ANALYTIC FUNCTIONS RELATED TO PRIMES

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1. *Introduction.* We show how to put certain number-theoretic problems connected with primes into analytic, or function-theoretic form. The general procedure is to construct analytic functions having special behaviour at the primes without, however, having the primes enter directly into the definition of the functions. It is thought that students knowing a little number theory as well as analysis might be interested in the ease with which the one type of problem can be converted into the other.

2. *Simple Functions.* By comparing the highest power of a prime contained in $(n - 1)!$ with the highest power contained in n, we find that n divides $(n - 1)!$ if n is any composite integer ≥ 6. Hence the function

$$p(z) = \sin \frac{\Gamma(z)}{z} \pi \tag{1}$$

has a simple zero at every composite integer ≥ 6 and no other zeros for integer values of z. In a similar manner we see, by using Wilson's theorem, that the function

$$q(z) = \sin \frac{\Gamma(z) + 1}{z} \pi \tag{2}$$

has simple zeros at the primes and no other integer zeros.[1]

In the expression

$$r_n(z) = \prod_{k=2}^{n} \sin \frac{\pi z}{k} \tag{3}$$

for $|z| \leq n$, all factors are different from zero if z is not an integer; just one factor is zero if z is a prime; and at least three factors are zero if z is composite. Hence there are simple zeros at the primes, higher order zeros at the composite integers, and no other zeros with $|z| \leq n$.

The series

$$s(z) = \sum_{k=2}^{\infty} \frac{1 - (z/k)^2}{\sin \pi z/k} \frac{e^k}{k!} \tag{4}$$

is uniformly convergent if z is bounded away from the integers, and hence represents an analytic function. Also if z is a prime, then only

[1] It has been brought to the author's attention that functions of the type (2) are actually well known; cf. Dickson's *History of Number Theory.*

one term of the sum will have a vanishing denominator, and that term will have a zero in the numerator. If z is composite, however, there is a simple pole for each term with k a factor of z. The sum of the residues for the poles of these separate terms cannot be zero, since e is transcendental. We thus conclude that $s(z)$ has simple poles at the composite integers and no other finite singularities.

3. *Primes in general.* From the above we see that the only real positive zeros of $q^2(z) + \sin^2 \pi z$ occur at the primes, and hence

$$\lim_{y \to 0} \int_{C_n} \frac{2q(z)q'(z) + \pi \sin 2\pi z}{q^2(z) + \sin^2 \pi z} dz = v(n) + 1 \tag{5}$$

if $v(n)$ is the number of primes $\leq n$. Here C_n is any simple closed contour in the right half plane containing the integers 1, 2, ... n but no others. The notation $y \to 0$ means that the contour is to contract down upon the real axis. Similarly

$$(1 - z^2) \frac{r_n(z)}{\sin \pi z} = \sum_{k=0}^{\infty} R_{nk} z^k \tag{6}$$

has its first pole at the first prime $\geq n$, so that its radius of convergence $\lim |R_{nk}|^{-1/n}$ is equal to this function of n.

The function

$$(1 - z^2)^2 \frac{r_n(z)}{\sin^2 \pi z} \tag{7}$$

has simple poles at the primes and no others with $|z| \leq n$; the function

$$\frac{p(z)}{\sin \pi z} \tag{8}$$

has simple poles at the primes and no other singularities for $R(z) \geq 6$. On the other hand the expressions

$$\frac{q(z)}{\sin \pi z}, \quad s(z) \tag{9}$$

have simple poles at the composite integers; and the former has no other singularities in the right half plane, the latter no others in the whole plane. Problems connected with the distribution of primes, then, can be formulated in terms of the limit (5), the radius of convergence (6), or the distribution of singularities in (7) - (9).

One may modify the functions to suit the particular problem, of course; for example, if we were interested in primes of the form $m^2 + 1$ we should be led to the expression

$$\frac{p(z^2 + 1)}{\sin \pi z} \tag{10}$$

which has simple poles at these primes and no other singularities for $R(z) \geq 3$. The classical conjecture is thus put into analytic form, viz., to show that (10) has infinitely many poles in the right half plane. A formulation can be obtained in terms of elementary functions by the expression

$$z^2(1 - z^2)\frac{r_n(z^2 + 1)}{\sin^2 \pi z}, \tag{11}$$

which has simple poles at primes $m + 1$ and no other singularities in $|z| \leq \sqrt{n} - 1$. Similar analytic statements can be given for the Goldbach conjecture and for other outstanding problems in the theory of primes.

4. *Twin primes.* A question of this sort is the conjecture that there are infinitely many twin primes, that is, infinitely many prime pairs like (17,19) or (29,31) which differ by 2. From the above remarks or by inspection of the functions we see that

$$z^3 \frac{r_n(z - 1)r_n(z + 1)}{\sin^3 \pi z} \tag{12}$$

has simple poles at the twin primes and no other singularities in $|z| \leq n - 1$, while

$$\frac{p(z - 1)p(z + 1)}{\sin \pi z} \tag{13}$$

has simple poles at the twin primes and no others for $R(z) \geq 6$. The same sort of thing can be done with $s(z - 1)s(z + 1)$.

To show that a function has infinitely many poles one may multiply by another function $\theta(z)$, having at most a finite number of poles, and integrate around a simple closed contour. If there are a finite number of poles the integral will eventually be constant, as the contour gets larger and larger; but if there are infinitely many poles, then there will be a $\theta(z)$ which will make the sequence of integrals diverge. Thus, a necessary and sufficient condition that there be infinitely many twin primes is that there exist an integral function $\theta(z)$ and a sequence of simple closed contours C_n free of points on the negative real axis, such that the set of numbers

$$\int_{C_n} \frac{p(z - 1)p(z + 1)}{\sin \pi z}\,\theta(z)\,dz$$

is unbounded.

By estimating the maximum and minimum residues, and comparing the minimum estimate for the pole having largest $|z|$ with the sum of maximum estimates for the other poles, one can choose a function $\theta(z)$ that certainly increases fast enough to give divergence if there are infinitely many poles. After carrying out these estimates of residues in (12), for example, one finds that $\phi(z) = e^{z^3}$ is good enough, with the following result: A necessary and sufficient condition that there be infinitely many twin primes is that the set of numbers

$$\int_{C_n} \frac{r_n(z - 1)r_n(z + 1)}{\sin^3 \pi z} e^{z^3} dz$$

be unbounded, if the contour C_n includes the integers, 0, ±1, ±2, ... $\pm(n - 1)$ and no others.

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