# Inseparable Galois Cohomology

BY

PIERRE CARTIER

The following report is a brief and elementary account of a generalized Galois cohomology for generalized algebraic groups which takes into account the inseparability as well. We assume a ground field $k$ of characteristic $p \neq 0$, because our theory reduces entirely to the classical one in the characteristic zero case. Much more general results have been obtained by M. Artin and A. Grothendieck (see [**1**]).

1. **Definition of an algebraic group.** Our algebraic groups are the same as the affine group schemes of finite type considered by A. Grothendieck and his school. We use the functorial point of view to define them. Namely, let $\mathfrak{U}_k$ be the category of commutative $k$-algebras. An algebraic group $G$ is a covariant functor from $\mathfrak{U}_k$ to the category of groups which is "representable" in the following sense: *there exists a pair* $(A_0, g_0)$, *where* $A_0$ *is some finitely generated algebra in* $\mathfrak{U}_k$ *and* $g_0$ *an element of the group* $G(A_0)$ *such that, for any object* $A$ *of* $\mathfrak{U}_k$ *and any* $g$ *in* $G(A)$, *there exists a unique homomorphism* $\sigma$ *from* $A_0$ *into* $A$ *such that* $G(\sigma)$ *maps* $g_0$ *into* $g$. We simplify the notation by writing $\tau \cdot g$ instead of $G(\tau) \cdot g$ when $g$ is in $G(A)$ and $\tau$ is an algebra homomorphism from $A$ to $B$.

EXAMPLES. (a) Let $V_k$ be any finite-dimensional vector space over $k$, and $V_A$ denote the additive group $V_k \otimes_k A$ for any object $A$ of $\mathfrak{U}_k$. The functor $V$ is called the *vector group associated to* $V_k$.

(b) If $E_k$ is any finite-dimensional $k$-algebra, commutative or not, we define $E_A^\times$ as the multiplicative group of the $k$-algebra $E_A = E_k \otimes_k A$. This defines the multiplicative group $E^\times$ of the "algebra-variety" $E$.

(c) Let $n$ be an integer. For any object $A$ in $\mathfrak{U}_k$, let $\mathrm{GL}_n(A)$ denote the group of invertible $n$ by $n$ matrices with coefficients in $A$. Besides the algebraic group $\mathrm{GL}_n$ thus defined, one can define in the same way the symplectic group, or the orthogonal group of a quadratic form with coefficients in $k$.

(d) The algebraic group $\mu_n$ associates to any $A$ the multiplicative group consisting of the elements $a$ in $A$ with $a^n = 1$.

The algebraic group $G$ is called commutative if the groups $G(A)$ are commutative. It can be shown that the commutative algebraic groups form an *abelian category*.

2. **Definition of the cohomology groups.** Let $K$ be a finite-dimensional commutative algebra over $k$. For any $A$ in $\mathfrak{U}_k$, let us define $X_A$ as the set of algebra

homomorphisms from $K$ to $A$. Moreover, let there be given a commutative algebraic group $G$. By definition, an *n-cochain* $c$ is a collection of functions

$$c_A : X_A \times \cdots \times X_A \to G_A \quad (n + 1 \quad \text{factors } X_A) \tag{1}$$

where $A$ runs over the objects of $\mathfrak{U}_k$, subjected to the condition

$$c_B(\sigma\sigma_0, \cdots, \sigma\sigma_n) = \sigma \cdot c_A(\sigma_0, \cdots, \sigma_n) \tag{2}$$

for any algebra homomorphism $\sigma: A \to B$. The coboundary $\delta c$ of the $n$-cochain $c$ is the $(n + 1)$-cochain defined by

$$(\delta c)_A(\sigma_0, \cdots, \sigma_{n+1}) = \sum_{0 \leqq i \leqq n+1} (-1)^i \sigma_i \cdot c_A(\sigma_0, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{n+1}). \tag{3}$$

As usual, we have $\delta\delta c = 0$ for any $n$-cochain $c$. We can therefore define cohomology groups in the standard fashion; they will be denoted $H^n(K/k, G)$. This definition includes as particular cases the ordinary Galois cohomology in case $K/k$ is a finite Galois extension and $G$ an "ordinary" algebraic group, and also the Amitsur cohomology when $K/k$ is a finite algebraic extension and $G$ is 'the' multiplicative group $G_m = \mathrm{GL}_1$.

The cohomology groups $H^n(K/k, G)$ depend functorially on $G$ and also on the pair $(K, k)$ in the sense that any commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\phi} & K' \\ \cup & & \cup \\ k & \xrightarrow{\alpha} & k' \end{array}$$

gives rise to a homomorphism $\alpha^n$ from $H^n(K/k, G)$ to $H^n(K'/k', G)$ *independent* of $\phi$.

We can define the absolute cohomology groups $H^n(k, G)$ in two equivalent ways. The first is to replace $K$ by the algebraic closure $\bar{k}$ of $k$ in the previous definitions (the fact that $K$ is finite-dimensional over $k$ played no role); the second consists in taking the direct limit of the groups $H^n(K/k, G)$ when $K$ runs over the finite algebraic subextensions of $\bar{k}$.

The group $G(k)$ consists of the "rational points" of $G$ and will also be denoted $\Gamma(G)$. The functor $\Gamma$ maps the category of commutative algebraic groups into the category of abelian groups; the derived functors $R^n\Gamma$ of $\Gamma$ are therefore defined. It turns out [2] that $R^n\Gamma(G)$ is nothing else than $H^n(k, G)$, which fact entails among other properties the existence of an exact sequence of cohomology associated to any short exact sequence of algebraic groups.

3. **Some particular cases** ($K$ finite algebraic extension of $k$). Two important results are the following:

(a) *For any vector space $V_k$ over $k$ and any $n \geqq 1$, one has* $H^n(K/k, V) = 0$.

(b) *For any commutative algebra $E_k$, one has* $H^1(K/k, E^\times) = 0$ (generalization of Hilbert's Theorem 90). The assumption of commutativity of $E$ can be dropped provided one defines the first cohomology group $H^1(K/k, G)$ for a noncommutative algebraic group $G$ as well, which causes no difficulty. Once this is done, one can prove for instance

$$H^1(K/k, \mathrm{GL}_n) = 0. \tag{4}$$

(c) *One has* $H^1(K/k, G_m) = 0$ *and* $H^2(K/k, G_m)$ *is the relative Brauer group of the field extension* $K/k$ (that is the group of similarity classes of normal $k$-algebras split by $K$).

Going to the limit over $K$, we get as a corollary:

(d) *One has* $H^1(k, G_m) = 0$ *and* $H^2(k, G_m)$ *is the Brauer group of* $k$.

Finally using the exact sequence

$$0 \to \mu_n \to G_m \xrightarrow{\nu} G_m \to 0$$

defining $\mu_n$ (with $\nu_A(x) = x^n$ for every $A$) and the associated exact sequence of cohomology, we get the following information:

(e) *The group* $H^1(k, \mu_n)$ *is isomorphic to* $k^\times/(k^\times)^n$ *and* $H^2(k, \mu_n)$ *is the subgroup of the Brauer group of* $k$ *defined by the condition* $a^n = 1$.

4. **Comparison with standard cohomology.** Let us denote by $\bar{k}$ any algebraic closure of $k$, and by $k_s$ the maximal separable subextension of $\bar{k}$; the letter $g$ denotes the group of $k$-automorphisms of $\bar{k}$. If $G$ is any commutative algebraic group, the Galois group $g$ acts on $G(k_s)$ and $G(\bar{k})$ and corresponding cohomology groups $H^n(g, G(k_s))$ and $H^n(g, G(\bar{k}))$ are defined after Tate [4]. Moreover, there are canonical homomorphisms

$$H^n(g, G(k_s)) \xrightarrow{\alpha_G^n} H^n(k, G) \xrightarrow{\beta_G^n} H^n(g, G(\bar{k})).$$

Using recent results by Shatz [3], one can prove that $\alpha_G^n$ and $\beta_G^n$ are isomorphisms in each of the following cases (except possibly $\beta_G^n$ for $n \leqq 2$)

(a) *$k$ is perfect.*

(b) *$G$ is smooth,* that is the algebra $A_0 \otimes_k \bar{k}$ has no nilpotent element where $A_0$ is as in the definition of $G$ (§ 1).

(c) *The integer $n$ is distinct from* 1 *and* 2.

Moreover, for every $n$, the kernels and cokernels of $\alpha_G^n$ and $\beta_G^n$ are $p$-torsion groups and the different cohomology groups involved have no $p$-torsion for $n > 2$. Finally, $\alpha_G^1$ is injective and $\beta_G^2$ is surjective.

5. **Infinitesimal groups.** The algebraic group $G$ is called *infinitesimal* in case $G(K)$ is 0 for every field $K$; it suffices to assume $G(\bar{k}) = 0$. These groups enter as the kernels of the purely inseparable isogenies, and any information about their cohomology enables us via the exact sequence of cohomology to compare the cohomologies of any two purely inseparably isogeneous groups.

The basic result is again due to Shatz [3] and states that $H^n(k, G)$ is 0 for $n \neq 1, 2$ and isomorphic to $H^{n-1}(g, H^1(k_s, G))$ where the Galois group $g$ acts in the natural way on $H^1(k_s, G)$.

## References

1. M. Artin, *Grothendieck topologies*, Mimeographed notes for a Harvard Seminar, Spring 1962.

2. S. Shatz, *Cohomology of artinian group schemes over local fields*, Ann. of Math. **79** (1964), 411–449.

3. ———, *The cohomological dimension of certain Grothendieck topologies*, Ann. of Math. (to appear).

4. J. Tate, *Cohomology of compact totally discontinuous groups*. See a report by A. Douady in the Bourbaki Seminar, Dec. 1959, exp. 189.