A "theorem of Lie–Kolchin" for trees [102]

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Introduction

It is known that the automorphism group of a (sufficiently homogeneous) tree T behaves in some respects as SL_2 over a field with a non-archimedean valuation. In that analogy, the end space of T plays the role of the projective line. The Lie–Kolchin theorem [2, 7], as generalized à la Rosenlicht [9], asserts that a connected k-split solvable subgroup of $SL_n(k)$—for any field k—has a fixed point in the projective space $\mathbf{P}_{n-1}(k)$. Thus, our central—and quite easy—Corollary 2, according to which

a solvable fixed point free automorphism group of a tree T leaves invariant an end or a pair of ends of T

can be regarded as an analogue of the theorem of Lie–Kolchin for trees. Our main application of the above result (or rather of the somewhat stronger Corollary 1, dealing with nilpotent groups) aims, through the general Proposition 3, at showing that the group G of rational points of an algebraic almost simple group of relative rank ≥ 2 over a field k cannot operate without fixed point or fixed end on a tree; actually, for lack of complete knowledge of the structure of G, we can only state the above property in full generality for a "big" normal subgroup of G (cf. Corollary 4, but also Remark 4.3a). In the rank 1 case, treated in Section 5, it turns out, roughly speaking, that the only way for a group of the above type to act on a tree without fixed point or fixed end is the known one, that is, through a valuation of the "standard root datum" of the group in question (cf. [6, §6]). These results on the action of algebraic simple groups on trees are, in some sense, the substitutes of the main theorem of [5] on "abstract" homomorphisms of algebraic simple groups when the receiving group is replaced by the automorphism group of a tree. Note that our notion of tree is somewhat more general than the usual one, and devised so as to include the buildings of groups of rank 1 over fields with nondiscrete valuations.

The study of groups which cannot operate on trees without fixed point was initiated by J.-P. Serre [10]. The viewpoint we adopt in the presentation of our results is inspired by his. There is also some analogy between our methods of proofs and the techniques used in [10]. For trees in the usual sense, our "theorem of Lie–Kolchin" is implicitly contained in a paper of H. Bass [1]. Finally, it should be mentioned that much deeper results on the existence of fixed points for actions of arithmetic groups on trees have recently been obtained by G. A. Margulis (unpublished).

1. Trees

1.1 We call *segment* (resp. *half-line*; resp. *line*) of a metric space, the image of a closed interval $[0, d]$ (resp. of the half-line $[0, \infty)$; resp. of $\mathbf{R}$) by an isometric embedding α in that space. The *extremities* of the segment $\alpha([0, d])$ are the points $\alpha(0)$ and $\alpha(d)$; the *extremity* of the half-line $\alpha([0, \infty))$ is $\alpha(0)$. We define a *tree* as a nonempty, complete metric space containing no homeomorphic image of a circle and such that any two points are the extremities of a segment. That segment is then unique; indeed, if α and α' are two distinct isometric embeddings of $[0, d]$ in a metric space, with $\alpha(0) = \alpha'(0)$ and $\alpha(d) = \alpha'(d)$, if $t \in [0, d]$ is such that $\alpha(t) \neq \alpha'(t)$, and if t', t'' denote respectively the largest element of $[0, t]$ and the smallest element of $[t, d]$ on which α and α' coincide, then $\alpha([t', t'']) \cup \alpha'([t', t''])$ is homeomorphic to a circle. A subset of a tree which is itself a tree when endowed with the induced metric is called a *subtree*. Clearly, the intersection of any number of subtrees is empty or is a subtree. In the set of all half-lines of a tree, the relation "$A \cap B$ is a half-line" is an equivalence relation whose equivalence classes are called the *ends* of the tree. An end is said to belong to a subtree if the latter contains a representative of it.

In the sequel, T always denotes a tree, d its distance function, and $E = E(T)$ the set of its ends. The segment joining two points p, q of a tree is denoted by $[p, q]$.

1.2 *The intersection of any number of segments or half-lines with a common extremity p is a segment or a half-line having p as an extremity.*

This is obvious.

1.3 *If $p, q, r \in T$ are such that $[p, q] \cap [q, r] = \{q\}$, then $[p, r] = [p, q] \cup [q, r]$.*

Indeed, set $[p, q] \cap [p, r] = [p, x]$ and $[q, r] \cap [p, r] = [y, r]$. If one had $x \neq q$ or $y \neq q$, the set $[x, q] \cup [q, y] \cup [x, y]$ would be homeomorphic to a circle.

1.4 *Let A, B be two subtrees of T whose intersection consists of at most one point. Then, there exist unique points $p \in A$ and $q \in B$ such that $A \cap [p, q] = \{p\}$ and $[p, q] \cap B = \{q\}$. The segment $[p, q]$ is contained in every subtree which has a nonempty intersection with both A and B.*

Let C be any subtree having a nonempty intersection with A and B (such a subtree obviously exists: take for instance any segment joining a point of A and a point of B). Let $x \in A \cap C$ and $y \in B \cap C$ and set $[x, y] \cap A = [x, p]$ and $[x, y] \cap B = [q, y]$. It is clear that p, q have the property stated. Furthermore, if p', q' is any pair of points with that property, one has, by Section 1.3, $[p, p'] \cup [p', q'] = [p, q'] = [p, q] \cup [q, q']$, hence $p = p'$ and $q = q'$. Since $[p, q] \subset C$, the last assertion also holds.

1.5 From Sections 1.3 and 1.4, it readily follows that if $p \in T$ and $e \in E$, there is a unique representative of the end e with extremity p; we denote it by $[p, e)$. Similarly, two distinct ends e, f belong to a unique line, denoted by (e, f).

1.6 ***Lemma*** *Let $\{A_i \mid i \in I\}$ be a set of subtrees of T such that any two of them have a nonempty intersection. Then, $\bigcap_I A_i \neq \varnothing$ or all A_i's have a unique common end.*

(Compare with the last lemma of [10].)

Choose a point $p \in T$. For $i \in I$, let $q_i \in A_i$ be the unique point such that $[p, q_i] \cap A_i = \{q_i\}$. For $i, j \in I$ and $x \in A_i \cap A_j$, one has, by Section 1.4, $[p, q_i] \cup [p, q_j] \subset [p, x]$; hence either $[p, q_i] \subset [p, q_j]$ or $[p, q_j] \subset [p, q_i]$. It readily follows that the closure L of $\bigcup_I [p, q_i]$ is a segment or a half-line, and that $L \cap A_i \cap A_j \neq \varnothing$ for every $i, j \in I$. If $L \cap A_k$ is a segment for some $k \in I$, the sets $L \cap A_k \cap A_i$ are pairwise nondisjoint subsegments; therefore $\bigcap_I A_i \neq \varnothing$. Otherwise, all $L \cap A_i$ are half-lines representing the same end. Finally, if the A_i's have two distinct common ends e, f, one has $\bigcap_I A_i \supset (e, f) \neq \varnothing$.

2. Various fixed-point properties

2.1 By *automorphism* of a tree T, we mean an isometry of T onto itself. If X is a set of automorphisms of T or a subset of a group acting on T, we denote by T^X and E^X the fixed-point sets of X in T and $E = E(T)$. An end $e \in E^X$ is said to be *neutral* (resp. *attracting*; resp. *repulsing*) for X if, for every $x \in X$, there exists a half-line L representing e and such that $x(L) = L$ (resp. $\subsetneq L$; resp. $\supsetneq L$). (N.B. This terminology differs from that of [13], where we reserved the expression "fixed by X" to designate the neutral ends. Here, talking of a single end, we use the words "fixed," "invariant," and "stable" as synonymous; as usual, "fixed," applied to a set, means "pointwise fixed.")

2.2 Let G be a group acting on a tree T by isometries. We shall be concerned with the following mutually exclusive possibilities.

(P1) $T^G \neq \emptyset$ (in which case T^G is a tree);

(P2) $T^G = \emptyset$ and G has a neutral fixed end (in which case, E^G is reduced to that end);

(P3) $T^G = \emptyset$ and card $E^G \geq 2$ (in which case E^G consists of exactly two ends which are not neutral for G);

(P4) $T^G = E^G = \emptyset$ and there is a pair of ends invariant by G (that pair is then unique);

(P5) $T^G = \emptyset$ and E^G consists of a single end which is not neutral for G.

When there is no doubt as to which action of G on a tree T is being considered, we shall sometimes, by abuse of language, say that a subgroup H of G has property (Pn), or satisfies (Pn), if that property holds for the restriction of the given action to H.

The following assertions are obvious.

2.2.1 If G acting on T has property (Pn) ($1 \leq n \leq 5$), then, every subgroup H of G has property (Pm) for some $m \leq n$; more precisely, one has $(n, m) = (n, 1)$, (n, n), $(5, 2)$, $(4, 3)$ or $(5, 3)$. If $m = 1$, every end fixed by G belongs to the tree T^H.

2.2.2 Let H be a normal subgroup of G. If H has one of the properties (P2) and (P5) (resp. (P3) and (P4)), so does G; indeed, more generally, if H has a single invariant end (resp. pair of ends), so does G. If H has property (P1), then G satisfies (Pn) if and only if G/H acting on the tree T^H does.

2.2.3 If G has property (P2), every finitely generated subgroup of G has property (P1).

2.2.4 If (P3) or (P5) holds, G has a nontrivial homomorphism in $\mathbf{R}$. If (P4) holds, G has a homomorphic image which is the semidirect product of $\{\pm 1\}$

and a nontrivial subgroup of **R** on which $\{\pm 1\}$ acts by multiplication. In particular, G cannot be perfect in any of those cases.

2.3 For $n \in \{1, 2, 3, 4, 5\}$, we shall say that a group G is F_n, or is an *F_n-group,* if every action of G on a tree satisfies one of the conditions (Pm), with $1 \leq m \leq n$. Except for the difference in the definition of trees, properties F_1 and F_2 are respectively properties (FA) and (FA') of [10] and [1]. Clearly, $F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow F_4 \Rightarrow F_5$.

2.3.1 *Every finite group is* F_1, by [6, Lemme (3.2.3)].

2.3.2 *Every finitely generated F_2-group is* F_1, by Section 2.2.3.

2.3.3 *Every perfect F_5-group is* F_2, by Section 2.2.4.

2.3.4 *Let* $(m, n) = (1, 1), (2, 2), (2, 4)$, *or* $(5, 5)$. *Let G be a group and H a subgroup containing a normal subgroup N such that G/N is F_m. If, for a given action of G on a tree, H has one of the properties* (P1) *to* (Pn), *then so does G.*

We shall only consider the case $m = 2$, the other ones being similar but easier. By Section 2.2.1, if H has one of the properties (P1) to (Pn), N has property (Pn') for some $n' \leq n$. If $n' \neq 2$, our assertion readily follows from Section 2.2.2. If $n' = 2$, the unique fixed end of N is fixed by G and, being neutral for N, it must be neutral for G because G/N has no nontrivial homomorphism in **R** (otherwise, G/N would have a fixed point free action on **R**, in contradiction with the assumption that it is F_2).

From Section 2.3.4, we deduce:

2.3.5 *If* $n = 1, 2$, *or* 5, *every extension of an F_n-group by an F_n-group is F_n.*

2.3.6 *If* $n = 1, 2, 4$, *or* 5, *every group having an F_n-subgroup of finite index is F_n.*

3. Solvable groups

3.1 ***Proposition 1*** *Let f be an automorphism of a tree T. If $T^f \neq \varnothing$, every subtree of T invariant by f has a nonempty intersection with T^f and every end fixed by f belongs to T^f. If $T^f = \varnothing$, then E^f consists of exactly two ends, an attracting one a and a repulsing one r; the line (a, r) is contained in every subtree of T invariant by f.*

(Compare [13, Prop. 3.2].)

Suppose $T^f \neq \varnothing$. If T' was a subtree invariant by f and disjoint from T^f, the segment $[p, q]$ defined by $[p, q] \cap T^f = \{p\}$ and $[p, q] \cap T' = \{q\}$ (cf. Section 1.4) would be fixed by f, hence contained in T^f, a contradiction.

Further, if e is an end fixed by f, the half-line $[p, e)$ is fixed by f for any $p \in T^f$, therefore e belongs to T^f.

From now on, we assume that T^f is empty. Let $p \in T$ and set $[p, f^{-1}p] \cap [p, fp] = [p, q]$. Thus, $[q, f^{-1}p] \cap [q, fp] = \{q\}$. Since $q \in [f^{-1}p, p]$, we have $fq \in [p, fp]$. Similarly, $f^{-1}q \in [p, f^{-1}p]$. Suppose that $fq \in [p, q]$. Then $q \in [fp, fq]$, hence $f^{-1}q \in [p, q]$, and since $d(q, fq) = d(q, f^{-1}q)$, it follows that $fq = f^{-1}q$. Therefore, the segment $[q, fq]$ is invariant by f and so is its middle point, a contradiction. This shows that $fq \notin [p, q]$, hence $fq \in [q, fp]$. Similarly, $f^{-1}q \in [q, f^{-1}p]$. Consequently, one has $[f^{-1}q, q] \cap [q, fq] = \{q\}$ and, transforming by f^n, $[f^{n-1}q, f^n q] \cap [f^n q, f^{n+1}q] = \{f^n q\}$. From this relation, assertion 1.3, and the nonexistence of "circles" in T, it readily follows that the set $\bigcup_{n \in Z}[f^n q, f^{n+1}q]$ is a line whose two ends are invariant by f. Now, if e, e' are any two distinct ends invariant by f, one of them must be attracting and the other repulsing, because the line (e, e') is invariant and not fixed by f. Therefore, f can have at most two fixed ends. Finally, applying the result just proved to the restriction of f to any invariant subtree T', we see that the two ends fixed by f must belong to T'. The proof is complete.

3.2 ***Proposition 2*** *If the quotient of a group G by its center is* F_3, *then G itself is* F_3.

Let G operate on a tree T. We want to prove that one of the properties (P1), (P2), (P3) of Section 2.2 holds.

We first consider the case where G is commutative. If there exists $g \in G$ such that T^g is empty, then, by Proposition 1, E^g consists of one attracting and one repulsing end; it follows that G, which centralizes g, fixes E^g, and (P3) holds. Assume therefore that $T^g \neq \emptyset$ for every $g \in G$. For any $g, g' \in G$, the tree $T^{g'}$ is invariant by g (because g centralizes g'); therefore $T^g \cap T^{g'} \neq \emptyset$, by Proposition 1. We are now in a position to use Lemma 1.6: either $T^G = \bigcap_{g \in G} T^g \neq \emptyset$, which means that (P1) holds, or $T^G = \emptyset$ and the subtrees T^g have a common end, which is obviously neutral for G, and this is property (P2). The proposition is thus proved for a commutative group G.

Going over to the general case, we denote by C the center of G. If T^C is not empty, it is a tree on which G/C operates and our assertion follows from the hypothesis made on G/C. If T^C is empty, the special case of the proposition already proved implies that E^C consists either of two ends, or of a single one e which is neutral for C. All we have to show is that E^C is fixed by every element g of G and that, in the second case, e is neutral for g. But for this, it suffices to apply again the proposition already proved in the commutative case to the group generated by C and g.

3.3 ***Corollary 1*** *Every nilpotent group is* F_3.

3.4 ***Corollary 2*** *Every solvable group is* F_5.

(Use Corollary 1 and Section 2.3.5.)

3.5 ***Corollary 3*** *Every solvable torsion group is* F_2.

(Use Corollary 2 and Section 2.2.4.)

4. Algebraic simple groups of relative rank ≥ 2

4.1 ***Proposition 3*** *Let G be a group and X a subset generating a subgroup of finite index of G. For* $g \in G$, *let* $N(g)$ *denote the union of all normalizers of nilpotent subgroups containing g. Suppose that, for any* $x, y \in X$, *either* $y \in N(x)$ *or* $N(x) \cap N(y)$ *generates a subgroup of finite index of G. Then G is* F_4.

(N.B. The following proof shows that the proposition remains valid if one replaces "nilpotent" by "F_3" and "of finite index" by "containing a normal subgroup H of G such that G/H is F_2." Here, F_2 can be replaced by F_5 at the cost of changing the conclusion in: G is F_5.)

Let G operate on a tree T. We claim that one of the properties (P1)–(P4) holds for some subgroup of finite index of G. By Section 2.3.6, this will prove our proposition.

Suppose first that T^c is empty for some $c \in G$. Then, the unique line invariant by c (Section 3.1) is also the unique line invariant by every nilpotent subgroup containing c (Section 3.3). Therefore, it is also invariant by $N(c)$ and we are through since the hypotheses made on X imply, whether $N(c)$ contains X or not, that the group generated by $N(c)$ is of finite index in G.

Thus we may and shall assume that $T^x \neq \varnothing$ for all $x \in X$. Our next step is to show that, for $g, h \in G$,

$$\text{if} \quad T^g \neq \varnothing, \quad T^h \neq \varnothing, \quad \text{and} \quad h \in N(g), \qquad \text{then} \quad T^g \cap T^h \neq \varnothing. \quad (1)$$

Indeed, let H be a nilpotent subgroup containing g and normalized by h. By Section 3.3, either $T^H \neq \varnothing$, or H leaves invariant a unique line L or it fixes a unique end e. Since h normalizes H, it leaves invariant T^H, L, or e. But then, it follows from Proposition 1 that, according to the case, $T^g \cap T^h \supset T^H \cap T^h \neq \varnothing$, or $L \subset T^g \cap T^h$, or e belongs to both T^g and T^h, and (1) is proved.

If $T^x \cap T^y \neq \varnothing$ for every $x, y \in X$, it follows from Section 1.6 that either $T^X \neq \varnothing$, in which case T^X is a subtree fixed by the subgroup of finite index G' of G generated by X, or the subtrees T^x $(x \in X)$ have a common end, which is obviously neutral for G'; in both cases, our assertion is proved.

Suppose therefore that $T^b \cap T^c = \varnothing$ for some $b, c \in X$. Let $p, q \in T$ be defined by $[p, q] \cap T^b = \{p\}$ and $[p, q] \cap T^c = \{q\}$. From (1), it follows that $c \notin N(b)$ and that, for every $x \in N(b) \cap N(c)$, T^x has a nonempty intersection with both T^b and T^c, hence contains p and q, by Section 1.4. As a result, p and q are fixed by the subgroup generated by $N(b) \cap N(c)$, which is of finite index in G by hypothesis. This completes the proof.

4.2 *Corollary 4* *If $\mathscr{G}$ is an algebraic almost simple group of relative rank ≥ 2 over a field k, then the normal subgroup G of $\mathscr{G}(k)$ generated by all rational unipotent elements contained in k-split unipotent subgroups of $\mathscr{G}$ is F_2.*

(N.B. If card $k \geq 4$, G can also be described as the unique minimal noncommutative normal subgroup of $\mathscr{G}(k)$: cf. [12] and [4].)

Let $\mathscr{S}$ be a maximal k-split torus of $\mathscr{G}$ and Φ the system of nondivisible roots of $\mathscr{G}$ with respect to $\mathscr{S}$. For $a \in \Phi$, let $\mathscr{U}_{(a)}$ denote the corresponding unipotent "root group" (cf. [3, 5.2]) and set $U_{(a)} = \mathscr{U}_{(a)}(k)$. Then, the set $X = \bigcup_{a \in \Phi} U_{(a)}$ fulfills the conditions of Section 4.1. Indeed, it generates G (cf. [5, 6.2(v)]). Furthermore, let $x \in U_{(a)}$ and $y \in U_{(b)}$ $(a, b \in \Phi)$. If $b \neq -a$, x and y generate a nilpotent group and one has $y \in N(x)$. If $b = -a$, the same argument shows that $N(x) \cap N(y)$ contains all $U_{(c)}$ for $c \neq \pm a$, and hence generates G. The assertion now follows from the above proposition and Sections 2.3.3 and 2.3.1, since G is perfect or finite.

4.3 *Remarks* (a) In many cases, possibly always, $\mathscr{G}(k)/G$ is an abelian torsion group. Then, by Sections 2.3.5 and 3.5, $\mathscr{G}(k)$ itself is F_2.

(b) The argument proving Corollary 4 applies without change to the classical groups "of rank ≥ 2" over arbitrary division rings (cf. e.g. [6, §10]), the Ree groups of type ${}^2F_4$, and, using Corollary 3 of [11, p. 115], the groups $\mathscr{G}(R)$ where $\mathscr{G}$ is an almost simple Chevalley group-scheme of rank ≥ 2 and R a Euclidean ring. In particular, by Section 2.3.2, $\mathscr{G}(\mathbf{Z})$ is F_1 (compare [10, no. 5, remarque 2]). But Proposition 3 also applies to other arithmetic groups. It can for instance be used to prove that

if $\mathscr{G}$ is an almost simple Chevalley group scheme of rank ≥ 2, every subgroup of finite index of $\mathscr{G}(\mathbf{Z})$ is F_1,

which partially answers a question of J.-P. Serre ([10, no. 6]; a complete answer, based on methods similar to those of [8], has been announced by G. A. Margulis). Here, one takes for X the set of all unipotent elements of the subgroup in question which are central in a maximal unipotent subgroup of $\mathscr{G}(\mathbf{C})$; to prove that this X meets the requirements of Proposition 3 is beyond the scope of the present paper and will be done elsewhere†

5. The rank 1 case

5.1 Let G be a group, U_+ and U_- two proper subgroups generating G, S the intersection of their normalizers, and M the set of all $m \in G$ such that ${}^mU_+ = U_-$ and ${}^mU_- = U_+$. We assume that

(R) for every $u \in U_+ - \{1\}$, there exist unique elements $u', u'' \in U_-$ such that $m(u) = u'uu'' \in M$.

A typical example of that situation is the following: $G = SL_2(K)$ for some division ring K,

$$U_+ = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in K\right\}, \qquad U_- = \left\{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \middle| t \in K\right\},$$

$$S = \left\{\begin{pmatrix} t & 0 \\ 0 & t' \end{pmatrix}\right\} \cap G, \qquad M = \left\{\begin{pmatrix} 0 & t \\ t' & 0 \end{pmatrix}\right\} \cap G.$$

Another example (including the previous one if K is finite-dimensional over its center) is obtained by taking for U_+ and U_- the groups of k-rational points of the unipotent radicals of two distinct proper k-parabolic subgroups in an almost simple algebraic group of relative rank 1 over some field k.

5.2 Let η: $S \to \mathbf{R}$ be a nontrivial homomorphism such that $\eta(msm) = -\eta(s)$ for all $s \in S$ and $m \in M$, choose arbitrarily m_0 in M and define the function φ: $U_+ \to \mathbf{R} \cup \{\infty\}$ by

$$\varphi(u) = \begin{cases} \frac{1}{2}\eta(m(u) \cdot m_0) & \text{if } u \neq 1, \\ \infty & \text{if } u = 1. \end{cases}$$

We shall say that η is a *valuation* of the system (U_+, U_-) if for every real number r, $\varphi^{-1}([r, \infty])$ is a group, a condition which is clearly independent of the choice of m_0. (N.B. The terminology adopted here for the convenience of the exposition somewhat deviates from that of [6]; in particular, our valuations also include the "quasi-valuations" of [6, 6.2.3e].) In the case of $SL_2(K)$, with U_+ and U_- defined as above, the valuations are nothing else but the homomorphisms of the form

$$\begin{pmatrix} t & 0 \\ 0 & t' \end{pmatrix} \mapsto 2\omega(t) = -2\omega(t'),$$

† *Added in proof*: In view of results obtained after this paper was written (cf. *C. R. Acad. Sci. Paris* **283** (1976), 693–695), our statement concerning subgroups of finite index of $\mathscr{G}(\mathbf{Z})$ can now be deduced from Proposition 3 (whose full force is no longer necessary) in exactly the same way as Corollary 4. Also Serre's method [10] can be applied.

where ω denotes a non-archimedean valuation of K (cf. [6, 10.2.10]), and one has

$$\varphi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \omega(t) \qquad \left(\text{for} \quad m_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

5.3 To a valuation η of (U_+, U_-), one associates, exactly as in [6, §§7, 8], a *complete building* T_η, which is a tree on which G operates by isometries. We briefly recall the construction of T_η, without going into details. For $r \in \mathbf{R}$, let G_r denote the group generated by $\varphi^{-1}([r, \infty])$, $m_0 \cdot \varphi^{-1}([-r, \infty]) \cdot m_0^{-1}$, and Ker η. In $G \times \mathbf{R}$, let us introduce the equivalence relation

$$(g, r) \sim (g', r') \quad \Leftrightarrow \quad r = r' \quad \text{and} \quad g^{-1}g' \in G_r.$$

Then, G operates on the left on the quotient $T_\eta^0 = (G \times \mathbf{R})/\sim$, and there is a unique metric in T_η^0 which is invariant by G and such that $r \mapsto (1, r) \bmod \sim$ is an isometry of $\mathbf{R}$ into T_η^0. Finally, the tree T_η is the completion of the metric space T_η^0.

5.4 ***Proposition 4*** *Let G, U_+, U_- be as in Section* 5.1, *suppose U_+ and U_- are nilpotent, and let G act on a tree T without fixed point and without fixed end. Then, there exist a unique valuation η of (U_+, U_-) and a unique isometric embedding of the complete building T_η into T compatible with the actions of G onto T_η and T.*

(N.B. In that statement, "nilpotent" can be replaced by "F_3" as will be obvious from the proof.)

Since U_+ and U_- are nilpotent and conjugate, it follows from Section 3.3 that one of the following cases occurs:

(i) U_+ (resp. U_-) leaves invariant a unique line L_+ (resp. L_-), on which it induces a nontrivial group of translations;

(ii) the fixed-point sets T_+ and T_- of U_+ and U_- are not empty;

(iii) there is a unique end e_+ (resp. e_-) fixed by U_+ (resp. U_-) and neutral for it.

We shall treat them successively.

Case (*i*) Since G is generated by U_+ and U_- and has no fixed end, the lines L_+ and L_- have no common end; consequently, their intersection is empty or is a segment. Let u be an element of U_+ which does not fix L_+ and such that $L_+ \cap L_- \cap uL_- = \varnothing$ (since $L_+ \cap L_-$ is bounded, a sufficiently high power of any element of U_+ which does not fix L_+ has that property) and let u', u'', $m(u)$ be as in Section 5.1(R).

Suppose first that $L_+ \cap L_-$ is empty and let $p_+ \in L_+$ and $p_- \in L_-$ be defined by $[p_+, p_-] \cap L_\pm = \{p_\pm\}$. By Section 1.3, one has $[p_-, up_-] = [p_-, p_+] \cup [p_+, up_+] \cup [up_+, up_-]$, and

$$[p_-, up_-] \cap L_- = \{p_-\}, \qquad [p_-, up_-] \cap uL_- = \{up_-\}$$

(the reader is advised to draw a picture), hence, denoting by $d(X, Y)$ the distance of the sets X and Y and using Section 1.4, $d(L_-, uL_-) = d(p_-, up_-) > d(p_-, p_+)$. On the other hand,

$$d(L_-, uL_-) = d(u'L_-, u'uL_-) = d(L_-, m(u)L_-) = d(L_-, L_+) = d(p_-, p_+),$$

a contradiction.

Thus, $L_+ \cap L_-$ is a segment $[p, q]$. Consequently, one has $L_- \cap uL_- = L_- \cap u'^{-1}L_+ = u'^{-1}(L_- \cap L_+) \neq \varnothing$. It follows that the segment $[p, up]$, which joins a point of L_- and a point of uL_-, meets their intersection. Since $[p, up] \subset L_+$, this contradicts the hypothesis made on u, and we conclude that the first case cannot occur.

Case (*ii*) Since the intersection of the trees T_+ and T_- is fixed by G, it must be empty. Let $p_+ \in T_+$ and $p_- \in T_-$ be defined by $[p_+, p_-] \cap T_\pm = \{p_\pm\}$. The trees T_+, T_- and hence the points p_+, p_- are permuted by every element of the set M defined in Section 5.1. Let u be an element of U_+ which does not fix the middle point of $[p_+, p_-]$ and set $[p_+, p_-] \cap [p_+, up_-] = [p_+, q]$. Then, $d(p_-, q) > \frac{1}{2}d(p_-, p_+)$ and, by Section 1.3, $d(p_-, up_-) = d(p_-, q) + d(q, up_-) = 2d(p_-, q) > d(p_-, p_+)$. On the other hand, if u', u'', $m(u)$ are as in Section 5.1(R), one has $d(p_-, up_-) = d(u'p_-, u'up_-) = d(p_-, m(u)p_-) = d(p_-, p_+)$, a contradiction which shows that also the second case cannot occur.

Case (*iii*) Since G fixes no end, one has $e_+ \neq e_-$. The group S and the set M of Section 5.1 clearly preserve the line $A = [e_+, e_-]$: the ends e_+, e_- are permuted by M and fixed by S. Let $u \in U_+ - \{1\}$ and let u', u'', $m(u)$ be as in Section 5.1(R). The element u does not fix e_- (otherwise $m(u)$ would), therefore $T^u \cap A$ is a half-line $[q, e_+)$. We shall show that

$$q \text{ is fixed by } u' \text{ and } u'', \text{ hence by } m(u). \tag{1}$$

Suppose the contrary. Upon replacing u', u, u'' by u''^{-1}, u^{-1}, u'^{-1}, if necessary, we may assume that q is not fixed by u''. Setting $T^{u''} \cap A = [q', e_-)$, we then have $q' \notin [q, e_+)$, hence $[q, uq'] \cap A = \{q\}$, therefore $q \in [uq', e_-)$ and, transforming by u', $u'q \in [u'uq', e_-) = [m(u)q', e_-) \subset A$. This implies that $u'q = q$. But then, one has $[u''^{-1}q, e_-) = [m(u)^{-1}q, e_-) \subset A$, which clearly implies that q is fixed by u'', a contradiction establishing our assertion (1).

The elements $m(u)$ of M cannot all have the same fixed point in A, otherwise, as follows from (1), this point would be fixed by U_+, hence by U_- and finally by G. Therefore S $(= M \cdot M)$ does not fix A. Let us now choose an element m_0 in M and identify A with $\mathbf{R}$ isometrically in such a way that 0 is fixed by m_0 and that e_+ corresponds to $+\infty$. Let η: $S \to \mathbf{R}$ be the homomorphism defined by $st = t + \eta(s)$ for $s \in S$ and $t \in A = \mathbf{R}$. Then, it readily follows from (1) that the function φ: $U_+ \to \mathbf{R} \cup \{\infty\}$ of Section 5.2 is given by $\varphi(u) = \inf\{t \in \mathbf{R} \mid ut = t\}$. Therefore, η is a valuation and it is easy to see that the mapping $(g, r) \mapsto gr$ of $G \times \mathbf{R}$ into T factorizes through an isometry α: $T_\eta \to T$. The unicity of η and α are clear from the way they have been obtained.

5.5 Restated in loose form in the case of $SL_2(K)$, Proposition 4 essentially means that

Every fixed point free and fixed end free action of $SL_2(K)$ on a tree is through a well-defined non-archimedean valuation of K.

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