

Measuring and Hedging Financial Risks in Dynamical World

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Abstract

Financial markets have developed a lot of strategies to control risks induced by market fluctuations. Mathematics has emerged as the leading discipline to address fundamental questions in finance as asset pricing model and hedging strategies. History began with the paradigm of zero-risk introduced by Black & Scholes stating that any random amount to be paid in the future may be replicated by a dynamical portfolio. In practice, the lack of information leads to ill-posed problems when model calibrating. The real world is more complex and new pricing and hedging methodologies have been necessary. This challenging question has generated a deep and intensive academic research in the 20 last years, based on super-replication (perfect or with respect to confidence level) and optimization. In the interplay between theory and practice, Monte Carlo methods have been revisited, new risk measures have been back-tested. These typical examples give some insights on how may be used mathematics in financial risk management.

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1. Introduction

Financial markets have become an important component of people's life. All sorts of media now provide us with a daily coverage on financial news from all markets around the world. At the same time, not only large institutions but also more and more small private investors are taking an active part in financial trading. In particular, the e-business has led to an unprecedented increase in small investors direct trading. Given the magnitude of the potential impact a financial crisis can

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have on the real side of the economy, large, and even small, breakdowns have received particularly focal attention. Needless to say, the rapid expansion of financial markets calls upon products and systems designed to help investors to manage their financial risks. The financial risk business now represents more than \$15 trillion annually in notional. A large spectrum of simple contracts (futures, options, swaps, etc.) or more exotic financial products (credit derivatives, catastrophe bonds, exotic options, etc.) are offered to private investors who use them to transfer financial risks to specialized financial institutions in exchange for suitable compensation. A classic example is the call option, which provides a protection in case of a large increase in the underlying asset price¹. More generally, a derivative contract is an asset that delivers a payoff $H(\omega)$ at maturity date, depending upon the scenario ω .

As argued by Merton [19], the development of the financial risk industry would not have been possible without the support of theoretical tools. Mathematics has emerged as the leading discipline to address fundamental questions in financial risk management, as asset pricing models and hedging strategies, based on daily (infinitesimal) risk management. Mathematical finance, which relates to the application of the theory of probability and stochastic processes to finance, in particular the Brownian motion and martingale theory, stochastic control and partial differential equations, is now a field of research in its own right.

2. The Black & Scholes paradigm of zero-risk

It is surprising that the starting point of financial industry expansion is “the Brownian motion theory and Itô stochastic calculus”, first introduced in finance by Bachelier in this PhD thesis (1900, Paris), then used by Black, Scholes and Merton in 1973. Based on these advanced tools, they develop the totally new idea in the economic side that according to an optimal dynamic trading strategy, it is possible for option seller to deliver the contract at maturity without incurring any residual risk. At this stage, any people not familiar with stochastic analysis (as the majority of traders in the bank) may be discouraged. As Foellmer said in Bachelier Congress 2000, it is possible to reduce technical difficulties, and to develop arguments which are essentially probability-free.

A DYNAMICAL UNCERTAIN WORLD

The uncertainty is modelled via a family Ω of scenarios ω , i.e. the possible trajectories of the asset prices in the future. Such paths are described as positive continuous functions ω with coordinates $X_t = \omega(t)$, such that the continuous quadratic variation exists: $[X]_t(\omega) = \lim_n \sum_{t_i \leq t, t_i \in D_n} (X_{t_{i+1}} - X_{t_i})^2$ along the sequence of dyadic partitions D_n . The pathwise version of stochastic calculus yields to Itô's formula,

$$f(t, X_t)(\omega) = f(0, x_0) + \int_0^t f'_x(s, X_s)(\omega) dX_s(\omega) + \int_0^t f_t(s, X_s)(\omega) dt$$

¹A Call option provides its buyer with the right (and not the obligation) to purchase the risky asset at a pre-specified price (the exercise price) at or before a pre-specified date in the future (maturity date). The potential gain at maturity can therefore be written as $(X_T - K)^+$, where X_T denotes the value of the underlying asset at time T

$$+ \int_0^t \frac{1}{2} f''_{xx}(s, X_s)(\omega) d[X]_s(\omega).$$

The second integral is well defined as a Lebesgue-Stieljes integral, while the first exists as Itô's integral, defined as limit of non-anticipating Riemann sums, (where we put $\delta_t = F'_x(t, X_t)$), $\sum_{t_i \leq t, t_i \in D_n} \xi_{t_i}(\omega)(X_{t_{i+1}} - X_{t_i})$.

From a financial point of view, the Itô's integral may be interpreted as the cumulative *gain process* of trading strategies: δ_t is the number of the shares held at time t then the increment in the Riemann sum is the price variation over the period. The non anticipating assumption corresponds to the financial requirement that the investment decisions are based only on the past prices observations. The residual wealth of the trader is invested only in cash, with yield rate (short rate) r_t by time unit. The *self-financing condition* is expressed as:

$$dV_t = r_t(V_t - \delta_t X_t)dt + \delta_t dX_t = r_t V_t dt + \delta_t (dX_t - r_t X_t dt), \quad V_0 = z.$$

HEDGING DERIVATIVES: A SOLVABLE TARGET PROBLEM

Let us come back to the problem of the trader having to pay at maturity T the amount $h(X_T)(\omega)$ in the scenario ω ($(X_T(\omega) - K)^+$ for a Call option). This target has to be hedged (approached) in all scenarios by the wealth generated by a self-financing portfolio. The "miraculous" message is that, in Black & Scholes world, a perfect hedge is possible and easily computable, under the additional assumption: the short rate is deterministic and the quadratic variation is absolutely continuous $d[X]_t = \sigma(t, X_t) X_t^2 dt$. The (regular) strictly positive function $\sigma(t, x)$ is a key parameter in financial markets, called the *local volatility*.

Looking for the wealth as a function $f(t, X_t)$, we see that, given Itô's formula and self-financing condition,

$$\begin{aligned} df(t, X_t) &= f'_t(t, X_t)dt + f'_x(t, X_t)dX_t + \frac{1}{2} f''_{xx}(t, X_t)X_t^2 \sigma^2(t, X_t) dt \\ &= f(t, X_t)r_t dt + \delta(t, X_t)(dX_t - X_t r_t dt) \end{aligned}$$

By identifying the dX_t terms (thanks to assumption $\sigma(t, x) > 0$), $\delta(t, X_t) = f'_x(t, X_t)$, and f should be solution of the following partial differential equation, Pricing PDE in short,

$$f'_t(t, x) + \frac{1}{2} f''_{xx}(t, x)x^2 \sigma^2(t, x) + f'_x(t, x)xr_t - f(t, x)r_t = 0, \quad f(T, x) = h(x) \quad (2.1)$$

The derivative price at time t_0 must be $f(t_0, x_0)$, if not, it is easy to generated profit without bearing any risk (arbitrage). That is the *rule of the unique price*, which holds in a liquid market.

The PDE's fundamental solution $q(t, x, T, y)$ ($h(x) = \delta_y(x)$) may be interpreted in terms of Arrow-Debreu "states prices" density, introduced in 1953 by these Nobel Prize winners for a purely theoretical economical point of view and by completely different arguments. The pricing rule becomes: $f(t, x) = \int h(y)q(t, x, T, y)dy$. q is also called *pricing kernel*. When $\sigma(t, x) = \sigma_t$, the pricing kernel is the *log-normal density*, deduced from the Gaussian distribution by an explicit change of variable.

The closed formula for Call option price is the famous ² Black and Scholes formula, which is known by any practitioner in finance. The impact of this methodology was so important that Black (who already died), Scholes and Merton received the Nobel prize for economics in 1997.

In 1995, B.Dupire [9] give a clever formulation for the dual PDE (one dimensional in state variable) satisfied by $q(t, x, T, y)$ in the variables (T, y) . If $C(T, K)$ is the Call price with parameters (T, K) when market conditions are (t_0, x_0) , then

$$C'_T(T, K) = \frac{1}{2}\sigma^2(T, K) K^2 C''_{KK}(T, K) - rK C'_K(T, K), \quad C(t_0, x_0) = (x_0 - K)^+$$

In short, if $r_t = 0$, $C'_T(T, K) = \frac{1}{2}\sigma^2(T, K) K^2 C''_{KK}(T, K)$

3. Model calibration and Inverse problem

In practice, the main problem is the model calibration. Before discussing that, let me put the problem in a more classical framework. Following P. Lévy, asset price dynamics may be represented through a Brownian motion via stochastic differential equation (SDE)

$$dX_t = X_t(\mu(t, X_t)dt + \sigma(t, X_t)dW_t), \quad X_{t_0} = x_0$$

where the Brownian motion W may be viewed as a standardized Gaussian noise with independent increments.

The local expected return $\mu(t, X_t)$ is a trend parameter appearing for the first time in our propose. That is a *key point* in financial risk management. Since this parameter does not appear in the Pricing-PDE, the Call price does not depend on the market trend. It could seem surprising, since the first motivation of this financial product is to hedge the purchaser against underlying rises. By using dynamical hedging strategy, the trader (seller) may be also protected against this unfavorable evolution. For a statistical point of view, this point is very important, because this parameter is very difficult to estimate.

In the B & S model with constant parameters, the volatility square is the variance by time unit of the log return $\ln(X_t) - \ln(X_{t-h}) = R_t$. If the only available information is given by asset price historical data, the statistical estimator to be used is the empirical variance, computed on a more or less long time period, $\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=0}^{N-1} (R_{t_i} - \overline{R_N})^2$, where $\overline{R_N} = \frac{1}{N} \sum_{i=0}^{N-1} R_{t_i}$. This estimator is called *historical volatility*.

However, traders are reserved in using this estimator. Indeed, they argue that financial markets are not “statistically” stationary and that past is not enough to

²In the Black-Scholes model with constant coefficients, the Call option price $C^{BS}(t, x, K, T)$ is given via the Gaussian cumulative function $\mathcal{N}(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$, and $\theta = T - t$,

$$\left\{ \begin{array}{l} C^{BS}(t, x, t + \theta, K) = x\mathcal{N}(d_1(\theta, x/K)) - K e^{-r\theta}\mathcal{N}(d_0(\theta, x/K)) \\ d_0(\theta, x/K) = \frac{1}{\sigma\sqrt{\theta}} \text{Log} \left(\frac{x}{K e^{-r\theta}} \right) - \frac{1}{2}\sigma\sqrt{\theta}, \quad d_1(\theta, x/K) = d_0(\theta, x/K) + \sigma\sqrt{\theta} \end{array} \right.$$

Moreover $\Delta(t, x) = \partial_x C^{BS}(t, x, t + \theta, K) = \mathcal{N}(d_1(\theta, x/K))$.

explain the future. When it is possible, traders use additional information given by quoted option prices and translate it into volatility parametrization. The *implied volatility* is defined as: $C^{obs}(T, K) = C^{BS}(t_0, x_0, T, K, \sigma^{imp})$.

Moreover, the Δ of the replicating portfolio is $\Delta^{imp} = \partial_x C^{BS}(t_0, x_0, T, K, \sigma^{imp})$.

This strategy is used dynamically by defining the implied volatility and the associated Δ at any renegotiation dates. It was this specific use based on the hedging strategy that may explain the Black & Scholes formula success.

This attractive methodology has been appeared limited: observed implied volatilities are depending on the option parameters (time to maturity and exercise price) (implied volatility surface) in complete contradiction with B&S model assumptions. In particular, the market quotes large movements (heavy tail distribution) higher than in the log-normal framework. The first idea to take into account this empirical observation is to move to a model with local volatility $\sigma(t, x)$. The idea is especially attractive since the Dupire formula (2.2) gives a simple relation between "quoted Call option prices" and local volatility: $\sigma^2(K, T) = 2C_T'(T, K)(K^2 C_{KK}''(T, K))^{-1}$. The local volatility is computable from a continuum of *coherent* (without arbitrage) observed quoted prices. Unfortunately, the option market is typically limited to a relatively few different exercise prices and maturities; a naive interpolation yields to irregularity and instability of the local volatility.

ILL-POSED INVERSE PROBLEM

The problem of determining local volatility can be viewed as a function approximation non linear problem from a finite data set. The data set is the value $C_{i,j}$ of the solution at (t_0, x_0) of Pricing-PDE with boundary conditions $h_{i,j}(x) = (x - K_{i,j})^+$ at maturity T_i . Setting the problem as PDE's inverse problem yields to more robust calibration methods. These ideas appear for the first time in finance in 1997 [16], but the problem is not classical because of the strongly non linearity between option prices and local volatility; the data set is related to a single given initial condition.

Prices adjustment is made through a least square minimization program, including a penalization term related to the local volatility regularity.

$$G(\sigma) = \sum_{i,j} \omega_{i,j} (f(t_0, x_0, h_{i,j}, T_i, \sigma(\cdot, \cdot)) - C_{i,j}^{Obs})^2,$$

$$J(\alpha, \sigma) = \alpha \|\nabla \sigma\|^2 + G(\sigma) \rightarrow \min_{\sigma}.$$

Existence and uniqueness of solution is only partially solved [4]. Using large deviation theory, the asymptotic in small time of local volatility is expressed in terms of implied volatility: $\sigma^{implied}(K, t_0)^{-1} = \ln(\frac{K}{x_0})^{-1} \int_{x_0}^K \frac{d\xi}{\xi \sigma(\xi, t_0)}$.

Avellaneda & alii [2] have used another penalization criterion based on a stochastic control approach ; the control is the volatility parameter itself constrained to be very close to a prior volatility ($\eta(\sigma) = |\sigma(t, x) - \sigma_0(t, x)|^2$ for instance). The gradient criterion is replaced by $K(\sigma) = U(t_0, x_0, \sigma)$ where $U(T, x, \sigma) = 0$ and

$$U_t'(t, x) + \frac{1}{2}\sigma^2(t, x) x^2 U_{xx}''(t, x) + rx U_x'(t, x) - U(t, x) + \eta(\sigma(t, x)) = 0.$$

4. Portfolio, duality and incomplete market

In the previous framework, options market may be entirely explained by underlying prices. In economic theory, it corresponds to *market efficiency*: a security price contains all the information on this particular security. In option world, the observed statistical memory of historical volatility leads naturally to consider stochastic volatility models with specific uncertainty

$$dX_t = X_t(\mu(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dW_t^1), \quad dY_t = \eta(t, X_t, Y_t) + \gamma(t, X_t, Y_t) dW_t^2$$

where dW^1 and dW^2 are two correlated Brownian motions. γ is the volatility of the volatility. What does it change ? In fact, everything ! Perfect replication by a portfolio is not possible any more ; the notion of unique price does not exist any longer... But, such a situation is often the general case. What kind of answer may we bring to such a problem?

Super-replication and Robust Hedging

The option problem is still a target problem C_T , to be replicated by a portfolio $V_T(\pi, \delta) = \pi + \int_0^T \sum_i \delta_i^i dX_t^i$ depending on market assets X^i . Constraints (size, sign...) may be imposed on investment decisions (δ_i^i) ³. Let \mathcal{V}_T be the set of all derivatives, replicable at time T by an admissible portfolio. Their price at t_0 is the value of their replicating portfolio.

Super-replicating C_T is finding the smallest derivative $\hat{C}_T \in \mathcal{V}_T$ which is greater than C_T in all scenarios. The super-replication price is the price of such a derivative. The \hat{C}_T replicating portfolio is the C_T *robust hedging*.

There are several ways to characterise the super-replicating portfolio:

1) Dynamic programming on level sets: this is the most direct (but less recent) approach. This method proposed by Soner & Touzi [20] has led the way for original works in geometry by giving a stochastic representation of a class of mean curvature type geometric equations.

2) Duality: this second approach is based on the \mathcal{V}_T "orthogonal space", a set \mathcal{Q}_T of martingale measures to be characterised. The super-replication price is given by

$$\hat{C}_0 = \sup_{Q \in \mathcal{Q}_T} \mathbf{E}_Q[C_T].$$

We develop this last point, which is at the origin of many works.

Martingale measures

The idea of introducing a dual theory based on probability measures is due to Bachelier (1900), and above all to Harrison & Pliska (1987). The actual and achieved form is due to Delbaen & Schachermayer [7] and to the very active international group in Theoretical Mathematical Finance.

A *martingale measure* is a probability such that: $\forall V_T \in \mathcal{V}_T, \mathbf{E}_Q[V_T] = V_0$. Using simple strategies (discrete times, randomly chosen), this property is equivalent to prices of fundamental assets (X_t^i) are \mathbf{Q} -(local) martingales: the best X_{t+h}^i -estimated (w.r. to \mathbf{Q}) given the past at time t is X_t^i itself. The financial game is fair with respect to martingale measures.

³For the sake of simplicity, interest rates are assumed to be null

When \mathcal{V}_T contains all possible (path-dependent) derivatives, the market is said to be complete, and the set of martingale measures is reduced to a unique element \mathbf{Q} , often called risk-neutral probability. This is the case in the previous framework. Dynamics become $dX_t = X_t \sigma(t, X_t) dW_t^{\mathbf{Q}}$ where $W^{\mathbf{Q}}$ is a \mathbf{Q} -Brownian motion. This formalism is really efficient as it leads to the following path dependent derivative pricing rule: $\hat{C}_0 = \mathbf{E}_{\mathbf{Q}}(C_T)$.

Computing the replicating portfolio is more complex. In the diffusion case, the price is a deterministic function of risk factors and the replicating portfolio only depends on partial derivatives. The general case will be mentioned in the paragraph dedicated to Monte-Carlo methods.

INCOMPLETE MARKET

The characterization of the set \mathcal{Q}_T is all the more delicate so since there are many different situations which may lead to market imperfections (non-tradable risks, transaction costs ...).

An abstract theory of super-replication (and more generally of portfolio optimization under constraints) based on duality has been intensively developed. The super-replicating price process is showed [10],[15] to be a super-martingale with respect to any admissible martingale measure. Hence, by the generalization of the Doob-Meyer representation, the super-replicating portfolio is the “ \mathcal{Q}_T - martingale” part of the super-price Kramkov-decomposition.

Super-replication prices are often too expensive to be used in practice. However, they give an upper bound to the set of possible prices. In the previously described stochastic volatility model, the super-replication price essentially depends on possible values of stochastic volatility:

1. If the set is R^+ , then the super-replicating derivative of $h(X_T)$ is $\hat{h}(X_T)$ where \hat{h} is the concave envelop of h ; the replicating strategy is the trivial one: buying $\hat{h}'_x(x_0)$ stocks and holding them till maturity.
2. If the volatility is bounded (up and down relatively to 0), the super-replication price is a (not depending on y) solution of
$$\hat{h}'_t(t, x) + \frac{1}{2} \sup_y (\sigma^2(t, x, y) \hat{h}''_{xx}(t, x)) = 0, \quad \hat{h}(T, x) = h(x).$$

When h is convex, $\hat{h}(t, x)$ is convex and the super-replication price is the one calculated with the *upper volatility* (in y).

Calibration constraints may be easily taken into account without modifying this framework. We only have to assume that the terminal net cash flows of calibrating derivatives belong to \mathcal{V}_T or equivalently we have to add linear constraints to the dual problem: $\hat{C}_0^{\text{cal}} = \sup\{\mathbf{E}_{\mathbf{Q}}(C_T) ; Q \in \mathcal{Q}_T, \mathbf{E}_{\mathbf{Q}}((X_{T_i} - K_{ij})^+) = C_{ij}\}$.

Risk measures

When super-replicating is too expensive, the trader has to measure his market risk exposure. The traditional measure is the variance of the replicating error. But a new criterion, taking into account extreme events, is now used, transforming the risk management at both quantitative and qualitative levels.

VALUE AT RISK

The VaR criterion, corresponding to *the maximal level of losses* acceptable with

probability 95%, has taken a considerable importance for several years. Regulation Authorities have required a daily VaR computation of the global risky portfolio from financial institutions. Such a measure is important on the operational point of view, as it affects the provisions a bank has to hold to face market risks. VaR estimation (quantile estimation) and its links with extreme value theory [12] are widely debated in the market, just as by academics.

Moreover, a huge debate has been introduced by academics [1] on the VaR efficiency as risk measure. For instance, its non-additive property enables banks to play with subsidiary creations. This debate has received an important echo from the professional world, which is possibly planning to review this risk measure criterion. Sub-additive and coherent risk measures are an average estimation of losses with respect to a probability family: $\rho(X) = \sup_{Q \in \mathcal{Q}_T} \mathbf{E}_Q(-X)$.

This characterisation has recently been extended to convex risk measures, by adding a penalization term depending on probability density (entropy for instance).

RISK MEASURES AND RESERVE PRICE

A trader willing to relax the super-replication assumption is naturally thinking in terms of potential losses with given probability (level confidence) of his replicating error. It corresponds to the quantile hedging strategy. Other risk measures (quadratic, convex, entropic) may be used. Optimization theory is coming back when looking for the smallest portfolio, generating an acceptable loss. The initial value of this portfolio is called the *reserve price*. Mean-variance and entropic problems have now a complete solution [13]. More surprisingly (because of non-convexity), this also holds for the quantile hedging problem [14]. All these results are in fact sub-products of portfolio optimization in incomplete markets [6] or [8].

5. New research fields

Monte-Carlo methods

Dual version of super-replication problems, just as new risk measures, underline the interest to compute very well and quickly quantities such as $\mathbf{E}_Q(X)$ and more generally $\sup_{Q \in \mathcal{Q}_T} \mathbf{E}_Q(X)$. For small dimensional diffusions, these quantities may be computed as the solution of some linear PDE (for the expected value) and non-linear PDE (for the sup). However, the computational efficiency falls rapidly with the dimension. That increases the interest for the so-called *probabilistic methods*.

The fundamental idea of Monte-Carlo methods is the computation of $\mathbf{E}_Q(X)$ by simulation, i.e. by drawing a large number ($N \simeq 10^5$) of independent scenarios ω^i and taking the average value of the results $\frac{1}{N} \sum_{i=1}^N X(\omega^i)$. Of course, this method does not work very well when being too naive, but convergence may be accelerated by different techniques.

In the finance area, the important quantities are both the price and the sensitivities to different model parameters. Based on integration by parts, efficient methods have been developed to compute in a coherent manner prices and their derivatives [17]. In the case of path-dependent options, the derivative is taken with

respect to a Brownian motion perturbation (Malliavin Calculus).

Very original is the actual research on solving, by Monte-Carlo methods, optimization problem expressed as "sup" of a family of expected values (super-replication prices). These solutions are based on dynamic programming, enabling to turn a maximization in expected value into a pathwise maximization. The formulation in terms of Backward SDE's, introduced by Peng and Pardoux in 1987 and in finance⁴ in [11],[18] well describes this effect.

In all cases, the problem is to compute conditional expectation by Monte Carlo, using the function approximation theory, or more generally random variable approximation in the Wiener space (chaos decomposition).

Problems related to the dimension, and statistical modelling

Financial problems are usually multidimensional, but only few *liquid* financial products are depending on multi-assets. Even if the different market actors consider they can have a good knowledge of each individual asset behavior, the question is now to find a multidimensional distribution given each component distribution. This problem is a statistical one, known as the *copula theory*: Copula is a distribution function on $[0,1]^n$ with identity function as marginal. They are useful to give bounds to asset prices. Dynamically, the problem still to be solved is to find the local volatility matrix of multidimensional diffusion given the "Dupire" dynamics of each coordinate.

High dimensional problems arise when computing bank portfolio VaR (with a number of observations less than that of risk factors), or hedging derivatives depending on a large number of underlying assets. Main risk factors may be very different in a Gaussian framework or heavy tail framework (Lévy processes)[5]. Random matrix theory or other asymptotic tools may bring some new ideas to this question.

By presenting the most important tools of the financial risk industry, I have voluntarily left apart anything on financial asset statistical modelling, which may be the subject of a whole paper on its own. It is clear that the VaR criterion, the market imperfections are highly dependent on an accurate analysis of the real and historical world [3]. Intense and very innovating research is now developed (High-frequency data, ARCH and GARCH processes, Lévy processes with long memory, random cascades).

6. Conclusion

As a conclusion, applied mathematicians have been highly questioned by problems coming from the financial risk industry. This is a very active world, rapidly evolving, in which theoretical thoughts have often immediate and practical fallout.

⁴A BSDE solution is a couple of adapted processes (Y_t, Z_t) such that

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = C_T.$$

Thanks to comparison theorem, $\hat{Y}_t = \sup_{i \in I} Y_t^i$ is the solution of the BSDE with driver $\hat{f}(t, y, z) = \sup_{i \in I} f^i(t, y, z)$

On the other hand, practical constraints raise new theoretical problems. This paper is far from being an exhaustive view of the financial problems. It is more a subjective view conditioned by my own experience. Many exciting problems, from both theoretical and practical points of view, have not been presented. May active researchers in these fields forgive me.

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