\overset{i}{\Sigma}_n$ not on Σ_{n-1} or $\overset{i}{\Sigma}_{n-1}$. We take the orientations such that A followed by the vertices of Σ_{n-1} corresponds to the same as A_i followed by those of $\overset{i}{\Sigma}_{n-1}$.

In order that $\Sigma_n - C_n'$ be a cycle, it is necessary and sufficient that the boundaries on Σ_{n-1} cancel, or that

$$\Sigma_{n-1} - (\overset{1}{\Sigma}_{n-1} + \cdots + \overset{q}{\Sigma}_{n-1})$$

be a cycle. This reduces the verification from n to $n-1$, hence ultimately to $n=1$ for which it is immediate.

38. Let us return to two arbitrary defining complexes C_n, $\bar{C}_n$. Our customary approximations, when applied to $\bar{C}_n$ by means of C_n, cannot go farther than what is yielded by the Alexander-Veblen process. In this case it comes down to this: We subdivide C_n and $\bar{C}_n$ into C_n' and $\bar{C}_n'$, then establish a correspondence T whereby to each cell of C_n' is assigned a unique one of C_n'. The n-cells of C_n' are each covered positively by k_2 cells corresponding to positive cells of $\bar{C}_n'$ and negatively by k_1 such cells. Furthermore it is a property of the correspondence that $C_n' - T(\bar{C}_n')$ bounds, hence $k_2 - k_1$ is fixed for all n-cells of C_n, else this complex would have boundary cells exterior to the boundary of M_n and would not be a cycle. Moreover

$$T(\bar{C}_n') \sim \bar{C}_n' \sim \bar{C}_n \sim C_n \sim C_n' ;$$

therefore

$$C_n' - T(\bar{C}_n') = (1+k_1-k_2)C_n' \sim 0 .$$

When a subcomplex of a C_n bounds it bounds also such a subcomplex (Coll. Lect. p. 118). But C_n' has no $(n+1)$-cells, hence this homology can only be true if $1+k_1-k_2=0$, $k_2-k_1=1$. *Thus $T(\bar{C}_n')$ covers C_n' exactly once.* The importance of this will appear later.

39. **Invariance of the index.** It will be established by means of (18.1) that

$$(C_h \cdot \Gamma_{n-h}) = (-1)^h \cdot (\Gamma_{h-1} \cdot C_{n-h+1}), \tag{39.1}$$

where the conditions of No. 15 must be satisfied. Here the intersecting complexes are the two extremes, the cycles being their boundaries. We have, then, given C_h, C_{n-h}, their approximations C_h', C'_{n-h} by means of a first defining complex C_n, and $\bar{C}_h'$, $\bar{C}'_{n-h}$ by a second $\bar{C}_n$, the latter being, if we wish, approximations to C_h' and C'_{n-h} rather than to the given complexes (No. 37). We denote the indices as to C_n by the usual round parentheses, those as to $\bar{C}_n$ by square parentheses, and our object is to show that

$$(39.2) \qquad (C_h' \cdot C'_{n-h}) = [C_h' \cdot C'_{n-h}] \; .$$

Assuming first that neither h nor $n-h$ are zero, we shall replace (39.2) by a similar formula with $h-1$ in place of h. This will allow us to reduce everything to the case $h=0$ for which the proof is simple.

40. Suppose that there exists a polyhedral C'_{n-h+1} (until further notice straightness and the like are defined by reference to C_n) whose boundary

$$\Gamma'_{n-h} = C'_{n-h} + C''_{n-h}$$

where C'_h and C''_{n-h} do not meet. Then first

$$(C'_h \cdot C'_{n-h}) = (C'_h \cdot \Gamma'_{n-h}) \; .$$

Also assuming all due conditions satisfied, and denoting by Γ'_{h-1} the boundary of C'_h, we find from (39.1)

$$(40.1) \qquad (C'_h \cdot C'_{n-h}) = (-1)^h \cdot (\Gamma_{h-1} \cdot C'_{n-h+1}) \; .$$

The necessary conditions are fulfilled with ease. According to our very construction of approximating complexes, C'_h and C'_{n-h} intersect in a finite number of points, any one of which, say A_i, is on cells of maximum dimensionality of the two complexes and of C_n. Let in particular E^i_{n-h} be that of C'_{n-h}. We can construct a simplicial E^i_{n-h+1} with E^i_{n-h} on its boundary and positively related to it. Let this last cell count m_i times for C'_{n-h}. We choose

$$C'_{n-h+1} = \sum m_i E^i_{n-h+1} \; ;$$

C''_{n-h} is now what is left of the boundaries of the cells at the right when the cells $m_i E^i_{n-h}$ are removed from them. We may therefore manifestly so choose the cells E^i_{n-h+1} that C_{n-h} will contain no A point.

Without prejudice to what precedes, the cells of C''_{n-h} may be brought as near as we please to C'_{n-h}, hence we may so construct C'_{n-h+1} that C''_{n-h} intersects C_h in a finite number of points (none an A), each on h-, $(n-h+1)$- and n-cells of C'_h, C'_{n-h+1} and C_n. Let B be such a point. Remove from C_h'

a simplicial cell E_h containing B. If it is sufficiently small, it will contain no A and its bounding $(h-1)$-cells will intersect C'_{n-h+1} each at a single point of the cell that carries B. Operate similarly for all B's and let the new complex still be denoted by C'_{h-1} and its boundary by Γ'_{h-1}.

If the cells of C'_{n-h+1} and those removed from the initial C_h' are adequately small, the intersection of the new C'_h with C'_{n-h+1} will be a sum of isolated rectilinear segments each on an n-cell of C_n, and as between the two complexes the various restrictions of §3 are verified. Since the new C_h' does not meet C''_{n-h}, it gives rise to the same index $(C'_h \cdot C'_{n-h})$ as the initial one, so that we are assured that (40.1) holds with C_h' as the new approximation to C_h and Γ'_{h-1} as its boundary.

41. At this stage we introduce the second defining complex $\bar{C}_n$. All approximations by means of it are to be denoted by the same symbols barred.

We first construct $\bar{\Gamma}'_{h-1}$, $\bar{C}'_{n-h}$, $\bar{C}''_{n-h}$. On examining our approximating processes, it is seen that the sum of the last two complexes is a suitable $\bar{\Gamma}'_{n-h}$. The cycle $\bar{\Gamma}'_{h-1}$ bounds $C'_h+C^0_h$, where the second complex, introduced by the approximation, is as near as we please to Γ'_{h-1}. Since C'_h and C'_{n-h} were constructed in general relative position, the second does not meet the boundary Γ'_{h-1} of the first. With approximations sufficiently fine, it will not meet C^0_h either. Hence (No. 35, VI)

$$[(C'_h+C^0_h) \cdot C'_{n-h}] = [C'_h \cdot C'_{n-h}] \, .$$

Therefore in place of approximating C_h' we may approximate $C_h'+C^0_h$. To avoid more notations than necessary, we shall denote that approximation by $\bar{C}_h'$.

Similarly $\bar{\Gamma}'_{n-h}$ bounds a complex $C'_{n-h+1}+C^0_{n-h+1}$, the latter very near the cycle and therefore not intersecting Γ'_{h-1}, hence, as above, its approximation may take the place of that of C'_{n-h+1} and will then be denoted by $\bar{C}'_{n-h+1}$.

As previously, we may choose the approximations $\bar{C}'_h$ and $\bar{C}'_{n-h+1}$ such as to fulfil the conditions of §3 (Nos. 13, 15), for a well defined intersection. Even for $h=1$ is this the case. $\bar{C}_1'$ may then be so chosen that the points of Γ_0' are approximated by any in their vicinity. We have therefore in all cases where $h>0$

$$[\bar{C}'_h \cdot \bar{C}'_{n-h}] = (-1)^h [\bar{\Gamma}'_{h-1} \cdot \bar{C}'_{n-h+1}] \, .$$

By the definition of the index, each expression defines that of the approximated complexes, which are here the primed complexes. Hence

$$[C'_h \cdot C'_{n-h}] = (-1)^h \cdot [\Gamma'_{h-1} \cdot C'_{n-h+1}] \, .$$

42. From (40.1) and (41.1) follows that in place of (39.2) we need only prove

$$(\Gamma'_{h-1} \cdot C_{n-h+1}) = [\Gamma'_{h-1} \cdot C_{n-h+1}] ,$$

of the same type but with h replaced by $h-1$. Proceeding thus if necessary we shall reduce h to zero. The last step will consist in replacing C_1' by its set of terminal points Γ_0', the signs affixed to them being $+$ or $-$ according as they are positively or negatively related to the complex. $\overline{\Gamma}_0'$ will then consist of points in the same number as those of Γ_0', very near to them and with same signs attached.

For the other complex, we now have a C_n' and by subdividing C_n if necessary, we may assume that C_n' is a subcomplex of it. Furthermore, (No. 41), the points of Γ_0' may all be chosen on n-cells of C_n'.

Owing to the distributive law, we may assume in the last analysis that we have a unique point A with a definite sign affixed, and a unique cell E_n of C_n carrying the point A. We have seen that the sign of the point is to be independent of the approximation; let us assume that it is $+$. In the other possible case, the reasoning would be the same with perhaps some signs changed. We must then show that

$$(A \cdot E_n) = +1 = [A \cdot E_n] .$$

Since A is not on $C_n - E_n$, we may add the last complex to E_n without affecting the indices. We have to prove then, that

$$[A \cdot C_n] = +1 .$$

We may choose A on E_n not a cell of less than n dimensions of $\overline{C}_n$ without affecting our indices. Let A be its own approximation by means of $\overline{C}_n$. The approximation of C_n by means of $\overline{C}_n$ will be the sum of the cells of a certain subdivision of $\overline{C}_n$ (No. 40), which is the same as $\overline{C}_n$ itself. Finally then

$$[A \cdot C_n] = [A \cdot \overline{C}_n] = +1 ,$$

which completes the proof of the invariance of the index of two complexes.

43. The extension to the index of several complexes can be made along the same lines. The essential thing is to obtain a relation analogous to (40.1). It will be sufficient to derive it for three complexes C_h, C_k, C_l, $h+k+l=2n$, with the usual conditions as to their boundaries. Their approximations will then satisfy conditions analogous to those of No. 15. With obvious notations, let C'_{h+1} have for boundary $\Gamma'_h = C'_h + C''_h$ with C''_h not intersecting C_k' nor C_l'. As a matter of fact, it would be sufficient to have it not intersecting $C_k' \cdot C_l'$. Then

$$(C_h' \cdot C_k' \cdot C_l') = (\Gamma_h' \cdot C_k' \cdot C_l') . \tag{43.1}$$

Also by (16.1) and with Γ'_{k-1} the boundary of C_k',

$$C'_{h+1}\cdot C'_k\equiv(-1)^{n-k}\cdot\Gamma'_h\cdot C_k+C'_{h+1}\cdot\Gamma'_{k-1}\ ,$$

it being granted that conditions of No. 15 are duly fulfilled. If we now further assume that the boundary of C_l' does not meet the complex at the left, then (No. 19)

$$(-1)^{n-k}(\Gamma_h'\cdot C_k'\cdot C_l')+(C'_{h+1}\cdot\Gamma'_{k-1}\cdot C_l')=0\ .$$

Hence by (43.1)

$$(C_h'\cdot C_k'\cdot C_l')=(-1)^{n-k-1}(C'_{h+1}\cdot\Gamma'_{k-1}\cdot C_l')\ .$$

Similarly, with notations whose meaning is transparent,

$$C_h'\cdot C_k'\cdot C_l'=(-1)^{n-l-1}\cdot(C_h'\cdot C'_{k+1}\cdot\Gamma'_{l-1})\ .$$

Thus we can raise both h and k at the expense of l, until $h=k=n$, $l=0$. From this point on the details of the discussion differ in no material way from the case of two complexes and need not be given here.

We have then proved that *the index of any number of complexes, when any exists, is independent of the defining C_n of M_n and the related straightness.* It has of course the various properties given in No. 36.

44. **Invariance of the intersection cycle.** The reasoning is exactly the same whatever the number of complexes, so we need only take two, C_h, C_k. This time, with our previous notations, we have to show that

$$C_h'\cdot C_k'\sim\bar{C}_h'\cdot\bar{C}_k'\ .$$

The cycle $C_h'\cdot C_k'$ is carried by a C_l, of the simplest type of No. 2, sum of the distinct simplicial l-cells of a decomposition of the cycle into such cells. Let these cells be $E_l^1,\cdots,E_l^p$, oriented each in a definite way. We shall have

$$C_l=\sum E_l^i\ ;\quad C_h'\cdot C_k'=\sum t_iE_l^i\ ,$$

where the t's are non-zero integers.

Since the barred complexes are as near as we please to the non-barred, $\bar{C}_h'\cdot\bar{C}_k'$ is as near as we please to C_l. Hence (No. 22) the latter carries a Γ_l, subcomplex of a sufficiently fine subdivision, such that $\bar{C}_h'\cdot\bar{C}_k'-\Gamma_l$ bounds a C_{l+1} whose points are as near as we please to C_l. But Γ_l is homologous, mod C_l, to a subcomplex of C_l itself (Coll. Lect., p. 120). Hence we may assume that Γ_l itself is such a subcomplex without weakening the assertion as to C_{l+1}. We shall have

$$\bar{C}_h' \cdot \bar{C}_k' \sim \sum s_i E_l^i = \Gamma_l \; .$$

Owing to our usual approximating procedure, each E_l^i is on an n-cell of C_n. Hence, just as if C_n were an S_n, we can construct a simplicial E_{n-l} that intersects C_l at a unique point on E_l^i in such manner that

$$(E_l^i \cdot E_{n-l}) = +1 \; ; \qquad (E_l^j \cdot E_{n-l}) = 0 \; , \qquad j \neq i \; .$$

By taking E_{n-l} sufficiently small, we may dispose of it so that its boundary does not meet C_l, while the cell itself does not meet the boundaries of C_h' or C_k'. This is due to the fact that the latter do not intersect C_l while E_{n-l} is very near to this complex.

Let us now approximate by means of $\bar{C}_n$. We have two congruences such as in No. 31:

$$\bar{C}_{h+1} \equiv C_h' - \bar{C}_h' + \bar{C}_h^0 \; ,$$

$$\bar{C}_{k+1} \equiv C_k' - \bar{C}_k' + C_k^0 \; .$$

When the approximation is carried sufficiently far,

(*a*) the boundary of E_{n-l} will not meet C_{l+1};

(*b*) it will not go through any intersection of $\bar{C}_{h+1}$ with C_k' or $\bar{C}_k'$, nor of $\bar{C}_{k+1}$ with C_h' or $\bar{C}_h'$;

(*c*) E_{n-l} will not meet $\bar{C}_h^0$ nor $\bar{C}_k^0$.

Since C_{l+1} is as near as we please to C_l, (*a*) follows from the fact that the boundary does not intersect C_l. Now were the first of (*b*) untrue, since $\bar{C}_{h+1}$ is as near as we please to C_h, it would only mean that the boundary of E_{n-l} goes through a point of $C_h' \cdot C_k'$, and hence meets C_l which carries this cycle, and this is not the case. Similarly for the rest of (*b*). As to (*c*), it follows at once from the fact that the cell does not meet the boundaries of C_h' and C_k' to which $\bar{C}_h^0$ and $\bar{C}_k^0$ are very near.

Owing to (*a*) and to IV and I of No. 36, we have

$$((\Gamma_l - \bar{C}_h' \cdot \bar{C}_k') \cdot E_{n-l}) = 0 \; ;$$

therefore

$$\sum s_j (E_l^j \cdot E_{n-l}) = s_i = (\bar{C}_h' \cdot \bar{C}_k' \cdot E_{n-l}) \; .$$

Also

$$(C_h' \cdot C_k' \cdot E_{n-l}) = \sum t_j (E_l^i \cdot E_{n-l}) = t_i \; .$$

Again by IV and the first of (*b*),

$$((C_h' - \bar{C}_h' + \bar{C}_h^0) \cdot C_k' \cdot E_{n-l}) = 0 \; .$$

From (c) we have

$$(\bar{C}_h^0 \cdot C_k' \cdot E_{n-l}) = 0 .$$

Therefore

$$(C_h' \cdot C_k' \cdot E_{n-l}) = (\bar{C}_h' \cdot C_k' \cdot E_{n-l}) .$$

Similarly from the last of (b) and (c),

$$(\bar{C}_h' \cdot (C_k' - \bar{C}_k' + \bar{C}_k^0) \cdot E_{n-l}) = 0 .$$

Therefore

$$(\bar{C}_h' \cdot C_k' \cdot E_{n-l}) = (\bar{C}_h' \cdot \bar{C}_k' \cdot E_{n-l}) .$$

By comparing we have at once

$$t_i = (C_h' \cdot C_k' \cdot E_{n-l}) = (\bar{C}_h' \cdot \bar{C}_k' \cdot E_{n-l}) = s_i .$$

From this follows finally

$$C_h' \cdot C_k' \sim \bar{C}_h' \cdot \bar{C}_k' ,$$

and our invariance proof is complete.

45. Intersection cycles or Kronecker indices are then the same whatever the defining complex of M_n from which they are derived, provided as regards indices that various complexes have their orientations suitably related. Cycles and indices have all the properties established in No. 35 and the effect of a permutation of complexes in the symbols is the same as for cells.

§7. Fundamental sets on an orientable M_n without boundary

46. Let $\Gamma_h^i (i = 1, 2, \cdots, p_h)$, Γ_{n-h}^j $(j = 1, 2, \cdots, p_{n-h})$ be two fundamental sets (No. 7) for the cycles of the same dimensions, and consider the matrix

$$\| (\Gamma_h^i \cdot \Gamma_{n-h}^j) \| . \tag{46.1}$$

Any other fundamental set, say for the dimensionality h, can be derived from a given one by applying to its cycles a transformation of determinant unity. This is at once derivable from the result established by Veblen, loc. cit., p. 117. By such a transformation we mean that every cycle of the new set will be homologous to a sum of cycles of the old, the matrix of the coefficients of the homologies being ± 1.

According to the distributive law for indices, if we change fundamental sets the above matrix is merely multiplied to the right or to the left by a matrix of determinant ± 1. *Therefore the invariant factors of* (46.1) *are independent of the two fundamental sets*, and in fact *they are all equal to unity*, an important result just proved by Veblen,* which we shall also derive later in a new way (No. 65). It follows from this and from Poincaré's theorem on the equality of the numbers R_h, R_{n-h}, that we may select our sets so that

(*a*) for $j > R_h$, Γ_h^j and Γ_{n-h}^j are zero-divisors;

(*b*) if $h \neq n/2$ or if $h = n/2$ and $n/2$ is even,

$$(\Gamma_h^i \cdot \Gamma_{n-h}^i) = \begin{cases} +1 & \text{if } h \leqq n/2 \\ (-1)^{(n+1)h} & \text{if } h > n/2 \end{cases} \quad i \leqq R_h ,$$

all other indices being zero;

(*c*) if $h = n/2$ and $n/2$ is odd, when (46.1) is an alternate matrix,

$$(\Gamma_{n/2}^{2i-1} \cdot \Gamma_{n/2}^{2i}) = -(\Gamma_{n/2}^{2i} \cdot \Gamma_{n/2}^{2i-1}) = +1 , \; i \leqq \tfrac{1}{2} R_{n/2}$$

with all other indices again zero. In this case, of course, *since the matrix is alternate and of rank $R_{n/2}$ this last integer is necessarily even*, a generalization of the well known result for two-dimensional manifolds.

The possibility of choosing the sets as above is based upon well known theorems on the reduction of matrices with integer terms.†

Fundamental sets of the type just described will be called *canonical.*

PART II. TRANSFORMATION OF MANIFOLDS

§1. PRODUCT COMPLEXES‡

47. Let E_p, E_q be two cells, A a point of E_p or of its boundary, B a similar point for E_q. Consider the set of couples A, B which by definition vary continuously if either A or B so varies. I say that the set is an E_n, $n = p + q$, plus its boundary. This follows at once from the fact that if $x_1, x_2, \cdots, x_n$ are cartesian coördinates for an S_n, E_p, E_q with their boundaries, then the sets in question are respectively homeomorphic to the following three sets:

$$0 \leqq x_i \leqq 1, \quad i \leqq p; \quad x_{p+j} = 0 ;$$
$$x_i = 0, i \leqq p; \quad 0 \leqq x_{p+j} \leqq 1 ;$$
$$0 \leqq x_i \leqq 1, \quad i = 1, 2, \cdots, n .$$

* These Transactions, vol. 25 (1923), p. 540. See in the same connection my Monograph already quoted, p. 13. In two recent notes of the Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 99–103, J. W. Alexander generalizing this notion has been led to a new set of topological invariants.

† See the expository paper by Veblen and Franklin in the Annals of Mathematics, ser. 2, vol. 23.

‡ The term and corresponding notation are due to E. Steinitz, Sitzungsberichte der Berliner mathematischen Gesellschaft, vol 7 (1908). See also in this connection H. Tietze, Abhandlungen des mathematischen Seminars zu Hamburg, vol. 2 (1923), p. 37, H. Künneth, Mathematische Annalen, vol. 90 (1923), p. 65, and my paper in these Transactions, vol. 22 (1921), p. 362.

48. Let Π_p, Π_q be polyhedra in spaces S_r, S_t, by means of which straightness and distances are defined for E_p and E_q and let $x_1, \cdots, x_r$ and $y_1, \cdots, y_t$ be cartesian coördinates for S_r and S_t. We may refer an S_{r+t} to the set of cartesian coördinates x_i, y_j. Then if a point (x) describes Π_p and a point (y) describes Π_q the corresponding point (x, y) of S_{r+t} describes a Π_n homeomorphic to E_n plus its boundary. The geometry of this is very elementary and we leave it to the reader. This Π_n shall be chosen as basis for straightness and distances on E_n and its boundary.

The *point* A, B of E_n or its boundary shall be designated henceforth by $A\times B$. Let $AA_1 \cdots A_p$, $BB_1 \cdots B_q$ be indicatrices of E_p and E_q. We shall agree to sense E_n by the indicatrix

$$A\times B \; A_1\times B \cdots A_p\times B \; A\times B_1 \cdots A\times B_q ,$$

and the cell so sensed shall be denoted by $E_p\times E_q$ and called *product* of the two cells, its *factors*.* The notation $A\times B$ merely corresponds to $p=q=0$. At once, we have

$$E_p\times E_q = -(-E_p)\times E_q = -E_p\times(-E_q) = (-1)^{pq} E_q\times E_p .$$

Let a^i_{p-1}, b^j_{q-1} be the boundary cells of E_p and E_q, so sensed that

$$E_p \equiv \sum a^i_{p-1}, \qquad E_q \equiv \sum b^i_{q-1} .$$

Then

$$E_p\times E_q \equiv \sum \epsilon_i a^i_{p-1}\times E_q + \sum \eta_j E_p\times b^j_{q-1} , \tag{48.2}$$

where ϵ_i and η_j are ± 1. To determine their actual value assume that $BB_1 \cdots B_{q-1}$ is on b^j_{q-1}. Then $(-1)^q \cdot BB_1 \cdots B_{q-1}$ is an indicatrix of b^j_{q-1} as boundary cell of E_q. That of $E_p\times b^j_{q-1}$ is

$$(-1)^q A\times B \; A_1\times B \cdots A_p\times B \; A\times B_1 \cdots A\times B_{q-1} .$$

Comparing with the indicatrix of $E_p\times E_q$ we find $\eta_j=(-1)^q$. If we interchange E_p and E_q we shall find by applying this very relation $(-1)^q \epsilon_i = (-1)^q$, hence $\epsilon_i=1$, so that finally (48.2) becomes

$$E_p\times E_q \equiv \sum a^i_{p-1}\times E_q + (-1)^p \sum E_p\times b^i_{q-1} , \tag{48.3}$$

which describes the boundary of the product.

49. Let now C_p, C_q be any complexes, a^i_p a generic p-cell of the first, b^j_q one of the second. We define their product by the relation

$$C_p\times C_q = \sum a^i_p \times b^j_q .$$

* Concerning this symbolic product see the footnote to No. 7.

By applying (48.3) to each term at the right we find

$$C_p\times C_q \equiv \sum a_{p-1}^i \times b_q^j + (-1)^p \sum a_p^i \times b_{q-1}^j \tag{49.1}$$

where only the boundary $(p-1)$- and $(q-1)$-cells appear at the right with the very orientation that they possess in the boundary congruences of the complexes. Hence if C_{p-1} and C_{q-1} are these boundaries,

$$C_p\times C_q \equiv C_{p-1}\times C_q + (-1)^p C_p \times C_{q-1}\ . \tag{49.2}$$

From (49.2) we have the following:

I. *The product is orientable* (in the sense that a manifold is) *if and only if each factor is.*

II. *The product of two manifolds is a manifold.*

III. *The product of two complexes is without boundary if, and only if, the complexes themselves have none.* Observe that all this can be extended to singular complexes, hence

IV. *The product of two cycles of the factors is a cycle of the product.*

From polyhedra of the factors we derive as in No. 48 one for the product, hence corresponding definitions of straightness and distances.

50. Every Γ_k of $C_p\times C_q$ is homologous to a polyhedral cycle Γ_k' (Coll. Lect., p. 120). Let α be a point of the cell $a_p^i\times b_q^j$ not on Γ_k'. Draw a rectilinear segment from α to every point β of Γ_k' on the cell, to its intersection at γ with the boundary. The set of all segments $\beta\gamma$ constitutes a polyhedral C_{k+1}. The effect of subtracting its boundary from Γ_k' is to reduce the latter to a similar cycle without any points on $a_p^i\times b_q^j$. On proceeding thus with all such cells, then with the C_{n-1} made up with the cells of less than n dimensions of $C_p\times C_q$, and so on, we shall reduce Γ_k' to a homologous polyhedral cycle whose points are all on cells of at most k dimensions of the complex. From the theorem of the Colloquium Lectures just recalled, as applied to the C_k made up of the cells of at most k dimensions of our complex, follows that the reduced cycle, which we still call Γ_k', is a sum of k-cells of $C_p\times C_q$. This may also be shown by remarking that if it has points on an E_k the whole cell must belong to it, else it would have a boundary on it.

We conclude then that *every cycle of a product is homologous to a sum of products of cells of its two factors.*

Suppose that such a Γ_k bounds on $C_p\times C_q$. We may apply to the complex that it bounds the identical reasoning, the operations made not affecting the boundary, and we conclude that *if a cycle, sum of products of cells of the factors, bounds on the product, it bounds a complex which is also such a sum.*

Henceforth we shall use the notation γ_k, δ_k for the k-cycles of C_p and C_q, keeping Γ_k itself for those of the product $C_p \times C_q$.

51. As shown by Veblen* every E_k of C_p is expressible as a sum of multiples of certain cycles γ_p^i and certain cells a_k^j. Of course no sum of these particular a's is a γ, hence every γ_k of C_p is a sum of multiples of the cycles γ_k^j alone. It is not ruled out that some of the latter bound, but their set may be so chosen as to include a *fundamental set* for the k-cycles. There exists of course a similar set δ_k^i, b_k^j for C_q.

52. THEOREM. *The k-cycles which are products of cycles taken from fundamental sets of the factors constitute a fundamental set for the product.*

According to No. 50, for any Γ_k we have

$$\Gamma_k \sim \sum \gamma_\mu^i \times \delta_{k-\mu}^j + \sum \gamma_\mu^i \times b_{k-\mu}^j + \sum a_\mu^i \times \delta_{k-\mu}^j + \sum a_\mu^i \times b_{k-\mu}^j .$$

We must express the fact that the right side has no boundary. The boundary of $a_\mu^i \times b_{k-\mu}^j$ is a sum of terms $\gamma_{\mu-1} \times b_{k-\mu}^j$, $a_\mu \times \delta_{k-\mu-1}$ where $\gamma_{\mu-1}$, $\delta_{k-\mu-1}$ are sums of cycles described in the preceding number. The boundaries of an $a \times \delta$ or of a $\gamma \times b$ are both of type $\gamma \times \delta$. Hence the boundaries of the terms in the fourth sum can only cancel each other, which compels the sum to be similar to the third. Hence for the cycle we have an homology

$$\Gamma_k \sim \sum \gamma_\mu^i \times \delta_{k-\mu}^j + \sum \gamma_\mu^i \times b_{k-\mu}^j + \sum a_\mu^i \times \delta_{k-\mu}^j .$$

We may assume that the second sum cannot be split into two, one of which is a cycle, for it would then be of the same type as the first and could be merged with it. Similarly for the third sum. From this follows that if the second sum is absent so is the third and conversely. Also since the boundaries of the terms in the last two sums must cancel each other, any term $\gamma_\mu^i \times b_{k-\mu}^j$ in the second sum contains only such a γ as may come from the boundaries of terms in the third. γ_μ^i bounds then a certain $C_{\mu+1}$ of C_p. Let

$$b_{k-\mu}^j \equiv \sum \epsilon_{js} \delta_{k-\mu-1}^s .$$

At once,

$$C_{\mu+1} \times b_{k-\mu}^j \equiv \gamma_\mu^i \times b_{k-\mu}^j + (-1)^{\mu+1} \sum \epsilon_{js} C_{\mu+1} \times \delta_{k-\mu-1}^s \sim 0 \qquad (\text{mod } C_p \times C_q).$$

On subtracting this bounding cycle from the second expression for Γ_k the term $\gamma_\mu^i \times b_{k-\mu}^j$ will have been replaced by a sum of terms that will go

---

* Coll. Lect., p. 116. The result in question, not stated explicitly by him, is really derived in the course of the discussion which leads to the theorem there stated.

into the first or the third sum. Therefore by a previous remark, the third must then also disappear. Hence

$$\Gamma_k \sim \sum \gamma_\mu^i \times \delta_{k-\mu}^i .$$

If γ_μ^i bounds $C_{\mu+1}$ of C_p, then $\gamma_\mu^i \times \delta_{k-\mu}^j$ bounds $C_{\mu+1} \times \delta_{k-\mu}^j$; hence in the expression of Γ_k only the terms for which neither γ nor δ bound need to be preserved, which proves our theorem.

It is not difficult to show that *if two factor cycles do not bound their product does not bound,* but as this is unnecessary for the sequel we omit the proof. A corollary is that *no cycle of the fundamental sets obtained bounds on $C_p \times C_q$.*

Of more import is the following observation. If γ_μ is a zero-divisor for C_p, $\gamma_\mu \times \delta_{k-\mu}$ is a zero-divisor or else bounds and similarly with δ in place of γ. For if $t\gamma_\mu$ bounds $C_{\mu+1}$, $(t\gamma_\mu) \times \delta_{k-\mu}$ bounds $C_{\mu+1} \times \delta_{k-\mu}$. Hence *the theorem that we have proved holds even when fundamental sets with respect to the operation $\approx$ take the place of the others.* For we may now eliminate from the fundamental sets of the product all cycles of which a factor is a zero-divisor.*

53. **Product of orientable manifolds.** We know already (No. 49) that if M_p, M_q are orientable manifolds so is $M_p \times M_q$. We are now especially concerned with the question of Kronecker indices.

Let E_h, E_{p-h} be simplicial cells on M_p and suppose that they intersect at A. Let E_k', E_{q-k}' and B be similar elements for M_q. Then $E_h \times E_k'$ and $E_{p-h} \times E_{q-k}'$ intersect at $A \times B$ on $M_p \times M_q$ and the question is *to determine their index in terms of $(E_h \cdot E_{p-h})$, and $(E_k' \cdot E_{q-k}')$.*

Let the A's and B's of No. 48 now serve to determine indicatrices for the manifolds, as previously for the cells. As a matter of fact, the M's might simply be E's without changing anything.

We may now assume that the indicatrices are

$$AA_1 \cdot \cdot \cdot A_h , \quad AA_{h+1} \cdot \cdot \cdot A_p \quad \text{for } E_h \text{ and } E_{p-h} ,$$

$$BB_1 \cdot \cdot \cdot B_k , \quad BB_{k+1} \cdot \cdot \cdot B_q \quad \text{for } E_k \text{ and } E_{q-k} .$$

Hence for the above named two-cell products the indicatrices

$$A \times B \; A_1 \times B \cdot \cdot \cdot A_h \times B \; A \times B_1 \cdot \cdot \cdot A \times B_k ,$$

$$A \times B \; A_{h+1} \times B \cdot \cdot \cdot A_p \times B \; A \times B_{k+1} \cdot \cdot \cdot A \times B_q .$$

* From this follow the results of my Transactions paper and those of Künneth, concerning connectivity and torsion indices.

By comparing with the indicatrix of $M_p \times M_q$ and applying the rule for the determination of the index we obtain the following formula:

$$(53.1)\qquad (E_h \times E'_k \cdot E_{p-h} \times E'_{q-k}) = (-1)^{k(p-h)} (E_h \cdot E_{p-h})(E'_k \cdot E'_{q-k}) .$$

It holds even for $k=0$, $h=p$, when it yields the following result, verifiable directly with ease:

$$(53.2)\qquad (E_p \times B \cdot A \times E_q) = +1 .$$

The product $E_p \times B$ simply denotes an E_p on the product cell, and similarly for $A \times E_q$.

Let now C_h, C_{p-h} be complexes on M_p whose points of intersection are neither on their boundaries nor on that of the manifold, and let C'_k, C'_{q-k} be analogous for M_q. Polyhedral approximations to C_h and C_k' have for product such an approximation to $C_h \times C_k'$, and similarly for the other two complexes. If the approximations to C_h and C_{p-h} and those to C'_k and C'_{q-k} intersect at isolated ordinary points of the complexes, the approximations to the products will behave likewise. Hence from the definition of the index of two polyhedral complexes with well defined intersections, the extension to arbitrary complexes, and (53.1), (53.2), we have

$$(53.3)\qquad (C_h \times C'_k \cdot C_{p-h} \times C'_{q-k}) = (-1)^{k(p-h)} (C_h \cdot C_{p-h})(C'_k \cdot C'_{q-k}) ;$$

$$(53.4)\qquad (M_p \times B \cdot A \times M_q) = +1 .$$

The products in (53.4) represent complexes homeomorphic to M_p or M_q on $M_p \times M_q$. The approximations to A and B in that case consist in choosing points of their vicinity situated on p- and q-cells of the covering complexes of the manifolds which serve to define straightness and distances on them.

Two complexes C_h, C_{p-h-i} on M_p, without points on its boundary, may always be approximated by two that do not intersect. Hence if C'_k, C'_{q-k+i} are on M_q, without points on its boundary, there are non-intersecting polyhedral approximations to $C_h \times C_k'$ and $C_{p-h-i} \times C'_{q-k+i}$. Therefore

$$(53.5)\qquad (C_h \times C'_k \cdot C_{p-h-i} \times C_{q-k+i}) = 0 .$$

54. THEOREM. *Let M_p, M_q be without boundary and let γ^i_k, δ^j_k be the cycles of their canonical fundamental sets. Then except for the signs their products constitute such a set for $M_p \times M_q$.*

This follows at once from the relations derived from (53.3), (53.4) (53.5):

$$(54.1)\quad (\overset{i}{\gamma}_\lambda \times \overset{j}{\delta}_{k-\lambda} \cdot \overset{i}{\gamma}_{p-\lambda} \times \overset{j}{\delta}_{q-k+\lambda}) = (-1)^{(k-\lambda)(p-\lambda)} \cdot (\overset{i}{\gamma}_\lambda \cdot \overset{i}{\gamma}_{p-\lambda})(\overset{j}{\delta}_{k-\lambda} \cdot \overset{j}{\delta}_{q-k+\lambda}) \;;$$

$$(54.2)\quad (\overset{i}{\gamma}_\lambda \times \overset{j}{\delta}_{k-\lambda} \cdot \overset{i}{\gamma}_{p-\mu} \times \overset{j}{\delta}_{q-k+\mu}) = 0 \,, \qquad \lambda \neq \mu \,.$$

They indicate also the manner in which the cycles are to be associated.

Of great importance for transformations is the special case $p=q=k=n$. We have then

$$(54.3)\quad (\overset{i}{\gamma}_\lambda \times \overset{j}{\delta}_{n-\lambda} \cdot \overset{i}{\gamma}_{n-\lambda} \times \overset{j}{\delta}_\lambda) = (-1)^{n(\lambda+1)} \cdot (\overset{i}{\gamma}_\lambda \cdot \overset{i}{\gamma}_{n-\lambda}) \cdot (\overset{j}{\delta}_\lambda \cdot \overset{j}{\delta}_{n-\lambda}) \,,$$

and all other indices vanish.

55. Formulas (53.3), (53.4) are special cases of a more general one corresponding to cycles that are intersections of complexes, which is derived by similar considerations. As we shall not use it later we merely give it here without proof. If C_l and C_λ of M_p do not intersect each other's boundary, and have no intersection on the boundary of the manifold, if also C'_m, C'_μ behave likewise relatively to M_q, the formula in question is as follows:

$$C_l \times C'_m \cdot C_\lambda \times C'_\mu \sim (-1)^{(p-l)(q-\mu)} \cdot C_l \cdot C_\lambda \times C'_m \cdot C'_\mu \qquad (\text{mod } M_p \times M_q) \,.$$

§2. TRANSFORMATIONS OF A MANIFOLD WITHOUT BOUNDARY

56. Let M_n be the manifold, M_n' another copy of it, T a transformation of M_n into itself or part of itself, which we subject to this sole condition of a very general nature: *If A is any point of M_n, B the image of any transform A' of it on M_n', the set of all points $A \times B$ is an n-cycle Γ_n on $M_n \times M_n'$.* The inclusiveness of the class of transformations so defined becomes apparent when we remark that all continuous one-valued transformations (for each A only one A' varying continuously with A) belong to it. The non-singular C_n without boundary of which Γ_n is then the image in the sense of No. 6 is M_n itself. More generally k-valued continuous transformations are also of our type; the corresponding C_n is then kM_n.

Much of the rest of this paper will center around the determination of certain Kronecker indices and it becomes essential to define all orientations involved. Let C_n cover M_n and let C_n' be its image covering M_n'. Suppose that Π_n is a polyhedron which associated with C_n serves to define straightness and distances on M_n. The polyhedron Π_n has the same cell structure as C_n' and we shall agree to use it, associated with C_n', to define straightness and distances on M_n'. Then any rectilinear segment of M_n has for image a similar one on M_n' and both have the same length. To an indicatrix E_n on M_n will correspond an E_n' on M_n'. We shall name the vertices of E_n' in the same order as the corresponding vertices of E_n, and use the simplex so sensed as indicatrix for M_n'. In accordance with our previous conventions the orientation of $M_n \times M_n'$ is now perfectly determined. Owing to its importance for the sequel, it is well perhaps to characterize it more geometrically. Through any point $A \times B$ of the manifold there pass $M_n \times B$ and $A \times M_n'$. Let B be the image of A. To E_n with A as its first vertex corresponds E_n' with B as its first vertex. Let $\bar{E}_n$ be the image of the first on $M_n \times B$, $\bar{E}_n'$ that of the second on $A \times M_n'$. The E_{2n} indicatrix of the product manifold has its vertices named in the following order: $A \times B$, the other vertices of $\bar{E}_n$, the other vertices of $\bar{E}_n'$.

There remains the orienting of Γ_n. If one cycle is suitable for T so is its opposite. Of the two we shall select the one such that

$$(\Gamma_n \cdot A \times M_n') = \alpha_0 \geqq 0 ,$$

a perfectly definite condition since the integer is a Kronecker index of cycles.

57. Let us apply to M_n and M_n' the very notations of No. 54, so that δ_k^i is now the cycle of M_n' that corresponds to γ_k^i on M_n. We shall then have

$$\Gamma_n \sim \sum \overset{ij}{\epsilon_\mu} \cdot \overset{i}{\gamma_\mu} \times \overset{j}{\delta}_{n-\mu} \qquad (\text{mod } M_n \times M_n') . \tag{57.1}$$

The γ's and δ's are, it will be recalled, cycles of fundamental sets. In particular γ_0, δ_0 are merely points of M_n, M_n' while γ_n, δ_n are the manifolds themselves. The number of cycles of the fundamental sets is *one* for these two extreme cases.

The ϵ's are important characteristic integers of T. It will be remembered that two transformations are said to be of the same *class* if they belong to one and the same continuous family of transformations. Whenever T varies within its class, Γ_n is continuously deformed (a condition that might serve to define the class) and Γ_n remains homologous to a fixed cycle, so that the ϵ's are unchanged. Hence *to a given class of transformations corresponds a fixed set of ϵ's.* Whether the converse holds or not is as yet unknown.

58. We must now introduce the notion of the *transform of a cycle* of M_n by T. The necessity of defining such a notion with precision arises from the fact that while T is a point transformation a cycle is not a mere point set, but consists of a set of cells each taken with a certain multiplicity. The cycle is then really a symbol attached to a set of cells, and what is meant by its transform is by no means evident a priori.

Let γ_μ, $0 \leqq \mu \leqq n$, be any cycle of M_n. If it is not polyhedral we approximate it by one that is and that furthermore has been reduced in accordance with the theorem of No. 23. Then $\gamma_\mu \times M_n'$ behaves likewise and we may also approximate Γ_n in the same manner so that the two have a well defined intersection $\Gamma_n \cdot \gamma_\mu \times M_n' = \Gamma_\mu$. (We economize in notations by designating the approximations like the cycles themselves.) To every point $A \times B$ of Γ_μ corresponds a unique B on M_n' varying continuously when $A \times B$ so varies on Γ_μ. Hence B gives rise to a cycle $\bar{\delta}_\mu$ on M_n'. It is a singular complex of M_n', continuous image of Γ_μ in the sense of No. 6. The image $\bar{\gamma}_\mu$ of $\bar{\delta}_\mu$ on M_n is by definition the transform of γ_μ by T. It will be observed that $\bar{\delta}_\mu$ is *polyhedral*, hence so is $\bar{\gamma}_\mu$. For $\bar{\delta}_\mu$ is represented on the copy $A \times M_n'$ of M_n' by the cycle $A \times \bar{\delta}_\mu$ which is the locus of all intersections of $A \times M_n'$ with all manifolds $M_n \times B$ that meet $\Gamma_n \cdot \gamma_\mu \times M_n'$. In this fashion, however, $A \times \bar{\delta}_\mu$ appears as an intersection of polyhedral complexes, and it is therefore also polyhedral. If $A \times B\ A_1 \times B_1 \cdots A_\mu \times B_\mu$ is an indicatrix of Γ_μ then $BB_1 \cdots B_\mu$ is an indicatrix of $\bar{\delta}_\mu$.

The preceding definition is justified by the fact that for $\mu = 0$, when γ_0 consists of a finite number of points, the correct transforms of these points are obtained. Furthermore (No. 62) when T is the identity all cycles are left invariant, which is as it should be with any properly selected definition.

Let the cycle transformations be represented by homologies

$$\bar{\gamma}_\mu^{i} \sim \sum \alpha_\mu^{ij} \gamma_\mu^{j} \qquad (\text{mod } M_n) . \tag{58.1}$$

For any given T one is far more likely to possess information concerning the α's than concerning the ϵ's. Hence the interesting and important problem arises *to determine the ϵ's in terms of the α's*. This problem shall be solved partly in a more narrow form. Let us drop zero divisors throughout. In accordance with No. 52, with the R's denoting the connectivity indices of M_n, and assuming that among the cycles γ_μ^{i} the first R_μ are independent, we shall now have in place of (57.1) and (58.1), the following relations whose appearance is the same:

$$\Gamma_h \approx \sum \epsilon_\mu^{ij} \cdot \gamma_\mu^{i} \times \delta_{n-\mu}^{j} \qquad (\text{mod } M_n \times M_n') ; \tag{58.2}$$

$$\bar{\gamma}_\mu^{i} \approx \sum \alpha_\mu^{ij} \gamma_\mu^{j} \qquad (\text{mod } M_n) , \tag{58.3}$$

where as before μ runs from 0 to n, but, for each μ, i and j run from 1 to $R_\mu = R_{n-\mu}$. The problem that we shall completely solve is *the determination of the ϵ's within this range in terms of the α's similarly limited*. This will suffice to provide all that is needed in our applications to fixed points and coincidences.

59. **An important formula.** The solution of our problem is based upon the following formula, whose proof will now occupy us:

$$(59.1)\qquad (\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu}) = (-1)^\mu \cdot (\bar{\gamma}_\mu \cdot \gamma_{n-\mu}),$$

with indices computed at the left as to $M_n \times M_n'$ and at the right as to M_n. In place of (59.1) it will be found more convenient to prove the equivalent

$$(59.2)\qquad (\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu}) = (-1)^\mu \cdot (\bar{\delta}_\mu \cdot \delta_{n-\mu}),$$

where the last index is computed as to M_n', and to this we now turn our attention.

60. By means of our usual approximations as applied to Γ_n, γ_μ and $\delta_{n-\mu}$, we may so arrange matters that all cycles are polyhedral and have well defined intersections. Let $A \times B$ be an intersection of Γ_n with $\gamma_\mu \times \delta_{n-\mu}$. Then B is an intersection of $\bar{\delta}_\mu$ with $\delta_{n-\mu}$. Therefore we need only to show that the contributions of the two points to their respective indices have the ratio $(-1)^\mu$. Owing to the distributive law we may assume that all complexes are cells and that the points count for ± 1 in the indices, or what is the same, that the manifolds are linear spaces and T a projectivity. The method of matrices (No. 4) will be found most convenient for our purpose.

61. We begin with *the contribution of* $A \times B$ *to the left side of* (59.2). Let A, B, and $A \times B$ be origins of cartesian axes for the spaces M_n, M_n', and their product, the coördinates being $x_1, \cdots, x_n$ for M_n, $y_1, \cdots, y_n$ for M_n', $x_1, \cdots, x_n, y_1, \cdots, y_n$ for $M_n \times M_n'$. To the points (x), (y) of M_n, M_n' corresponds the point (x, y) of the product.

We can assume for M_n a matrix-indicatrix in the sense of No. 4, of type

$$(61.1)\qquad \left\| \begin{matrix} X_\mu, & 0 \\ 0, & X_{n-\mu} \end{matrix} \right\|$$

where the X's denote square matrices of order equal to the index with determinants equal to $+1$, and the zeros matrices whose terms all vanish. The meaning of (61.1) is clear. If A_i is the point of M_n whose coördinates are the terms in the ith row, then $AA_1 \cdots A_n$ is an indicatrix of M_n. We may so select it that $AA_1 \cdots A_\mu$ and $AA_{\mu+1} \cdots A_n$ are indicatrices of γ_μ and $\gamma_{n-\mu}$. This implies a definite orientation for them, but if we invert γ_μ or $\gamma_{n-\mu}$ we also invert $\bar{\delta}_\mu$ or $\delta_{n-\mu}$ and therefore (59.2) is unaffected, so that there is really no restriction. In the same sense as for M_n, we may say that the matrices

$$\|X_\mu, 0\|,\quad \|0, X_{n-\mu}\|$$

define indicatrices of γ_μ and $\gamma_{n-\mu}$.

To the images of the transforms of $M_n, \gamma_\mu, \gamma_{n-\mu}$, by T, on M_n', correspond matrix-indicatrices

$$\left\| \begin{matrix} Y_\mu \, , & Y \\ Y' \, , & Y_{n-\mu} \end{matrix} \right\| , \quad \| \, Y_\mu \, , \, Y \, \| , \quad \| \, Y' \, , \, Y_{n-\mu} \, \|$$

where Y, Y' are rectangular arrays and the two other Y's square arrays of order equal to their index. The first corresponds to $\pm M_n' = \epsilon M_n'$, where the sign is plus if T maintains the indicatrix on M_n, minus if it inverts it. The second and third matrices correspond to $\bar{\delta}_\mu$ and $\delta_{n-\mu}$. Since (61.1) defines an indicatrix of M_n' when each row is considered as the coördinates of a y point, and since its determinant, product of $|X_\mu|$ and $|X_{n-\mu}|$, is $+1$, ϵ has the sign of the determinant

$$\left| \begin{matrix} Y_\mu & , & Y \\ Y' & , & Y_{n-\mu} \end{matrix} \right| .$$

From the definition of Γ_n we conclude that

$$\left\| \begin{matrix} X_\mu \, , & 0 & , & Y_\mu & , & Y \\ 0 \, , & X_{n-\mu} & , & Y' & , & Y_{n-\mu} \end{matrix} \right\|$$

in a matrix-indicatrix of $\theta\Gamma_n$, $\theta = \pm 1$. To determine the sign of θ observe that since $A \times M_n'$ has for matrix-indicatrix

$$\left\| \begin{matrix} 0 & , & 0 & , & X_\mu & , & 0 \\ 0 & , & 0 & , & 0 & , & X_{n-\mu} \end{matrix} \right\| ,$$

the contribution of $A \times B$ to $(\theta\Gamma_n \cdot A \times M_n')$ has the sign of the determinant

$$\left| \begin{matrix} X_\mu & , & 0 & , & 0 & , & 0 \\ 0 & , & X_{n-\mu} & , & 0 & , & 0 \\ 0 & , & 0 & , & X_\mu & , & 0 \\ 0 & , & 0 & , & 0 & , & X_{n-\mu} \end{matrix} \right| = +1 \, .$$

Hence θ represents the contribution of the point to the integer $\alpha_0 = (\Gamma_n \cdot A \times M_n')$ of No. 56. For example, in the case of the identical transformation, every point of intersection of the two cycles must contribute $+1$, therefore $\theta = +1$. This is also true for a continuous 1 to n transformation, where the contributions to the index have a constant sign. That θ is not -1 in these two cases comes from the condition $\alpha_0 > 0$ by which we have definitely oriented Γ_n.

But this is a digression from our main topic to which we now return. It is readily seen that

$$\left\| \begin{matrix} X_\mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & X_{n-\mu} \end{matrix} \right\|$$

is a matrix-indicatrix for $\gamma_\mu \times \delta_{n-\mu}$. As $M_n \times M_n'$ has for matrix-indicatrix

$$\left\| \begin{matrix} X_\mu, & 0, & 0, & 0 \\ 0, & X_{n-\mu}, & 0, & 0 \\ 0, & 0, & X_\mu, & 0 \\ 0, & 0, & 0, & X_{n-\mu} \end{matrix} \right\|$$

whose determinant is $+1$, we conclude from the definition of the Kronecker index that the contribution of $A \times B$ to $(\theta\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu})$ has the sign of the determinant

$$\left| \begin{matrix} X_\mu, & 0, & Y_\mu, & Y \\ 0, & X_{n-\mu}, & Y', & Y_{n-\mu} \\ X_\mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & X_{n-\mu} \end{matrix} \right| = (-1)^\mu |Y_\mu| .$$

The contribution of $A \times B$ to $(\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu})$ has then the sign of $(-1)^\mu \theta |Y_\mu|$.

62. Let us now examine *the contribution of* B *to* $(\bar{\delta}_\mu \cdot \delta_{n-\mu})$. Suppose that we have found for $\Gamma_n \cdot \gamma_\mu \times M_n'$ a matrix-indicatrix

$$\| Z, Z' \|,$$

where Z, Z' are matrices with μ rows and n columns. Then Z' will define an indicatrix for $\bar{\delta}_\mu$, referred of course to the y coördinates. For any row $x_1, \cdots, x_n, y_1, \cdots, y_n$ represents a point $A' \times B'$ such that B' is the point of M_n' with the y coördinates. From this and the remark, No. 58, as to relations between indicatrices on Γ_n and $\bar{\delta}_\mu$, follows the property of Z'.

Now $\gamma_\mu \times (\epsilon M_n')$ has for matrix-indicatrix

$$\left\| \begin{matrix} X_\mu, & 0, & 0, & 0 \\ 0, & 0, & Y_\mu, & Y \\ 0, & 0, & Y', & Y_{n-\mu} \end{matrix} \right\|$$

which we may replace by

$$\left\| \begin{matrix} X_\mu, & 0, & Y_\mu, & Y \\ 0, & 0, & Y_\mu, & Y \\ 0, & 0, & Y', & Y_{n-\mu} \end{matrix} \right\|,$$

derived from it by addition of rows, as this change of indicatrix merely corresponds to an affine transformation of determinant $+1$ applied to the $S_{\mu+n}$ involved, and is therefore permissible. The first row defines now an indicatrix of $\zeta\theta\Gamma_n \cdot \gamma_\mu \times (\epsilon M_n')$, where $\zeta = \pm 1$. From our mode of defining sensed intersections (No. 12) it follows that ζ has the same sign as the determinant

$$\begin{vmatrix} X_\mu & , & 0 & , & Y_\mu & , & Y \\ 0 & , & X_{n-\mu} & , & Y' & , & Y_{n-\mu} \\ 0 & , & 0 & , & Y_\mu & , & Y \\ 0 & , & 0 & , & Y' & , & Y_{n-\mu} \end{vmatrix} = \begin{vmatrix} Y_\mu & , & Y \\ Y' & , & Y_{n-\mu} \end{vmatrix},$$

that is, the sign of ϵ. As both are unity in absolute value, $\zeta = \epsilon$, so that actually the matrix

$$\| X_\mu \quad , \quad 0 \quad , \quad Y_\mu \quad , \quad Y \|$$

defines an indicatrix of $\epsilon\,\theta\Gamma_n \cdot \gamma_\mu \times (\epsilon M_n') = \theta\Gamma_n \cdot \gamma_\mu \times M_n'$. According to what has been said at the beginning of this number, then

$$\| Y_\mu \quad , \quad Y \|$$

defines an indicatrix for $\theta\bar{\delta}_\mu$.

Since M_n' has a matrix-indicatrix of determinant $+1$, the contribution of B to $(\theta\bar{\delta}_\mu \cdot \delta_{n-\mu})$ has the sign of the determinant

$$\begin{vmatrix} Y_\mu & , & Y \\ 0 & , & X_{n-\mu} \end{vmatrix},$$

since the last line is a matrix-indicatrix for $\delta_{n-\mu}$. As the determinant has for value $|Y_\mu|$, the contribution of B to $(\bar{\delta}_\mu \cdot \delta_{n-\mu})$ has the same sign as $\theta|Y_\mu|$. Its ratio to that of $(\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu})$ is then $(-1)^\mu$, which suffices to prove (59.1).

Remark. One incidental result follows readily from our discussion. The indicatrix of $\bar{\delta}_\mu$ coincides with the image of the transform of that of γ_μ or with its opposite according as θ is positive or negative. Upon translating this back to M_n itself, we find the following result. *If* $A \times B$ *contributes* $+1$ *to* $(\Gamma_n \cdot A \times M_n')$ *then the indicatrix of the transform of any cycle through* A *is the transform of the indicatrix; it is the opposite of it if the contribution is* -1. For example in the case of a continuous sense preserving 1 to n transformation, the indicatrix of the transform is always the transform of the indicatrix. In particular for the identical transformation the two coincide, and $\bar{\gamma}_\mu$ not only coincides with γ_μ element for element, but also with preservation of orientation. This justifies in a sense our convention as to sensing Γ_n and our definition of the transform of a cycle (No. 58).

63. It is important to emphasize the fact that the extreme values $\mu=0$, n are not exceptional at all. For $\mu=0$, we have already imposed

$$(\Gamma_n \cdot A \times M_n') = \alpha_0 \ . \tag{63.1}$$

The interpretation given for this integer in No. 60 fits in perfectly with our discussion. Let α_n be the similar integer for T^{-1}. Its interpretation is then this:* An arbitrary E_n of M_n is covered with a certain number of cells of $T \cdot M_n$. Among these there will be, say, k' positive cells, k'' negative cells of M_n and $\alpha_n = k' - k''$. Here of course α_n may well be negative. Thus if T is a homeomorphism inverting the indicatrix, $\alpha_n = -1$. An example of this is the symmetry of a sphere with respect to a diametral plane. It is important to remember that the T considered is not the initial transformation but one that corresponds to a polyhedral approximation of its Γ_n.

If we compute the index as to $M_n' \times M_n$ we obtain then

$$(\Gamma_n \cdot B \times M_n) = (\Gamma_n \cdot M_n \times B) = \alpha_n \ .$$

If a manifold is inverted all corresponding indices must be changed in sign, an immediate corollary of their definition. As $M_n \times M_n' = (-1)^n M_n' \times M_n$, when the index is computed with reference to $M_n \times M_n'$,

$$(\Gamma_n \cdot M_n \times B) = (-1)^n \alpha_n \ . \tag{63.2}$$

This is what (59.1) becomes for $\mu = n$. Both (63.1) and (63.2) can also be derived with ease by means of matrix-indicatrices. Indeed this would merely involve repeating the discussion of Nos. 60, 61.

Observe that the notation α_0, α_n is in accord with the meaning given to a^{ij}_μ in No. 57. They correspond to the homologies describing the behavior of the zero-cycle (a point) and the n-cycle (M_n itself). Explicitly for $\mu=0$, n, i and j can only be unity, and $a_0^{11} = \alpha_0$, $a_n^{11} = \alpha_n$.

64. We now pass to the actual determination of the ϵ's. If we remember that in calculating Kronecker indices zero-divisors may be dropped, we see that in applying (59.1) we may substitute from formulas (58.2) and (58.3) in place of (57.1) and (58.1). Let us then substitute in (59.1) for Γ_n its expression as given by (58.2), for γ_μ, $\delta_{n-\mu}$ the cycles γ^h_μ, $\delta^k_{n-\mu}$, a nd for $\bar{\gamma}^h_\mu$ its expression (58.3). We obtain

$$\sum_{i,j=1}^{R_\mu} \epsilon^{ij}_{n-\mu}(\gamma^i_{n-\mu} \times \delta^j_\mu \cdot \gamma^h_\mu \times \delta^k_{n-\mu}) = (-1)^\mu \sum_{j=1}^{R_\mu} a^{hj}_\mu (\gamma^j_\mu \cdot \gamma^k_{n-\mu}) \ ,$$

* This is the same as Brouwer's *degree* of T. See Mathematische Annalen, vol. 71 (1911), p. 105. He limits himself to the case where every point has a unique transform, which is a special case of transformations with $\alpha_0 = 1$.

all other terms having dropped out in accordance with (54.1). Transforming the left side by means of (53.3) and (53.4) this becomes

$$(-1)^{\mu(n+1)} \sum_{i,j=1}^{R_\mu} \overset{ij}{\epsilon}_{n-\mu}(\overset{h}{\gamma}_\mu \cdot \overset{j}{\gamma}_{n-\mu})(\overset{j}{\gamma}_\mu \cdot \overset{k}{\gamma}_{n-\mu}) = \sum_{j=1}^{R_\mu} \overset{hj}{\alpha}_\mu (\overset{j}{\gamma}_\mu \cdot \overset{k}{\gamma}_{n-\mu}).$$

For each μ we have here a set of R_μ^2 linear equations in the B_μ unknowns $\overset{ij}{\epsilon}_{n-\mu}$. The determinant of their coefficients is a power of the determinant

$$|(\overset{i}{\gamma}_\mu \cdot \overset{j}{\gamma}_{n-\mu})|,$$

whose value is actually ± 1, as may be deduced from Veblen's theorem mentioned in Part I, §6. These equations may therefore be solved even explicitly if need be, a task which presents little interest. However, if, *as we shall assume from now on, the fundamental sets on M_n are canonical the solution can be carried through in an instant.*

For if we substitute this time the indices as given in Part I, §6, and as they must be applied here, we have at once

(*a*) $\mu \neq \frac{1}{2}n$ *or* $= \frac{1}{2}n =$ *an even integer.*

Then

$$\overset{hk}{\epsilon}_{n-\mu} = \overset{hk}{\alpha}_\mu, \qquad \mu > \tfrac{1}{2}n;$$

$$\overset{hk}{\epsilon}_{n-\mu} = (-1)^{\mu(n+1)} \overset{hk}{\alpha}_\mu, \qquad \mu \leqq \tfrac{1}{2}n.$$

(*b*) $\mu = \frac{1}{2}n =$ *an odd integer.*

Then

$$\overset{2h-1,k}{\epsilon}_{n/2} = -\overset{2h,k}{\alpha}_{n/2}, \qquad \overset{2h,k}{\epsilon}_{n/2} = \overset{2h-1,k}{\alpha}_{n/2}.$$

We obtain for Γ_n the following relations:

(*a*) $n \not\equiv 4$, mod 2.

$$\Gamma_n \approx \sum_{0 \leqq \mu < n/2} \sum_{i,j=1}^{R_\mu} \overset{ij}{\alpha}_{n-\mu} \cdot \overset{i}{\gamma}_\mu \times \overset{j}{\delta}_{n-\mu} + \sum_{\mu \geqq n/2} (-1)^{(n+1)\mu} \sum_{i,j=1}^{R_\mu} \overset{ij}{\alpha}_{n-\mu} \cdot \overset{i}{\gamma}_\mu \times \overset{j}{\delta}_{n-\mu}.$$

(*b*) $n \equiv 4$, mod 2.

$$\Gamma_n \approx \sum_{0 \leqq \mu < n/2} \sum_{i,j=1}^{R_\mu} (\overset{ij}{\alpha}_{n-\mu} \cdot \overset{i}{\gamma}_\mu \times \overset{j}{\delta}_{n-\mu} + (-1)^\mu \overset{ij}{\alpha}_\mu \cdot \overset{i}{\gamma}_{n-\mu} \times \overset{j}{\delta}_\mu) + \sum_{h=1}^{\frac{1}{2}R_{n/2}} \sum_{k=1}^{R_{n/2}} (\overset{2h-1}{\alpha}_{n/2} \cdot \overset{2h}{\gamma}_{n/2} \times \overset{k}{\delta}_{n/2} - \overset{2h,k}{\alpha}_{n/2} \cdot \overset{2h-1}{\gamma}_{n/2} \times \overset{k}{\delta}_{n/2}).$$

65. **A new proof of Veblen's theorem on fundamental sets.** We refer to the theorem of Part I, §6, according to which the invariant factors of the matrix

$$\| (\gamma_\mu^i \cdot \gamma_{n-\mu}^j) \|$$

for any two fundamental sets are all unity. Let them be in any case $e_1, e_2, \cdots, e_{R_\mu}$. That their number is R_μ follows from other considerations that need not detain us.* The reduction to canonical sets will be accomplished as before, only now

$$(\gamma_\mu^i \cdot \gamma_{n-\mu}^i) = \pm e_i ,$$

the same indices being zero as previously. Therefore we have now

$$\epsilon_{n-\mu}^{hk} \cdot e_h e_k = \pm a_\mu^{hk} \cdot e_k .$$

Hence e_h is a factor of a_μ^{hk}, and in particular of a_μ^{hh} *for every* T. But for the identical transformation $a_\mu^{hh} = 1$, hence $e_h = 1$ which is precisely Veblen's theorem.

§3. Coincidences and fixed points of transformations of a manifold

66. A *coincidence* of two transformations T, T' is a pair of points A, A', of M_n, such that A' is a transform of A, by both T and T'. A *fixed point* of T is a coincidence for T and the identical transformation. Let Γ_n, Γ_n' be the cycles corresponding to T and T', and let B be the image of A' on M_n'; $A \times B$ is an intersection of Γ_n with Γ_n' and conversely to such an intersection corresponds a coincidence of T and T'. The determination of the *number of coincidences and fixed points* is then reduced to a question of intersections of cycles. The actual numbers are not definite, may even vary for transformations of the same class, become infinite, and so on. Not so, however, with the attached Kronecker index $(\Gamma_n \cdot \Gamma_n')$, whose determination alone is usually possible. Its interpretation is simple enough. If we consider suitable approximations to T and T' there will be only a finite number of intersections. Some of these, say k', shall be counted positively, others, say k'', counted negatively according to a definite rule, and the difference $k' - k''$ is independent of the mode of approximation, in a sense sufficiently clear in the light of Part I, §5. From what has been established there, it follows that *the number of coincidences to be obtained in the sequence is an actual topological invariant of the transformations involved*, the same being

* See my Monograph, p. 13; also Veblen's paper on this question quoted in Part I, §6.

true for *the number of fixed points* counted in an analogous manner to the coincidences.

67. We shall now endeavor to characterize topologically the coincidences according to the signs of their contributions to $(\Gamma_n \cdot \Gamma_n')$. We select two polyhedral approximations that intersect in a finite number of points only, in such a way that if $A\times B$ is one of them it has neighborhoods on the cycles and on $M_n \times M_n'$ that are interiors of simplexes. We continue to call the approximations Γ_n, Γ_n'. Each intersection $A\times B$ counts for ± 1 in computing the index, and as far as its neighborhood alone and those of A and B on M_n and M_n' are concerned, everything is as if all complexes involved were linear spaces. The matrix-indicatrix method will then be again the most convenient.

To Γ_n and Γ_n' we may make correspond matrices

$$\| X_n \,,\, Y_n \| \,,\, \| X_n' \,,\, Y_n' \| \,.$$

Regarding X_n and X_n' we may replace them by any others whose determinants have the same signs. This is an immediate corollary of the definition of the indicatrix. Let us assume our coördinate system such that the unit matrix of order n, I_n, corresponds to an indicatrix for M_n as referred to the x coördinates. Then I_n will play the same part for M_n' and the y coördinates, I_{2n} for $M_n \times M_n'$ and the coördinates x, y. Under the circumstances, as we have shown in No. 60, if θ, θ' denote the contributions of $A\times B$ to the indices $(\Gamma_n \cdot A\times M_n')$, $(\Gamma_n' \cdot A\times M_n')$, then the determinants in question have the sign of θ and θ'. We conclude that there will be suitable indicatrices for $\theta\Gamma_n$ and $\theta'\Gamma_n'$ of type

$$\| I_n \,,\, Y_n \| \quad , \quad \| I_n \,,\, Y_n' \| \,.$$

Now the contribution of $A\times B$ to $(\theta\Gamma_n \cdot \theta'\Gamma_n')$, in absolute value equal to unity, has the sign of

$$\begin{vmatrix} I_n \,,\, Y_n \\ I_n \,,\, Y_n' \end{vmatrix} ,$$

since I_{2n} is a matrix-indicatrix for $M_n \times M_n'$. This determinant is equal to $|Y_n' - Y_n|$, and therefore the contribution of $A\times B$ to $(\Gamma_n \cdot \Gamma_n')$ has the sign of $\theta\theta' |Y_n' - Y_n|$. The determinant factor is certainly $\neq 0$, else $A\times B$ would not be an isolated intersection of the Γ's. Furthermore owing to the degree of arbitrariness in the choice of these cycles we may always assume that the equation of the nth degree

$$f(t) = | Y_n' - tY_n | = 0$$

(characteristic equation of $Y_n' - tY_n$) has only distinct roots. We know already that $f(1) \neq 0$.

Let $e = \pm 1$ be the same integer as in No. 60 ($+1$ if T maintains an indicatrix of M_n at A, -1 if it inverts it). Then as shown there

$$\epsilon \mid Y_n \mid > 0 .$$

Since the determinant Y_n is the leading coefficient of $f(t)$, $f(1)$ has the sign of $(-1)^\nu \epsilon$, where ν is the number of real roots of $f(t)$ that are less than unity. Therefore the contribution of $A \times B$ to $(\Gamma_n \cdot \Gamma_n')$ *is equal to* $(-1)^\nu \epsilon \theta \theta'$.

68. It is decidedly desirable and worth while to find a geometric interpretation for ν. The transformation T acts as an affine transformation whereby the vector V whose x components are $\xi_1, \xi_2, \cdots, \xi_n$ is changed into a vector whose components are defined by the matrix product.

$$\| \xi_1 , \xi_2 , \cdots , \xi_n \| \cdot Y_n ,$$

and similarly for T' and Y_n'. Let t_i be a real root of $f(t)$. Then there is a V_i whose transforms by T and T' are collinear and in the ratio $1 : t_i$. Let $V_1, V_2, \cdots, V_\nu$ be the similar vectors for all real roots <1. The vectors

$$u_1 V_2 + \cdots + u_\nu V_\nu ; \qquad 0 \leqq u_i \leqq 1 , \sum u_i = 1$$

fill up a simplex Σ_ν^0 of M_n with a vertex at A. Its transforms by T and T' are simplexes Σ_ν, Σ_ν' with the common vertex A', the second being entirely on the first, with all cells through A' situated in the same linear spaces, and ν is the largest integer for which such simplexes exist. This is the desired interpretation for ν.

69. We now proceed with the determination of $(\Gamma_n \cdot \Gamma_n')$.

Referring to No. 54, and recalling the special relations for canonical sets, we find that the only indices $(\gamma_\lambda^i \times \bar{\delta}_{n-\lambda}^j \cdot \gamma_\mu^h \times \delta_{n-\mu}^k) \neq 0$ are those for which $\lambda + \mu = n$ and whose values are given below.

(*a*) $\mu \neq \frac{1}{2} n$ *or else* $= \frac{1}{2} n$ *an even integer.*

$$\begin{aligned} (\gamma_\mu^i \times \delta_{n-\mu}^j \cdot \gamma_{n-\mu}^i \times \delta_\mu^j) &= (-1)^{n(\mu+1)} (\gamma_\mu^i \cdot \gamma_{n-\mu}^i)(\gamma_\mu^j \cdot \gamma_{n-\mu}^j) \\ &= (-1)^{n(\mu+1)}; \quad 1 \leqq i , j \leqq R_\mu , \end{aligned}$$

for with canonical systems, the two indices dropped are equal, their common value being ± 1.

(*b*) $\mu = \frac{1}{2} n$ *an odd integer*. For entirely similar reasons the indices below alone are left, with values as given:

$$(\gamma^{2h-1} \times \delta^{2k-1} \cdot \gamma^{2h} \times \delta^{2k}) = (\gamma^{2h} \times \delta^{2k} \cdot \gamma^{2h-1} \times \delta^{2k-1}) = -1 ,$$

$$(\gamma^{2h-1} \times \delta^{2k} \cdot \gamma^{2h} \times \delta^{2k-1}) = (\gamma^{2h} \times \delta^{2k-1} \cdot \gamma^{2h-1} \times \delta^{2k}) = +1 .$$

To facilitate the reading of these formulas we have omitted the lower indices whose common value is $\frac{1}{2}n$.

70. The application to $(\Gamma_n \cdot \Gamma_n')$ is immediate. Let, throughout, the η's and β's play the same part for Γ_n' or T' as the ϵ's and α's for Γ_n or T. The distributive law for indices gives

$$(\Gamma_n \cdot \Gamma_n') = \sum_{i,j,\mu} \overset{ij}{\epsilon}_{n-\mu} \overset{hk}{\eta}_\mu \ (\overset{i}{\gamma}_\mu \sim \overset{i}{\delta}_{n-\mu} \cdot \overset{h}{\gamma}_{n-\mu} \times \overset{k}{\delta}_\mu) \ .$$

Therefore for $n \not\equiv 2$, mod 4, and with μ replaced by $n-\mu$,

$$(70.1) \qquad (\Gamma_n \cdot \Gamma_n') = \sum_{\mu=0}^{n} (-1)^{n\mu} \sum_{i,j=1}^{R\mu} \overset{ij}{\epsilon}_{n-\mu} \overset{ij}{\eta}_\mu \ ,$$

while for $n \equiv 2$, mod 4, after some simplifications,

(70.2) $(\Gamma_n \cdot \Gamma_n') =$ (same sum as in (70.1) except that μ does not take the value $\frac{1}{2}n$)

$$+ \sum_{h,k=1}^{\frac{1}{2}R_{n/2}} \Big(\overset{2h-1,2k}{\epsilon_{n/2}} \overset{2h,2k-1}{\eta_{n/2}} + \overset{2h,2k-1}{\epsilon_{n/2}} \overset{2h-1,2k}{\eta_{n/2}} - \overset{2h-1,2k-1}{\epsilon_{n/2}} \overset{2h,2k}{\eta_{n/2}} - \overset{2h,2k}{\epsilon_{n/2}} \overset{2h-1,2h-1}{\eta_{n/2}} \Big) \ .$$

In terms of the α's and β's, by means of No. 64 we find, if $n \not\equiv 2$, mod 4,

$$(70.3) \qquad (\Gamma_n \cdot \Gamma_n') = \sum_{\mu=0}^{n} (-1)^{\mu} \sum_{i,j=1}^{R\mu} \overset{ij}{\alpha}_\mu \overset{ij}{\beta}_{n-\mu} \ ,$$

and if $n \equiv 2$, mod 4,

(70.4) $(\Gamma_n \cdot \Gamma_n') =$ (same sum as in (70.3) except that μ does not take the value $\frac{1}{2}n$)

$$+ \sum_{h,k=1}^{\frac{1}{2}R_{n/2}} \Big(\overset{2h-1,2k}{\alpha_{n/2}} \overset{2h,2k-1}{\beta_{n/2}} + \overset{2h,2k-1}{\alpha_{n/2}} \overset{2h-1,2k}{\beta_{n/2}} - \overset{2h-1,2k-1}{\alpha_{n/2}} \overset{2h,2k}{\beta_{n/2}} - \overset{2h,2k}{\alpha_{n/2}} \overset{2h-1,2k-1}{\beta_{n/2}} \Big) \ .$$

71. To obtain the number of invariant points of T, always counted with a certain sign, it is convenient to replace throughout T by the identical transformation, whose cycle we denote by Γ_n^0, and T' by T. The homologies for the identical transformation are (No. 62)

$$\overset{h}{\bar{\gamma}}_\mu \sim \overset{h}{\gamma}_\mu \ ,$$

hence $\alpha_\mu^{ii}=1$, $\alpha_\mu^{ij}=0$ if $i\neq j$. On replacing the α's by these values and the β's by the α's both (70.3) and (70.4) reduce to the very simple formula

$$(\Gamma_n^0 \cdot \Gamma_n) = \sum_{\mu=0}^{n} (-1)^\mu \sum_{i=1}^{R_\mu} \alpha_\mu^{ij} . \tag{71.1}$$

Remarkably enough *this formula is still correct if the fundamental sets cease to be canonical.* For if $\gamma_\mu'^i$ is a new fundamental set, we have

$$\gamma_\mu'^i \approx \sum_j A_{ij}\gamma_\mu^j \qquad (\text{mod } M_n) ,$$

where the A's are integers whose determinant

$$|A_{ij}| = \pm 1 .$$

Hence the transformation matrix of the γ''s that takes the place of

$$||\alpha_\mu^{ij}||$$

is obtained by transforming the latter by means of the matrix of the A's, an operation which leaves unchanged the sum of the terms in the principal diagonal, which is the term corresponding to μ in (71.1).

A particularly noteworthy case is when the effect of T on any cycle is merely to increase it by a zero-divisor, which includes as a special case deformations. Then *all* the α's in (71.1) are equal to $+1$ and

$$(\Gamma_n \cdot \Gamma_n) = \sum_{\mu=0}^{n}(-1)^\mu R_\mu .$$

This expression is the well known Euler-Poincaré *characteristic number of* M_n (difference between the number of cells of even dimensionality of any covering C_n and the number of the rest). Since $R_\mu = R_{n-\mu}$ its value is zero for n odd, hence this very neat proposition : *for a T of the preceding type, in particular of the same class as the identity, the number of invariant points counted with their signs is equal to the Euler-Poincaré characteristic. It is zero when n is odd.*

Thus for $n=2$ the number is $2-2p$ as found by Birkhoff, who however confined himself to analytic transformations.

Fixed points for transformations of hyperspheres have proved of importance in many questions. There are then no cycles of dimensions other than 0 or n so that (71.1) becomes

$$(\Gamma_n^0 \cdot \Gamma_n) = \alpha_0 + (-1)^n \alpha_n ,$$

a result obtained by Brouwer for $\alpha_0 = 1$.