Mr. Poisson, in his researches into definite integrals, determined a number of properties of $\vartheta_4(x,q)$. The delicate methods of this noted geometer find delightful verification in the theory of elliptic functions. For example, in issue 19 of the *Journal of the Polytechnic School*, Mr. Poisson established the following identity:

$$x^{-1/2} = \frac{1+2\sum_{n=1}^{\infty} e^{-n^2\pi x}}{1+2\sum_{n=1}^{\infty} e^{-n^2\pi/x}} = \frac{\vartheta_4(x,-q)}{\vartheta_4(x^{-1},-q)}$$

If we let $x = \frac{\mathrm{K}'}{\mathrm{K}}$ and transform $k \mapsto k' = \sqrt{1-k^2}$, then we transform $x \mapsto \frac{\mathrm{K}}{\mathrm{K}'} = x^{-1}$. Starting with

$$\begin{aligned} \sqrt{\frac{2\mathrm{K}}{\pi}} &= 1+2\sum_{n=1}^{\infty} q^{n^2} \\ &= 1+2\sum_{n=1}^{\infty} e^{-n^2\pi x} \end{aligned}$$

and transforming $k \mapsto k'$, we obtain

$$\sqrt{\frac{2\mathrm{K}'}{\pi}} = 1+2\sum_{n=1}^{\infty} e^{-n^2\pi/x}.$$

Dividing these two results immediately gives Mr. Poisson's identity.

Definition de theta_4 (page 2)

The elliptic functions can be replaced by the new transcendental function ϑ_4 defined by the following series expansion:[1]

$$\vartheta_4(x,q) := 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nx.$$