ICM 2002 · Vol. II · 471–481

Characteristic Classes of Flat Bundles and Determinant of the Gauss-Manin Connection

Hélène Esnault*

2000 Mathematics Subject Classification: 14C22, 14C25, 14C40, 14C35, 14C99.

1. Introduction

The purpose of this note is to give a survey on recent progress on characteristic classes of flat bundles, and how they behave in a family.

2. Characteristic classes

Let X be a smooth algebraic variety over a field k. In [13] and [15], we defined the ring

$$\begin{aligned} AD(X) &= \oplus_n AD^n(X) \\ &= \oplus_n \mathbb{H}^n(X, \mathcal{K}_n^M \xrightarrow{d\log} \Omega^n_{X/k} \xrightarrow{d} \ldots \to \Omega^{2n-1}_{X/k}) \end{aligned} \tag{2.1}$$

of algebraic differential characters. Here the Zariski sheaf $\mathcal{K}_n^M$ is the kernel of the residue map from Milnor K-theory at the generic point of X to Milnor K-theory at codimension 1 points. More precisely, $\mathcal{K}_n^M$ satisfies a Gersten type resolution (see [16] and [18])

$$\begin{aligned} \mathcal{K}_n^M &\overset{\cong}{\to} (i_{k(X),*}K_n^M(k(X)) \xrightarrow{\text{Res}} \oplus_{x\in X^{(1)}} i_{x,*}K^M_{n-1}(\kappa(x)) \to \\ &\ldots \oplus_{x\in X^{(a)}} i_{x,*}K^M_{n-a}(\kappa(x)) \to \ldots \to \oplus_{x\in X^{(n)}} i_{x,*}K^M_0(\kappa(x))). \end{aligned}$$

Here $X^{(a)}$ means the free group on points in codimension a, while $i_x : x \to X$ is the embedding. The map $d\log(\{a_1, \ldots, a_n\}) = d\log a_1 \wedge \cdots \wedge d\log a_n$ from $K_n^M(k(X))$

*Mathematik, Universität Essen, FB6, Mathematik, 45117 Essen, Germany. E-mail: esnault@uni-essen.de

to $\Omega^n_{k(X)/k}$ carries $\mathcal{K}^M_n$ to

$$\Omega^n_{X/k} = \mathrm{Ker}(\Omega^n_{k(X)} \xrightarrow{\mathrm{res}} \oplus_{x\in X^{(1)}}\Omega^{n-1}_{x/k}).$$

This defines the map $d\log : \mathcal{K}^M_n \to \Omega^n_{X/k}$.

By the Gersten resolution, $H^n(X, \mathcal{K}^M_n) = CH^n(X)$, the Chow group of codimension n points. Thus one has a forgetful map

$$AD^n(X) \xrightarrow{\mathrm{forget}} CH^n(X). \tag{2.2}$$

The restriction map to the generic point $\mathrm{Spec}(k(X))$ fulfills

$$\begin{aligned} AD^1(X) &\overset{\cong}{\to} H^0(X, \Omega^1_{X/k}/d\log \mathcal{O}^\times_X) \subset AD^1(k(X)) \\ AD^n(X) &\to H^0(X, \Omega^{2n-1}_{X/k}/d\Omega^{2n-2}_{X/k}) \subset \Omega^{2n-1}_{k(X)/k}/d\Omega^{2n-2}_{k(X)/k}, \quad \text{for } n \geq 2 \end{aligned} \tag{2.3}$$

(see [2]). It is no longer injective for $n \geq 2$.

The Kähler differential $d : \Omega^{2n-1}_{X/k} \to \Omega^{2n}_{X/k,\mathrm{clsd}}$ defines

$$AD^n(X) \xrightarrow{d} H^0(X, \Omega^{2n}_{X/k,\mathrm{clsd}}). \tag{2.4}$$

The ring $AD(X) = \oplus_n AD^n(X)$ contains the subring

$$\begin{aligned} AD(X)_{\mathrm{clsd}}(X) &= \oplus_n AD^n_{\mathrm{clsd}}(X) \oplus_n \mathbb{H}^n(X, \mathcal{K}^M_n \xrightarrow{d\log} \Omega^n_{X/k} \to \ldots \to \Omega^{\dim(X)}_{X/k}) \\ &= \mathrm{Ker}(AD(X) \xrightarrow{d} \oplus_n H^0(X, \Omega^{2n}_{X/k,\mathrm{clsd}})). \end{aligned} \tag{2.5}$$

We call them the closed characters. The restriction map to $\mathrm{Spec}(k(X))$ fulfills

$$\begin{aligned} AD^1(X)_{\mathrm{clsd}} &\overset{\cong}{\to} H^0(X, \Omega^1_{X/k,\mathrm{clsd}}/d\log \mathcal{O}^\times_X) \subset AD^1(k(X))_{\mathrm{clsd}} \\ AD^n(X)_{\mathrm{clsd}} &\to H^0(X, \mathcal{H}^{2n-1}_{DR}) \subset H^{2n-1}_{DR}(k(X)/k), \quad \text{for } n \geq 2 \end{aligned} \tag{2.6}$$

(see [2]). Here $\mathcal{H}^p_{DR}$ is the Zariski sheaf of de Rham cohomology.

If k is the field of complex numbers $\mathbb{C}$, one can change from the Zariski topology to the analytic one. This yields a map

$$\begin{aligned} AD(X) &= \oplus_n AD^n(X) \xrightarrow{\iota} \\ D(X) &= \oplus_n D^n(X) = \oplus_n \mathbb{H}^{2n}(X_{\mathrm{an}}, \mathbb{Z}(n) \to \mathcal{A}^0_X \xrightarrow{d} \ldots \to \mathcal{A}^{2n-1}_X). \end{aligned} \tag{2.7}$$

Here $D(X)$ is the ring of differential characters defined by Cheeger-Simons ([10]). One has

$$\iota(AD(X)_{\mathrm{clsd}}) \subset \oplus_n H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n)) \subset D(X). \tag{2.8}$$

It is classical that $AD^1(X)$ is the group of isomorphism classes of lines bundles L with connection ∇. If $\xi_{ij} \in \Gamma(U_{ij}, \mathcal{O}^\times_X)$ is a cocycle of L in a local frame e_i of

L on U_i, and $\nabla(e_i) = \alpha_i \in \Gamma(U_i, \Omega^1_{X/k})$ is the local form of the connection, then $d\log \xi_{ij} = \alpha_j - \alpha_i = \delta(\alpha)_{ij}$ defines the Cech cocycle of $c_1((L,\nabla))$. In [12], [13], [15], we generalize this class.

Theorem 2.1 ([12], [13], [15]). *Associated to an algebraic bundle E with connection ∇ (resp. with integrable connection), one has characteristic classes $c_n((E,\nabla)) \in AD^n(X)$ (resp. $\in AD^n(X)_{\rm clsd}$). These classes satisfy the following properties:*

(1) *The classes $c_*((E,\nabla)) \in AD^*(X)$ are functorial and additive.*
(2) *$c_1((E,\nabla))$ is the isomorphism class of $(\det(E), \det(\nabla))$.*
(3) *forget $(c_n((E,\nabla))) = c_n(E) \in CH^n(X)$ is the Chern class of the underlying algebraic bundle E in the Chow group.*
(4) *$d(c_n((E,\nabla))) = c_n(\nabla^2) \in H^0(X, \Omega^{2n}_{X/k,\rm clsd})$ is the Chern-Weil form which is the evaluation of the invariant polynomial c_n on the curvature ∇^2.*
(5) *The restriction to the generic point $c_n((E,\nabla)|_{k(X)})$ is the algebraic Chern-Simons invariant $CS_n((E,\nabla))$ defined in [4]. It has values in $H^0(X, \Omega^1_{X/k}/d\log\mathcal{O}^\times)$ (resp.$H^0(X,\Omega^1_{X/k,\rm clsd}/d\log\mathcal{O}^\times)$) for $n=1$, and in $H^0(X, \Omega^{2n-1}_{X/k}/d\Omega^{2n-2}_{X/k})$ (resp.$H^0(X,\mathcal{H}^{2n-1}_{DR})$) for $n \geq 2$.*
(6) *If $k \subset \mathbb{C}$, then $\iota(C_n((E,\nabla))) \in D^n(X)$(resp. $\in H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(n))$) is the differential character defined by Chern-Cheeger-Simons, denoted by $c_n(E^{\nabla}_{\rm an}) \in H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(n))$ if ∇ is integrable.*

If $k = \mathbb{C}, \nabla$ is flat and the underlying monodromy is finite, then the existence of $c_n((E,\nabla))$immediately implies that the Chern-Simons classes in $H^{2n-1}(X_{\rm an}, \mathbb{Q}(n)/\mathbb{Z}(n))$ are in the smallest possible level of the coniveau filtration ([14]).

If X is complex projective smooth, ∇ is integrable and $n \geq 2$, we relate $CS_n((E,\nabla))$ for $n \geq 2$ to the (generalized) Griffiths' group $\mathrm{Griff}^n(X)$. It consists of cycles which are homologous to 0 modulo those which are homologous to 0 on some divisor([4], definition 5.1.1). For $n = 2$, [2] implies that $\mathrm{Griff}^2(X)$ is the classical Griffiths' group. For $n \geq 2$, Reznikov's theorem ([19]) (answering positively Bloch's conjecture [3]), together with the existence of the lifting $c_n((E,\nabla))$, imply that the classes $CS_n((E,\nabla))$ lie in the image Im of the global cohomology $H^{2n-1}(X,\mathbb{Q}(n))$ in $H^0(X,\mathcal{H}^{2n}(\mathbb{Q}(n))$. This subgroup Im maps to Griff^n(X). One has

Theorem 2.2 ([4], Theorem 5.6.2).

$$\text{image of } \; CS_n((E,\nabla)) \in \mathrm{Griff}^n(X) \otimes \mathbb{Q}$$

is the Chern class $c_n^{\rm Griff}(E) \otimes \mathbb{Q}$ of the underlying algebraic bundle E. Moreover, $CS_n((E,\nabla)) = 0$ if and only if $c_n^{\rm Griff}(E) \otimes \mathbb{Q} = 0$.

A relative version $AD(X/S)$ of $AD(X)$ is defined in [6]. We give here an example of application.

Theorem 2.3([6], Corollary 3.15). *Let $f : X \to S$ be a smooth projective family of curves over a field k. Let $(E, \nabla_{X/S})$ be a bundle with a relative connection. Then there are classes $c_2((E,\nabla)) \in AD^2(X/S) := \mathbb{H}^2(X, \mathcal{K}_2 \xrightarrow{d\log} \Omega^1_S \otimes^1_{X/S})$ lifting the classes $c_2(E) \in CH^2(X)$. There is a trace map $f_* : AD^2(X/S) \to AD^1(S)$ compatible with the trace map on Chow groups $f_* : CH^2(X) \to CH^1(S)$. Thus*

$f_*c_2(E,\nabla)$ is a connection on the line bundle $f_*c_2(E)$, which depends functorially on the choice of $\nabla_{X/S}$ on E.

We now study the behavior at ∞ of $CS_n((E,\nabla))$ for $n \geq 2$ in characteristic 0. Let $j : X \to \bar{X}$ be a smooth compactification of X. Recall that a de Rham class $\in H^q_{DR}(k(X)/k)$ at the generic point is called *unramified* if it lies in $H^0(\bar{X}, \mathcal{H}^q_{DR} \subset H^q_{DR}(k(X)/k)([2])$.

Theorem 2.4. *We assume k to be of characteristic 0, and ∇ to be integrable. Then $CS_n((E,\nabla))$ is unramified for $n \geq 2$.*

Proof. If (E,∇) is regular singular, this is shown in [4], theorem 6.1.1. In general, one may argue as follows. One has

$$H^0(\bar{X}, \mathcal{H}^{2n-1}_{DR}) = \mathrm{Ker}(H^0(X, \mathcal{H}^{2n-1}_{DR}) \xrightarrow{\mathrm{Res}} \oplus_{x \in X^{(1)}} H^{2n-2}_{DR}(x/k)).$$

Thus it is enough to show that the residue map at each generic point at ∞ dies. At a smooth point of a divisor D at ∞, the residue depends only on the formal completion of X along D. So we may assume that ∇ is a connection on $\mathcal{O}_X = k(D)[[x]]$, integrable relative to k. By a variant (see [1], proposition 5.10.) of Levelt's theorem ([17]) for absolute flat connections, there are a finite extension $K \supset k(D)$, and a ramified extension $\pi : K[[x]] \subset K[[y]], y^N = x$ for some $N \in \mathbb{N}\backslash\{0\}$ such that $\pi^*(\nabla) = \oplus(L \otimes U)$. Here L is integrable of rank 1 and U is integrable with logarithmic poles along y=0. Since $\mathrm{Res}_{y=0}\pi^*(\alpha) = N\pi^*(\mathrm{Res}_{x=0}(\alpha))$, and $H^p_{DR}(k(D)/k) \subset H^p_{DR}(K/k)$, functoriality and additivity of the classes imply that we may assume $\nabla = L \otimes U$ on $K((x))$ with $K = k(D)$, In a local frame we have the equations $U = \Gamma\frac{dx}{x} + \Sigma$ where $\Gamma \in GL(r, K[[x]]), \Sigma \in M(r, K[[x]]) \otimes \Omega^1_K$, and $L = d(f) + \lambda\frac{dx}{x} + \beta$ where $f \in K((x)), \lambda \in k, \beta \in \Omega^1_K$ with $d\beta = 0$. The explicit formula Res $\mathrm{Tr}(A(d(A))^{n-1}) \in H^{2n-1}_{DR}(K)$ of $CS_n((E,\nabla)) \in H^{2n-1}_{DR}(K((x)))$ and [4], prop. 5.10 in the logarithmic case, imply that

$$CS_n((E,\nabla)) = \mathrm{Tr}(d(f) + \lambda\frac{dx}{x} + \beta)(d(\Gamma)\frac{dx}{x} + d\Sigma)^{n-1}.$$

This is the sum of 2 terms with rational coefficients, Res $\mathrm{Tr}(d(f) + \beta)(d(\Sigma)^{n-2} d(\Gamma)\frac{dx}{x})$ and Res $\mathrm{Tr}(\lambda\frac{dx}{x}(d(\Sigma)^{n-1}))$. Both terms are obviously exact.

Discussion 2.5. We assume here $k = \mathbb{C}$, ∇ is integrable and $n \geq 2$. We consider the image $c_n(E^{\nabla}_{\mathrm{an}}|_{\mathbb{C}(X)}) \in H^0(\bar{X}, \mathcal{H}^{2n-1}(\mathbb{C}/\mathbb{Z}(n)))$ of $CS_n((E,\nabla)) \in H^0(\bar{X}, \mathcal{H}^{2n-1}_{DR})$. When X is not compact, there is Deligne's unique algebraic (E,∇) with regular singularities at ∞ with the given underlying local system E^{∇}_{an}([11]), but there are many irregular connections (E,∇). The topological class $C_n(E^{\nabla}_{\mathrm{an}}) \in H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$ is not, a priori, extendable to $\bar{X}$, but we have seen that its restriction to $\mathrm{Spec}(\mathbb{C}(X))$ is unramified.

There is on X a fundamental system of Artin neighborhoods U which are geometrically successive fiberings in affine curves. Topologically they are $K\pi_1$ and their fundamental group is a successive extension of free groups in finitely many letters. On such an open U, the class $c_n(E^{\nabla}_{\mathrm{an}})$ lies in $H^{2n-1}(U_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n)) = H^{2n-1}(\pi_1(U_{\mathrm{an}}, u), \mathbb{C}/\mathbb{Z}(n))$.

If U is such that $\pi_1(U_{\rm an}, u)$ is isomorphic as an abstract group to $\pi_1(V_{\rm an}, v)$, where V is an Artin neighborhood on a rational variety, then $E^{\nabla}_{\rm an}|_U$ becomes a representation of $\pi_1(V_{\rm an}, v)$, and since then $H^0(\bar{V}, \mathcal{H}^{2n-1}_{DR}) = 0$ for $n \geq 2$ and $V \subset \bar{V}$ a good compactification, one obtains $c_n(E^{\nabla}_{\rm an}|_{\mathbb{C}(X)}) = 0, n \geq 2$ in this case. Such an example is provided by a product of smooth affine curves of any genus. It has the fundamental group of a product of $\mathbb{P}^1$ minus finitely many points.

Question 2.6. In view of the previous discussion, we may ask what complex smooth varieties X are dominated by $h : Y \to X$ proper, with Y smooth, such that Y has an Artin neighborhood, the fundamental group of which is the fundamental group of an Artin neighborhood on a rational variety, or more generally of a variety for which $H^0(\text{smooth compactification}, \mathcal{H}^n_{DR}) = 0$ for $n \geq 1$. We have seen that this would imply vanishing modulo torsion of $c_n(E^{\nabla}_{\rm an}|_{\mathbb{C}(X)})$, $n \geq 2$, or equivalently $CS_n((E, \nabla)) \in H^0(\bar{X}, \mathcal{H}^{2n-1}(\mathbb{Q}(n)))$.

On the other hand, if X is projective smooth, Reznikov's theorem ([19]) shows vanishing modulo torsion of the Chern-Cheeger-Simons classes in $H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Q}(n))$. It is a consequence of Simpson's nonabelian Hodge theory on smooth projective varieties. Our classes $CS_n((E, \nabla))$ live at the generic point of X. We don't have a nonabelian mixed Hodge theory at disposal. Yet one may ask whether it is always true that $CS_n((E, \nabla)) \in H^0(\bar{X}, \mathcal{H}^{2n-1}(\mathbb{Q}(n)))$ for $n \geq 2$, even if many X don't have the topological property explained above.

3. The behavior of the algebraic Chern-Simons classes in families in the regular singular case

The algebraic Chern-Simons invariants $CS_n((E, \nabla))$ have been studied in a family in [5]. Given $f : X \to S$ a proper smooth family, and (E, ∇) a flat connection on X, the Gauß-Manin bundles

$$R^i f_*(\Omega^{\bullet}_{X/S} \otimes E, \nabla_{X/S})$$

carry the Gauß-Manin connection $GM^i(\nabla)$. We give a formula for the invariants $CS_n((GM^i(\nabla) - \text{rank } (\nabla) \cdot GM^i(d)))$ on S, as a function of $CS_n((E, \nabla))$ and of characteristic classes of f. Here (O, d) is the trivial connection.

More generally, we may assume that f is smooth away from a normal crossings divisor $T \subset S$ such that $Y = f^{-1}(T) \subset X$ is a normal crossings divisor with the property that $\Omega^1_{X/S}(\log Y)$ is locally free. Then (E, ∇) has logarithmic poles along $Y \cup Z$ where $Y \cup Z \subset X$ is a normal crossings divisor, still with the property that $\Omega^1_{X/S}(\log(Y + Z))$ is locally free. That is Z is the horizontal divisor of singularities of ∇. The formula involves the top Chern class $c_d(\Omega^1_{X/S}(\log(Y + Z)) \in \mathbb{H}^d(X, \mathcal{K}_d \to \oplus_i \mathcal{K}_{Z_{i,d}})$, rigidified by the residue maps $\Omega^1_{X/S}(\log(Y + Z)) \to \mathcal{O}_{Z_i}$, as defined by T. Saito in [20]. One of its main features is that $CS_n((GM^i(\nabla) - \text{rank } (\nabla) \cdot GM^i(d)))$ vanishes if $CS_n((E, \nabla))$ vanishes. It is

Theorem 3.1 ([5], Theorem 0.1).

$$\begin{aligned} & CS_n(\sum(-1)^i(GM^i(\nabla) - \mathrm{rank}(\nabla)\cdot GM^i(d))) \\ = \ & (-1)^{\dim(X/S)} f_* c_{\dim(X/S)}(\Omega^1_{X/S}(\log(Y+Z)),\mathrm{res})\cdot CS_n((E,\nabla)). \end{aligned}$$

Here $\cdot$ is the cup product of the algebraic Chern-Simons invariants with this rigidified class, which is well defined, as well as the trace f_* to S.

Discussion 3.2. One weak point of the method used in [5] is that it does not allow to understand a formula for the whole invariants $c_n((E,\nabla))$, but only for $CS_n(E,\nabla)$. Indeed, we use the explicit formula studied in [4] to compute it, which can't exist for the whole class in $AD(X)$, as it in particular involves the Chern classes of the underlying algebraic bundle E in the Chow group.

4. The determinant of the Gauss-Manin connection: the irregular rank 1 case

Now we no longer assume that (E,∇) is regular singular at ∞. In the next two sections, we reduce ourselves to the case where $f: X \to S$ is a family of curves, and we consider only the determinant of the Gauß-Manin connection. That is we consider

$$\begin{aligned} \det(GM) \ &:= \ \sum_i (-1)^i c_1(GM^i) \\ &\in \ \mathbb{H}^1(S, \mathcal{O}_S^\times \xrightarrow{d\log} \Omega^1_S) \subset \Omega^1_{k(S)}/d\log(k(S))^\times. \end{aligned}$$

Since the determinant is recognized at the generic point of S, we replace S by its function field $K := k(S)$ in the next two sections. In other words, X/K is an affine curve. Let $\bar{X}/K$ be its smooth compactification.

In this section, we assume that the integrable connection (L,∇) we start with on X has rank 1. Following Deligne's idea (see his 1974 letter to Serre published as an appendix to [7], we first reduce the problem of computing the determinant of the cohomology of ∇ on the curve to the one of computing the determinant of cohomology of an integrable invariant connection, still denoted by ∇, on a generalized Jacobian. More precisely, ∇ has a divisor (with multiplicities) of irregularity $\sum_i m_i p_i$, where $m_i - 1$=irregularity of ∇ in $p_i \in \bar{X}\setminus X$. On the Jacobian $G = \mathrm{Pic}(\bar{X}, \sum_i m_i p_i)$ of line bundles trivialized at the order m_i at the points p_i, there is an invariant connection which pulls back to ∇ via the cycle map. On the torsor $p^{-1}(\omega_{\bar{X}}(\sum_i m_i p_i))$ under the affine group $p^{-1}(\mathcal{O}_{\bar{X}})$, where $p: G \to \mathrm{Pic}(\bar{X})$, one considers the hypersurface $\Sigma : \sum_i \mathrm{res}_{p_i} = 0$. We show that the relative invariant connection $\nabla_{/K}$ on G, while restricted to $\Sigma \subset p^{-1}(\omega_{\bar{X}}(\sum_i m_i p_i))$, acquires exactly one zero which is a K-rational point κ of G. Restricting ∇ to this special point yields a connection $\nabla|_\kappa$ on K. The formula then says that the determinant of the Gauß-Manin connection is the sum of this connection $\nabla|_\kappa$ and of a 2-torsion term, which we describe now. In a given frame of L at a singularity p_i, the local equation of the connection is $\alpha_i = a_i \frac{dt_i}{t_i^{m_i}}$+lower order terms, where t_i is a local parameter and $a_i \in k^\times$. Then the 2-torsion connection $\frac{m_i}{2} d\log a_i \in \Omega^1_{K/k}/d\log K^\times$ does not depend on the choices and is well defined. The 2-torsion term is the sum over the irregular points of these 2-torsion connections. Summarizing, one has

Theorem 4.1 ([7], Theorem 1.1).

$$\det\left(\sum_i (-1)^i H^i(X, (\Omega^\bullet_{X/K} \otimes L, \nabla_{X/K}), GM^i(\nabla)\right)$$
$$= \quad (-1)\nabla|_\kappa + \sum_{i, m_i \geq 2} \frac{m_i}{2} d\log a_i \in \Omega^1_{K/k}/d\log K^\times.$$

Discussion 4.2. The formula described above is global. As such, it has a spirit which is different from Deligne's formula describing the global ϵ-factor of an ℓ-adic character over a curve over a finite field as a product of local ϵ-factors. However, choosing another K-rational point $\kappa' \in p^{-1}(\omega_{\bar{X}}(\sum_i m_i p_i))$, it is easy to write the difference $\nabla|_{\kappa'} - \nabla|_\kappa$ as a sum of explicitly given connections on K. One has $u \cdot \kappa' = \kappa$, where $u = \prod u_i \in p^{-1}\mathcal{O}_{\bar{X}} = \prod_i (\mathcal{O}_{\bar{X},p_i}/\mathfrak{m}_{\mathfrak{p}_i})^\times/K^\times$. Then $\nabla_{\kappa'} = \nabla|_\kappa - \sum_i \mathrm{Res}_{p_i} d\log u_i \wedge \alpha_i$. Correspondingly, one may write the right hand side of the formula above as

$$(-1)\nabla|_{\kappa'} + \sum_i \left(\sup\,(1, \frac{m_i}{2}) d\log a_i + \mathrm{Res}_{p_i} d\log u_i \wedge \alpha_i\right).$$

In particular, the choice of some differential form $\nu \in \omega_{\bar{X}/K} \otimes K(X)$, generating $\omega_{\bar{X}/K}(\sum m_i p_i)$ at the point p_i, defines a trivialization of $\omega_{\bar{X}/K}(\sum_i m_i p_i)$ thus a point $\kappa(\nu)$. We write $\alpha_i = g_i \nu$ with $g_i \in (\mathcal{O}_{X,p_i}/\mathfrak{m}_{\mathfrak{p}_i})^\times$. Then the formula reads

Theorem 4.3 ([7], Formula 5.4).

$$\det\left(\sum_i (-1)^i (H^i(X, (\Omega^\bullet_{X/K} \otimes L, \nabla_{X/K}), GM^i(\nabla)\right)$$
$$= \quad (-1)\det(\nabla)|_{\kappa(\nu)} + \sum_i \left(\sup\,(1, \frac{m_i}{2}) d\log a_i + \mathrm{Res}_{p_i} dg_i g_i^{-1} \wedge \alpha_i\right).$$

5. The determinant of the Gauss-Manin connection: the irregular higher rank case

We assume in this section that we have an affine curve X over $K = k(S)$, k of characteristic zero as in the rank 1 case. The integrable connection (E, ∇) we are given on X has higher rank r.

In the rank one case, for any rank one bundle contained in j_*E, the equation of the connection in a local formal frame at a singular point is of the shape $\alpha = a\frac{dt}{t^m} + \frac{\beta}{t^{m-1}}$, where $m \in \mathbb{N}, a \in K[[t]]^\times$, $\beta \in \Omega^1_K \otimes K[[t]]$. In particular, $(m-1)$ is the irregularity of the connection ([11]). Here t is a local parameter. If $r > 1$, it is no longer the case that j_*E necessarily contains a rank r bundle such that in a local formal frame of this bundle, the local equation has the shape $A_i = a_i \frac{dt_i}{t_i^m} + \frac{\beta_i}{t_i^{m-1}}$, with $a_i \in GL(r, K[[t_i]]), \beta_i \in \Omega^1_K \otimes M(r, K[[t_i]])$. We call an integrable connection (E, ∇) with this existence property an *admissible* connection.

Even if (E, ∇) is admissible, its determinant connection $\det(E, \nabla)$ might have much lower order poles (for example trivial). This indicates that one can not extend directly in this form the formula 4.1. However, assuming (E, ∇) to be admissible and choosing some $\nu \in \omega_{\bar{X}/K} \otimes K(X)$ which generates $\omega(\sum_i m_i p_i)$ at p_i as for formula 4.3, the right hand side of 4.3 makes sense, if one replaces $d\log a_i$ by $d\log\det(a_i)$. Using global methods inspired by the Higgs correspondence between Higgs fields and connections on complex smooth projective varieties ([21]), one is able to prove the "same" formula as 4.3 in the higher rank case on $\mathbb{P}^1$.

Theorem 5.1 ([8], Theorem 1.3). *If (E, ∇) is admissible and has at least one irregular point, and if $\nu \in \omega_{\bar{X}/K} \otimes K(X)$ generates $\omega(\sum_i m_i p_i)$ at the points p_i, then*

$$\det\left(\sum_i (-1)^i (H^i(X, (\Omega^\bullet_{X/K} \otimes L, \nabla_{X/K}), GM^i(\nabla))\right)$$
$$= \quad (-1)\nabla|_{\kappa(\nu)} + \sum_i \left(\sup\left(1, \frac{m_i}{2}\right) d\log\det(a_i(p_i)) + \mathrm{Tr}\ \mathrm{Res}_{p_i} dg_i g_i^{-1} \wedge A_i\right).$$

The connection $\mathrm{Res}\ \mathrm{Tr}_{p_i} dg_i g_i^{-1} \wedge A_i \in \Omega^1_K / d\log K^\times$ is well defined, as well as the 2-torsion connection $\sup\left(1, \frac{m_i}{2}\right) d\log\det(a_i(p_i))$.

However, one needs a different method in order to understand the contribution of singularities in which (E, ∇) is not admissible.

We describe now the origin of the method contained [1]. It is based on the idea that Tate's method ([22]) applies for connections.

Locally formally over the Laurent series field $K((t))$, E becomes a r-dimensional vector space over $K((t))$. The relative connection $\nabla_{K((T))/K} : E \to \omega_{K((t))/K} \otimes E$ is a Fredholm operator. This means that $H^i(\nabla_{X/K}), i = 0, 1$ are finite dimensional K-vector spaces, and that $\nabla_{X/K}$ carries compact lattices to compact lattices. Let $E \cong \oplus_1^r K((t))$ be the choice of a local frame. A compact lattice is a K-subspace of E which is commensurable to $\oplus_1^r K[[t]]$. Given $0 \neq \nu \to\in \omega_{K((t))/K}$, one composes $\nabla_{K((t))/K,\nu} := \nu^{-1} \circ \nabla_{K((t))/K} : E \to E$ to obtain a Fredholm endomorphism. To a Fredholm endomorphism $A : E \to E$, one associates a 1-dimensional K-vector space $\lambda(A) = \det(H^0(A)) \otimes \det(H^1(A))^{-1}$ together with the degree $\chi(A) = \dim H^0(A) - \dim H^1(A)$. We call this a *super-line.* It does not refer to the topology defined by compact lattices. Then one measures how A moves a compact lattice $L \subset E$. First for 2 lattices L and L', one takes a smaller compact lattice $N \subset L \cap L'$ and defines $\det(L : L') := \det(L/N) \cdot \det(L'/N)^{-1}$, where $\cdot$ is the tensor product of super-lines and $\det(L/N)$ has degree $\dim(L/N)$. This does not depend on the choice of N. Then one defines asymptotic superlines. The compact one is $\lambda_c(A) = \det(A(L) : L) \cdot \det(L \cap \mathrm{Ker}(A))$ and the discrete one is $\lambda_d(A) = \det(L : A^{-1}(L)) \cdot \det(V/(L + A(V))$. They do not depend on the choice of

L. Taking $0 \neq \nu \in \omega_{\bar{X}/K} \otimes K(X)$ a rational differential form, and $E_{\min}$ the minimal extension of E, X the complement of the singularities of ν and $\nabla_{X/K}$, one has

$$\det H^*(X/K, E_{\min}) = \otimes_{X \in \bar{X} \setminus X} \lambda_d(\nabla_{K((t))/K,\nu}) \cdot \det(E^{\nabla})^{-1}$$

as a product of discrete lines.

On the other hand, one has the relation $\lambda(A) = \lambda_d(A) \cdot \lambda_c(A)^{-1}$. One easily computes that $\chi(\nabla_{K((t))/K,\nu}) = 0, \lambda(\nabla_{K((t))/K,\nu}) = 1$. Setting

$$\epsilon_x(\nabla_{K((t))/K,\nu}) = \lambda_c(\nabla_{K((t)/K,\nu}) \cdot \det(E^{\nabla})^{-1},$$

this implies immediately the product formula.

Theorem 5.2 ([1], (1.3.1)).

$$\det H^*(X/K, E_{\min}) = \oplus_{x \in \bar{X} \setminus X} \epsilon_x(\nabla_{K((t))/K,\nu}).$$

It remains to endow the local ϵ lines with a connection, compatible with the Gauß-Manin connection on the left. One chooses a section of the vector fields $T_{K/k} \subset T_{X/k}$ and a relative differential form ν which is annihilated by this section. One applies Grothendieck's definition of a connection. The Gauß-Manin connection is given by the infinitesimal automorphism $p_1^* \det H^*(X/K, E_{\min}) \to p_2^* \det H^*(X/K, E_{\min})$ on $K \otimes_k K$, induced by $\tau : p_1^* E_{\min} \to p_2^* E_{\min}, \tau \in T_{K/k}$, which by the choice commutes to $\nabla_{X/K}$. By functoriality of the ϵ lines, this defines a $K \otimes_k K$ homomorphism $p_1^*\epsilon \to p_2^*\epsilon$. This is the ϵ-connection.

Theorem 5.3 ([1], Theorem 5.6). *For an admissible connection of local equation* $A = a\frac{dt}{t^m} + \frac{\beta}{t^{m-1}}$, *with* $m \geq 2$, *the local* ϵ*-connection is*

$$\begin{aligned} \epsilon\left(\nabla_{K((t))/K}, \frac{dt}{t^m}\right) &= \mathrm{TrRes}_{t=0} daa^{-1}A + \frac{m}{2} d\log\det(a(t=0)) \\ &\in \Omega^1_{K/k}/d\log K^{\times}. \end{aligned}$$

The restriction on the choice of ν given by the commutativity constraint with some lifting of vector fields of K is not necessary. The construction is more general.

Given a relative connection $\nabla_{K((t))/K}$, the ϵ-lines for $0 \neq \nu \in \omega^{\times}_{K((t))/K}$ build a super-line bundle on the ind-scheme $\omega^{\times}_{K((t))/K}$. The line bundle obeys a connection relative to K on $\omega^{\times}_{K((t))/K}$. Formula 5.2 identifies line bundles with connections relative to K, where the left hand side carries the constant connection. The choice of an integrable lifting ∇ of $\nabla_{X/K}$ yields a lifting of the relative connection on the ϵ line to an integrable connection relative to k. Formula 5.2 identifies line bundles with integrable connections where the left hand side carries the Gauß-Manin connection ([1], 1.3).

The ϵ lines and connections are additive in exact sequences and compatible with push-downs. By a variant of Levelt's theorem for integrable formal connections, this allows to show that all connections are induced from admissible ones, for which we have the formula 5.3.

Question 5.4. We don't know how to precisely relate the algebraic group viewpoint developed to treat the rank 1 case, and the special rational point found there, with the polarized Fredholm line method which works in general.

Acknowledgements. I thank the mathematicians I have worked with on the material exposed in those notes. A large part of it has been jointly developed with Spencer Bloch. It is a pleasure to acknowledge the impact of his ideas on a programme I had started earlier and we continued together. I thank Alexander Beilinson. An unpublished manuscript of his and David Kazhdan allowed me to understand completely one of the two constructions explained in [15]. His deep viewpoint reflected in [1] changed the understanding of the formula we had as explained in [8]. I thank Pierre Deligne, whose ideas on epsilon factors have shaped much of my thinking. His letter to Serre on the rank 1 case is published as an appendix to [7], but the content of his seminar at the IHES in 1984 has not been available to me. I thank Takeshi Saito for his willingness to explain different aspects of the ℓ -adic theory.

References

[1] Beilinson A., Bloch S., Esnault H., ε-factors for Gauß-Manin Determinants, preprint 2001, 62 pages.

[2] Bloch S., Ogus A., Gersten's conjecture and the homology of schemes, Ann. Sc. Éc.Normale Sup. **IV**, sér. 7 (1974), 181–201.

[3] Bloch S., Applications of the dilogarithm function in algebraic K-theory and algebraic geometry, Proc. int. Symp. on Alg. Geom., Tokyo 1977, 103–114 (1977).

[4] Bloch S., Esnault H., Algebraic Chern-Simons theory. Am. J. of Mathematics **119** (1997), 903–952.

[5] Bloch S.,Esnault H., A Riemann-Roch theorem for flat bundles, with values in the algebraic Chern-Simons theory, Annals of Mathematics **151** (2000), 1–46.

[6] Bloch S., Esnault H., Relative Algebraic Characters, preprint 1999, 25, appears in the Irvine Lecture Notes.

[7] Bloch S., Esnault H., Gauß-Manin determinants of rank 1 irregular connection on curves, Math.Ann. **321** (2001), 15–87, with an addendum: the letter of P. Deligne to J.-P.Serre (Feb.74) on ϵ-factors, 65–87.

[8] Bloch S., Esnault H., A formula for Gauß-Manin determinants, preprint 2000, 37.

[9] Chern S., Simons J., Characteristic forms and geometric invariants, Ann. of Maths II.ser **99** (1974), 48–69.

[10] Cheeger, J., Simons J., Differential characters and geometric invariants, Geometry and Topology, Proc. Special Year College Park/Md. 1983/1984, Lect. Notes Math. **1167** (1985), 50–80.

[11] Deligne P., Équations Différentielles à Points Singuliers Réguliers, Lect. Notes in Mathematics, **163** (1970), Springer-Verlag.

[12] Esnault H., Characteristic classes of flat bundles. Topology **27** (1988), 323–352.

[13] Esnault H., Characteristic classes of flat bundles, II. K-theory, **6** (1992), 45–56.

[14] Esnault H., Coniveau of Classes of Flat Bundles Trivialized on a Finite Smooth Covering of a Complex Manifold, K-Theory, **8** (1994), 483–497.
[15] Esnault H., Algebraic differential characters, College Park/Md. 1983/1984, Lect. Notes Math. **1167** (1985), 50–80. in Regulators in Analysis, Geometry and Number Theory, Progress in Mathematics, Birkhäuser Verlag, **171** (2000), 89–117.
[16] Kato K., Milnor K-theory and the Chow group of zero cycles, Applications of algebraic K-theory to algebraic geometry and number theory, Proc. AMS-IMS-SIAM Joint Summer Research Conf. Boulder/Colo.(1983), Part 1, Contemp. Math., **55** (1986), 241–253.
[17] Levelt G., Jordan decomposition for a class of singular differential operators, Ark. Math. **13** (1975), 1–27.
[18] Rost M., Chow groups with coefficients, Doc. Math., J. DMV, **1** (1996), 209–214.
[19] Reznikov A., All regulators of flat bundles are torsion, Ann. Math., (2) **141** (1995), 373–386.
[20] Saito T., ϵ-factor of a tamely ramified sheaf on a variety, Inventiones Math. **113** (1993), 389–417.
[21] Simpson C., Higgs bundles and local systerms, Inst. Hautes Études Sci. Publ. Math., **75** (1992), 5–95.
[22] Tate J., Residues of differentials on curves, Ann. Sci. École Norm. Sup., sér. 4, **1** (1968), 149–159.