Theorem IX. *In any of the formal systems mentioned in Theorem VI,*[53] *there are undecidable problems of the restricted functional calculus*[54] (that is, formulas of the restricted functional calculus for which neither validity nor the existence of a counterexample is provable).[55]

This is a consequence of

Theorem X. *Every problem of the form* $(x)F(x)$ (*with recursive* F) *can be reduced to the question whether a certain formula of the restricted functional calculus is satisfiable* (that is, for every recursive F, we can find a formula of the restricted functional calculus that is satisfiable if and only if $(x)F(x)$ is true).

By formulas of the restricted functional calculus (r. f. c.) we understand expressions formed from the primitive signs $\overline{}$, $\vee$, (x), $=$, x, y, ... (individual variables), $F(x), G(x, y), H(x, y, z), \ldots$ (predicate and relation variables), where (x) and $=$ apply to individuals only.[56] To these signs we add a third kind of variables, $\phi(x)$, $\psi(x, y)$, $\chi(x, y, z)$, and so on, which stand for functions of individuals (that is, $\phi(x)$, $\psi(x, y)$, and so on denote single-valued functions whose arguments and values are individuals).[57] A formula that contains variables of the third kind in addition to the signs of the r. f. c. first mentioned will be called a formula in the extended sense (i. e. s.).[58] The notions "satisfiable" and "valid" carry over immediately to formulas i. e. s., and we have the theorem that, for any formula A i. e. s., we can find a formula B of the r. f. c. proper such that A is satisfiable if and only if B is. We obtain B from A by replacing the variables of the third kind, $\phi(x), \psi(x, y), \ldots$, that occur in A with expressions of the form $(\imath z)F(z, x)$, $(\imath z)G(z, x, y), \ldots$, by eliminating the "descriptive" functions by the method used in *PM* (I, $*14$), and by logically multiplying[59] the formula thus obtained by an expression stating about each $F, G, \ldots$ put in place of

[54]See *Hilbert and Ackermann 1928*. In the system P we must understand by formulas of the restricted functional calculus those that result from the formulas of the restricted functional calculus of *PM* when relations are replaced by classes of higher types as indicated on page 153 above.

[55]In *1930* I showed that every formula of the restricted functional calculus either can be proved to be valid or has a counterexample. However, by Theorem IX the existence of this counterexample is *not* always provable (in the formal systems we have been considering).

[56]Hilbert and Ackermann (*1928*) do not include the sign $=$ in the restricted functional calculus. But for every formula in which the sign $=$ occurs there exists a formula that does not contain this sign and is satisfiable if and only if the original formula is (see *Gödel 1930*).

[57]Moreover, the domain of definition is always supposed to be the *entire* domain of individuals.

[58]Variables of the third kind may occur at all argument places occupied by individual variables, for example, $y = \phi(x)$, $F(x, \phi(y))$, $G(\psi(x, \phi(y)), x)$, and the like.

[59]That is, by forming the conjunction.

some $\phi, \psi, \ldots$ that it holds for a unique value of the first argument [for any choice of values for the other arguments].

We now show that, for every problem of the form $(x)F(x)$ (with recursive F), there is an equivalent problem concerning the satisfiability of a formula i. e. s., so that, on account of the remark just made, Theorem X follows.

Since F is recursive, there is a recursive function $\Phi(x)$ such that

$$F(x) \sim [\Phi(x) = 0],$$

and for Φ there is a sequence of functions, $\Phi_1, \Phi_2, \ldots, \Phi_n$, such that $\Phi_n = \Phi$, $\Phi_1(x) = x + 1$, and for every Φ_k $(1 < k \leq n)$ we have either

$$\begin{aligned} 1. \qquad & (x_2, \ldots, x_m)[\Phi_k(0, x_2, \ldots, x_m) = \Phi_p(x_2, \ldots, x_m)], \\ & (x, x_2, \ldots, x_m)\{\Phi_k[\Phi_1(x), x_2, \ldots, x_m] = \\ & \qquad \Phi_q[x, \Phi_k(x, x_2, \ldots, x_m), x_2, \ldots, x_m]\}, \\ & \text{with } p, q < k,^{59a} \end{aligned} \tag{18}$$

or

$$2. \qquad (x_1, \ldots, x_m)[\Phi_k(x_1, \ldots, x_m) = \Phi_r(\Phi_{i_1}(\mathfrak{x}_1), \ldots, \Phi_{i_s}(\mathfrak{x}_s))], \\ \text{with } r < k, \quad i_v < k \quad (\text{for } v = 1, 2, \ldots, s),^{60} \tag{19}$$

or

$$3. \qquad (x_1, \ldots, x_m)[\Phi_k(x_1, \ldots, x_m) = \Phi_1(\Phi_1(\ldots(\Phi_1(0))\ldots))]. \tag{20}$$

We then form the propositions

$$(x)\overline{\Phi_1(x) = 0} \,\&\, (x, y)[\Phi_1(x) = \Phi_1(y) \rightarrow x = y], \tag{21}$$

$$(x)[\Phi_n(x) = 0]. \tag{22}$$

In all of the formulas (18), (19), (20) (for $k = 2, 3, \ldots, n$) and in (21) and (22) we now replace the functions Φ_i by function variables ϕ_i and the number 0 by an individual variable x_0 not used so far, and we form the conjunction C of all the formulas thus obtained.

The formula $(Ex_0)C$ then has the required property, that is,

[59a] [The last clause of footnote 27 was not taken into account in the formulas (18). But an explicit formulation of the cases with fewer variables on the right side is actually necessary here for the formal correctness of the proof, unless the identity function, $I(x) = x$, is added to the initial functions.]

[60] The $\mathfrak{x}_i$ $(i = 1, \ldots, s)$ stand for finite sequences of the variables $x_1, x_2, \ldots, x_m$; for example, x_1, x_3, x_2.

1. If $(x)[\Phi(x) = 0]$ holds, $(Ex_0)C$ is satisfiable. For the functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ obviously yield a true proposition when substituted for $\phi_1, \phi_2, \ldots, \phi_n$ in $(Ex_0)C$.

2. If $(Ex_0)C$ is satisfiable, $(x)[\Phi(x) = 0]$ holds.

Proof: Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be the functions (which exist by assumption) that yield a true proposition when substituted for $\phi_1, \phi_2, \ldots, \phi_n$ in $(Ex_0)C$. Let $\mathfrak{I}$ be their domain of individuals. Since $(Ex_0)C$ holds for the functions Ψ_i, there is an individual a (in $\mathfrak{I}$) such that all of the formulas (18)–(22) go over into true propositions, (18′)–(22′), when the Φ_i are replaced by the Ψ_i and 0 by a. We now form the smallest subclass of $\mathfrak{I}$ that contains a and is closed under the operation $\Psi_1(x)$. This subclass ($\mathfrak{I}'$) has the property that every function Ψ_i, when applied to elements of $\mathfrak{I}'$, again yields elements of $\mathfrak{I}'$. For this holds of Ψ_1 by the definition of $\mathfrak{I}'$, and by (18′), (19′), and (20′) it carries over from Ψ_i with smaller subscripts to Ψ_i with larger ones. The functions that result from the Ψ_i when these are restricted to the domain $\mathfrak{I}'$ of individuals will be denoted by Ψ_i'. All of the formulas (18)–(22) hold for these functions also (when we replace 0 by a and Φ_i by Ψ_i').

Because (21) holds for Ψ_1' and a, we can map the individuals of $\mathfrak{I}'$ one-to-one onto the natural numbers in such a manner that a goes over into 0 and the function Ψ_1' into the successor function Φ_1. But by this mapping the functions Ψ_i' go over into the functions Φ_i; and, since (22) holds for Ψ_n' and a,

$$(x)[\Phi_n(x) = 0],$$

that is, $(x)[\Phi(x) = 0]$, holds, which was to be proved.[61]

Since (for each particular F) the argument leading to Theorem X can be carried out in the system P, it follows that any proposition of the form $(x)F(x)$ (with recursive F) can in P be proved equivalent to the proposition that states about the corresponding formula of the r. f. c. that it is satisfiable. Hence the undecidability of one implies that of the other, which proves Theorem IX.[62]

4

The results of Section 2 have a surprising consequence concerning a consistency proof for the system P (and its extensions), which can be stated as follows:

[61] Theorem X implies, for example, that Fermat's problem and Goldbach's problem could be solved if the decision problem for the r. f. c. were solved.

[62] Theorem IX, of course, also holds for the axiom system of set theory and for its extensions by recursively definable ω-consistent classes of axioms, since there are undecidable propositions of the form $(x)F(x)$ (with recursive F) in these systems too.

Discussion on providing a foundation for mathematics (*1931a*)

* * *

According to the formalist view one adjoins to the meaningful propositions of mathematics transfinite (pseudo-)assertions, which in themselves have no meaning, but serve only to round out the system, just as in geometry one rounds out a system by the introduction of points at infinity. This view presupposes that, if one adjoins to the system S of meaningful propositions the system T of transfinite propositions and axioms and then proves a theorem of S by making a detour through theorems of T, this theorem is also contentually correct, hence that through the adjunction of the transfinite axioms no contentually false theorems become provable. This requirement is customarily replaced by that of consistency. Now I would like to point out that one cannot, without further ado, regard these two demands as equivalent. For, if in a consistent formal system A (say that of classical mathematics) a meaningful proposition p is provable with the help of the transfinite axioms, there follows from the consistency of A only that not-p is not formally provable *within* the system A. Nonetheless it remains conceivable that one could ascertain not-p through some sort of contentual (intuitionistic) considerations that are *not* formally representable in A. In that case, despite the consistency of A, there would be provable in A a proposition whose falsity one could ascertain through finitary considerations. To be sure, as soon as one interprets the notion "meaningful proposition" sufficiently narrowly (for example, as restricted to finitary numerical equations), something of that kind cannot happen. However, it is quite possible, for example, that one could prove a statement of the form $(Ex)F(x)$, where F is a finitary property of natural numbers (the negation of Goldbach's conjecture, for example, has this form), by the transfinite means of classical mathematics, and on the other hand could ascertain by means of contentual considerations that all numbers have the property not-F; indeed, and here is precisely my point, this would still be possible even if one had demonstrated the consistency of the formal system of classical mathematics. For of no formal system can one affirm with certainty that all contentual considerations are representable within it.

* * *

(Assuming the consistency of classical mathematics) one can even give examples of propositions (and in fact of those of the type of Goldbach or Fermat) that, while contentually true, are unprovable in the formal system of classical mathematics. Therefore, if one adjoins the negation of such a proposition to the axioms of classical mathematics, one obtains a consistent system in which a contentually false proposition is provable.

* * *

Postscript

I have been invited by the editors of *Erkenntnis* to give a synopsis of the results of my *1931*, which has recently appeared in *Monatshefte für Mathematik und Physik 38*, but was not yet available at the time of the Königsberg conference. That paper deals with problems of two kinds, namely: (1) the question of the completeness (decidability) of formal systems of mathematics; (2) the question of consistency proofs for such systems. A formal system is said to be complete if every proposition expressible by means of its symbols is formally decidable from the axioms, that is, if for each such proposition A there exists a finite chain of inferences, proceeding according to the rules of the logical calculus, that begins with some of the axioms and ends with the proposition A or the proposition not-A. A system $\mathfrak{S}$ is said to be complete with respect to a certain class of propositions $\mathfrak{K}$ if at least every statement of $\mathfrak{K}$ is decidable from the axioms of $\mathfrak{S}$. What is shown in the work cited above is that there is no system with finitely many axioms that is complete even with respect only to arithmetical propositions.[1] Here by "arithmetical propositions" are to be understood those propositions in which no notions occur other than $+$, $\cdot$, $=$ (addition, multiplication, identity, with respect to just the natural numbers), as well as the logical connectives of the propositional calculus and, finally, the universal and existential quantifiers, restricted, however, to variables whose domains are the natural numbers. (In arithmetical propositions, therefore, no variables other than those for natural numbers can occur at all.) Even in systems that have infinitely many axioms, there are always undecidable arithmetical propositions if only the "axiom scheme" satisfies certain (very general) assumptions. In particular, it follows from what has just been said that in all the well-known formal systems of mathematics—for example, *Principia mathematica* (together with the axioms of reducibility, choice and infinity), the Zermelo–Fraenkel and von Neumann axiom systems for set theory, and the formal systems

[1]Under the assumption that no false (that is, contentually refutable) arithmetical propositions are derivable from the axioms of the system in question.

represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality.

However, the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. What, however, perhaps more than anything else, justifies the

[40]Note that there is a close relationship between the concept of set explained in footnote 14 and the categories of pure understanding in Kant's sense. Namely, the function of both is "synthesis", i.e., the generating of unities out of manifolds (e.g., in Kant, of the idea of *one* object out of its various aspects).

acceptance of this criterion of truth in set theory is the fact that continued appeals to mathematical intuition are necessary not only for obtaining unambiguous answers to the questions of transfinite set theory, but also for the solution of the problems of finitary number theory[41] (of the type of Goldbach's conjecture),[42] where the meaningfulness and unambiguity of the concepts entering into them can hardly be doubted. This follows from the fact that for every axiomatic system there are infinitely many undecidable propositions of this type.

It was pointed out earlier (page 265) that, besides mathematical intuition, there exists another (though only probable) criterion of the truth of

1974. But for the usual systems and the various non-constructive extensions that have been considered, it is both much more natural and of greater generality to follow the lead of *Löb 1955*, in which quite elegant abstract derivability conditions (modifying those of Hilbert and Bernays) proved to be the appropriate means for settling the status of various self-referential statements and reflection principles in such systems. Löb's results have been put in an even more general logical context through the work of Solovay (*1976*) on the completeness of certain modal logics under the provability interpretation of the necessity operator.[j] Still, to study the question of applicability of Löb's derivability conditions, one must consider how formal systems may be presented within themselves. Here, as Kreisel has often stressed (see for example his *1965*, page 154), dealing with the question of what constitutes a *canonical presentation* of a formal system becomes the central concern. One solution has been provided in *Feferman 1982*.

One final technical point concerns incompleteness theorems for systems (much) weaker than arithmetic, for example those such as *PRA* which are quantifier-free. Gödel points out that his "most general" version of the second incompleteness theorem can be extended to apply to such systems. For the technical tools needed to deal with related versions of the theorem, see *Jeroslow 1973*.

Solomon Feferman

Remark 2

This remark begins with what Gödel terms "another version of the first undecidability theorem", which concerns the degree of complexity (or "complication", in Gödel's words) of axioms needed to settle problems of "Goldbach type" of high complexity. Gödel had also referred to problems of this type in *1964*, and he explained there (in footnote 42) that by such he meant "universal propositions about integers which can be decided in each individual instance".[k] Most generally, then, such propositions are statements of the form $\forall x R(x)$ with R general recursive (or effectively decidable, by Church's thesis). It is shown in recursion theory that every such statement is equivalent to one of the same form with R primitive recursive, and by definition these comprise the class of Π^0_1 statements. In fact, it is known through the work of

[j]See also *Boolos 1979*.

[k]See p. 269 above. Goldbach's own statement, dating from his 1742 letter to Euler, is the still unsettled conjecture that every even integer is the sum of two primes. (For Goldbach and Euler, 1 was a prime.)

Matiyasevich that every Π^0_1 statement is equivalent to one of the form $\forall x_1 \ldots \forall x_n[p(x_1, \ldots, x_n) \neq q(x_1, \ldots, x_n)]$, where p and q are polynomials with integer coefficients and $n \leq 13$.[l]

Gödel here takes the degree of complexity $d(A)$ of a formula A (in a given language) to be the number of basic symbols occurring in it. In other words, if, for a given basic stock of symbols $s_1, \ldots, s_m$, the formula is written as a concatenation $A = s_{i_1} \ldots s_{i_k}$, then $d(A)$ is defined to be k. For S a finite set of (distinct) formulas $A_1, \ldots, A_n$, considered as a system of non-logical axioms, the degree $d(S)$ is defined to be $d(A_1) + \cdots + d(A_n) + (n-1)$. The theorem stated informally by Gödel is that in order to solve all problems A of Goldbach type of a "certain" degree k, one needs a system of axioms S with degree $d(S) \geq k$, "up to a minor correction". It is not clear what kind of minor correction Gödel intended here, so we do not know just how he would have stated this as a precise result. After examining this question more closely, the authors have arrived at some results of the same character as Gödel's, but not quite as strong as what would be suggested by a first reading of his assertion; we have not, however, been able to establish the latter itself. These various statements and their status are explained as follows.

Let $\mathcal{L}$ be a language with a finite stock $s_1, \ldots, s_m$ of basic symbols, including logical symbols such as '$\neg$', '$\wedge$', '$\forall$', a constant symbol '$\mathbf{0}$', the successor symbol '$'$', a means for systematically forming variable symbols 'v_i' for $i = 0, 1, 2, \ldots$ from the basic symbols,[m] the equality symbol '$=$', and parentheses '(', ')'. $\mathcal{L}$ should also contain symbols, either directly or by definition, for a certain number of primitive recursive functions $f_0, \ldots, f_j$, where f_0 and f_1 are $+$ and $\cdot$, respectively. It is assumed that we have a consistent finite axiom system S_0 in $\mathcal{L}$ which contains (or proves) defining equations for $f_0, \ldots, f_j$, and enough of the axiom system of primitive recursive arithmetic for these functions in order to carry out Gödel's first incompleteness theorem. In particular, S_0 should be consistent and complete for Σ^0_1 sentences (and hence correct for Π^0_1 sentences). For the assertion of Gödel's being examined here, only those systems S are considered which are consistent and contain S_0. Then the following theorem can be proved:

[l]See *Davis, Matiyasevich and Robinson 1976*. Matiyasevich later showed that one could take $n \leq 9$; see his *1977*.

[m]One obvious way to do this is to identify v_i with $v\,\mathbf{0}\underbrace{''\ldots'}_{i}$ where 'v' is a new basic symbol; this makes $d(v_i) = i + 2$. However, there are somewhat more efficient ways of building v_i from basic symbols, so that $d(v_i) = \log_2 i + O(\log_2 \log_2 i)$; we shall assume that such an encoding is being used in the discussion that follows.

(∗) There are positive integers c_1, c_2 such that for all $k > c_2$ and $k_1 = (k - c_2)/c_1$, no finite consistent extension S of S_0 with $d(S) \leq k_1$ proves all true Π^0_1 statements A having $d(A) \leq k$.

The proof of (∗) rests on an examination of Gödel's construction in his *1931* (for the first incompleteness theorem) of a true Π^0_1 statement G_S which is not provable from S for any finite consistent S extending S_0. G_S can be regarded as a statement which expresses that $\text{Conj}(S) \rightarrow G_S$ is not logically provable, where $\text{Conj}(S) = A_1 \wedge \cdots \wedge A_n$ for $S = \{A_1, \ldots, A_n\}$. This construction is uniform in S; that is, for a suitable Π^0_1 formula $B(v_0)$ with at most v_0 free, we have G_S equivalent to $B(\ulcorner \text{Conj}(S) \urcorner)$, where $\ulcorner \text{Conj}(S) \urcorner$ is the numeral in $\mathcal{L}$ for a Gödel number of $\text{Conj}(S)$. Using this, it may be shown that G_S can be chosen with $d(G_S) \leq c_1 d(S) + c_2$, where c_1 is a constant depending on the efficiency of the Gödel numbering of expressions. It turns out that one can take $c_1 = [\log_2 m] + 1$, where m is the number of basic symbols in $\mathcal{L}$.[n] For the usual logical systems m is between 8 and 16, hence $c_1 = 4$. But a first reading of Gödel's assertion under consideration would put $c_1 = 1$ in (∗); call that assertion (‡). (If (‡) holds, Gödel's "minor correction" would simply be c_2.)

The remainder of Gödel's remark does not depend essentially on whether one can obtain (‡) or not, but only that we at least have (∗). For Gödel's way of measuring complexity, the crucial thing is that the degree of complexity of axiom systems needed to establish true Π^0_1 sentences A increases roughly in direct proportion c_1 to the complexity of A, where c_1 is small.

We now pass from these technical questions to Gödel's discussion of their significance. This shifts, in effect, to systems of set theory. The reason is that all of present-day mathematics can be formalized in a relatively simple finite system S_1 of set theory (for example, the Bernays–Gödel system of sets and classes). According to Gödel, it follows from the result (‡), or (∗), that in order to solve problems of Goldbach type which can be formulated in a few pages, the axioms of S_1 "will have to be supplemented by a great number of new ones or by axioms of great complication." Naturally, one would be led to accept as axioms only those statements that are recognized to be evident, though not necessarily immediately so for the intended interpretation (that being, in the case of BG, sets in the cumulative hierarchy together with arbitrary classes of sets). Thus Gödel says that one may be led to doubt "whether evident

[n]Gödel's own numbering of expressions in *1931* is rather inefficient and gives a comparatively large value for c_1. The proof that $c_1 = [\log_2 m] + 1$ suffices relies particularly on the more efficient coding of variables mentioned in footnote m.

axioms in such great numbers (or of such great complexity) can exist at all, and therefore the theorem mentioned might be taken as an indication for the existence of mathematical yes or no questions undecidable for the human mind."

In response to such doubts, Gödel points out "the fact that there *do* exist unexplored series of axioms which are analytic in the sense that they only explicate the content of the concepts occuring in them". As his main example, he cites the axioms of infinity in set theory, "which assert the existence of sets of greater and greater cardinality or of higher and higher transfinite types" and "which only explicate the content of the general concept of set." Here Gödel repeats ideas broached in *1947* and more fully in its revised version *1964*.[o] There he said that the axioms for set theory "can be supplemented without arbitrariness by new axioms which only unfold the concept of set ..." (*1964*, page 264). Moreover, the axioms of set theory are recognized to be correct by a faculty of mathematical intuition, which Gödel says is analogous to that of sense perception of physical objects: "... we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true" (*1964*, page 271). He goes on to note there that "mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned." In *1964* that point is elaborated by reference to Kantian philosophy. But at the end of the present remark, Gödel puts the matter in a way that is supported by the working experience of set theorists who have been led to accept axioms of infinity, namely: "These principles show that ever more (and ever more complicated) axioms appear during the development of mathematics. For, in order only to understand the axioms of infinity, one must first have developed set theory to a considerable extent." The implicit but unstated conclusion of all this is that such axioms of increasing complexity can be used to settle more and more complicated problems of "Goldbach type". In other words, *despite* results such as (∗) (or even (‡), if true) "mathematical yes or no questions undecidable for the human mind" need *not* exist, in principle.

There is one essential difference of aim in the discussions of *1964* and of the present remark, concerning the possible utility of axioms of infinity. In *1964*, Gödel thought that such axioms could be used to decide *CH*, whereas here he aims to use them to solve number-theoretic problems. The study of the so-called axioms of infinity goes back to Hausdorff (*1908*), followed by several publications by Mahlo (*1911, 1912, 1913*). After that, there was only scattered work in the subject until the late

[o]See particularly *1964*, pp. 264–265 and 271–272. Gödel first touched on axioms of infinity in footnote 48a of his *1931* and in *1932b*.

Some remarks on the undecidability results (*1972a*)

1. *The best and most general version of the unprovability of consistency in the same system.*[1] Under the sole hypothesis that Z (number theory) is recursively one-to-one translatable into S, with demonstrability preserved in this direction, the consistency (in the sense of non-demonstrability of both a proposition and its negation), even of very strong systems S, *may* be provable in S, and even in primitive recursive number theory. However, what can be shown to be unprovable in S is the fact that the rules of the equational calculus applied to equations demonstrable in S between primitive recursive terms yield only correct numerical equations (provided that S possesses the property which is asserted to be unprovable). Note that it is necessary to prove this "outer" consistency of S (which for the usual systems is trivially equivalent with consistency) in order to "justify" the transfinite axioms of a system S in the sense of Hilbert's program. ("Rules of the equational calculus" in the foregoing means the two rules of substituting primitive recursive terms for variables and of substituting one such term for another one to which it has been proved equal.)

This theorem remains valid for much weaker systems than Z. With insignificant changes in the wording it even holds for any recursive translation of the primitive recursive equations into S.

2. *Another version of the first undecidability theorem.* The situation may be characterized by the following theorem: In order to solve all problems of Goldbach type of a certain degree of complication k one needs a system of axioms whose degree of complication, up to a minor correction, is $\geq$ k (where the degree of complication is measured by the number of symbols necessary to formulate the problem (or the system of axioms), of course with inclusion of the symbols occurring in the definitions of the non-primitive terms used). Now all of present day mathematics can be derived from a handful of rather simple axioms about a very few primitive terms. Therefore, even if only those problems are to be solvable which can be formulated in a few pages, the few simple axioms being used today will have to be supplemented by a great number of new ones or by axioms of great complication. It may be doubted whether evident axioms in such great numbers (or of such great complexity) can exist at all, and therefore the theorem mentioned might be taken as an indication for the existence of mathematical yes or no questions undecidable for the human mind. But

[1] This has already been published as a remark to footnote 1 of the translation (*1967*, p. 616) of my *1931*, but perhaps it has not received sufficient notice.

Vienna, 19 October 1930

Dear Mr. Behmann,

Dr. Gödel is right now with me and he raises the following objections against my constructivity claim and your proof of it:

He claims to be able to construct examples where clearly an existential claim is proved although one cannot give a construction. The simplest example mentioned by Gödel—which also serves in principle as representative for all the remaining examples—is the following:

Let there be given a one-to-one mapping between the natural numbers and certain rational numbers from the interval $(0, 1)$. Then one can prove in the usual fashion that the given sequence of rational numbers has an accumulation point, although it would not in general be possible to give it explicitly. Let us consider an example where one certainly cannot give an accumulation point explicitly. A number is called Goldbachian if all smaller even numbers are sums of two prime numbers. The sequence of rational numbers is now defined as follows: if n is Goldbachian the number $1/n$ is associated to it; if n is not Goldbachian the number $1 - 1/n$ is associated to it. Then one can prove that this sequence has an accumulation point (and indeed a rational one), that is, either 0 or 1. However, without a solution of Goldbach's problem no accumulation point can be given explicitly.

Gödel has also discussed the issue with Carnap who finds the objection made by Gödel very plausible.

I would be very grateful if you could let me know your reaction to this objection, and in the meantime I will also rack my brains over it.

Gödel also adds that although in this case it is a question of a disjunction between two possibilities, this is not essential; the disjunction between infinitely many cases could also remain undecided.

With most cordial greetings,

Yours,

Felix Kaufmann

Vienna, 19 October 1930

Postscript

Despite the objections presented, I do not doubt at all the correctness of the constructivity theorem. It seems to me that the way to invalidate the objections made lies in the direction that one shows that, in the case where the existential claim resolves into a disjunction of claims, there is no longer an existential claim at all after terminological abbreviations

Introductory note to *1932*

In this short note on intuitionistic propositional logic (**H**),[a] Gödel shows that

(1) **H** cannot be viewed as a system of many-valued logic[b]

(that is to say, we cannot find a finite set M of truth values, with a subset $D \subset M$ of designated values, plus an interpretation of $\rightarrow$, $\wedge$, $\vee$ by binary operations on M and an interpretation of $\neg$ by a unary operation on M, such that $\mathbf{H} \vdash A$ if and only if, for all valuations ϕ in M, $\phi(A) \in D$) and that

(2) there is an infinite descending chain of logics intermediate in strength between **A** (classical propositional logic) and **H**.

From Gödel's argument one sees that one can take for this chain

$$\mathbf{A} = \mathbf{L}_2 \supset \mathbf{L}_3 \supset \mathbf{L}_4 \supset \ldots,$$

where $\mathbf{L}_n$ is the set of propositional formulas identically valid on the

[a]For the formal systems designated by *A* and *H* in Gödel's text we use bold face **A** and **H**, respectively, for greater typographical clarity.

[b]For an introduction to many-valued logics, see, e.g., *Rautenberg 1979*, Chapter III.

Zum intuitionistischen Aussagenkalkül
(*1932*)

[In Beantwortung einer von Hahn aufgeworfenen Frage] für das von A. Heyting[1] aufgestellte System *H* des intuitionistischen Aussagenkalküls gelten folgende Sätze:

[1] *Heyting 1930.*

n-element linearly ordered Heyting algebra (pseudo-Boolean algebra). A finite axiomatization of $\mathbf{L}_n$ ($n \geq 2$) was first given by I. Thomas (*1962*), based on Dummett's (*1959*) axiomatization of the logic **LC** of tautologies on the linear Heyting algebra of order type ω. **LC** can be axiomatized by adding to **H** the axiom $(P \to Q) \vee (Q \to P)$, or, equivalently, the following characterization of $\vee$:

$$(A \vee B) \leftrightarrow [((A \to B) \to B) \wedge ((B \to A) \to A)].$$

$\mathbf{L}_n$ is then axiomatizable as $\mathbf{LC} + F_{n+1}$, where F_{n+1} is as defined in Gödel's note.

It is to be noted that (1) is in fact a consequence of (2), since it is not difficult to show that any propositional logic characterized by a finite set of truth values (in the sense indicated above) and containing **H** has only finitely many proper strengthenings.

Gödel's second result may be regarded as the first contribution to the topic of intermediate propositional logic.[c]

In the last line of his note Gödel stated the disjunction property for **H**; a proof was given by Gentzen in his *1935*.

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[c]There is now an extensive literature on the subject. A survey of the literature up till 1970 may be found in *Hosoi and Ono 1973*. *Minari 1983* presents an extensive bibliography with historical comments. The reasons for studying intermediate logics are mainly technical; for example, intermediate logics give rise to interesting algebraic theories.

On the intuitionistic propositional calculus (*1932*)

[Answer to a question posed by Hahn:] For the system H, set up by Heyting,[1] of the intuitionistic propositional calculus the following theorems hold:

[1] *Heyting 1930*.

I. *There is no realization with* finitely *many elements (truth values) for which the formulas provable in H, and only those, are satisfied (that is, yield designated values for an arbitrary assignment).*

II. *Infinitely many systems lie between H and the system A of the ordinary propositional calculus, that is, there is a monotonically decreasing sequence of systems all of which include H as a subset and are included in A as subsets.*

The proof results from the following facts: Let F_n be the formula

$$\sum_{1 \leq i < k \leq n} (a_i \supset\subset a_k),$$

where Σ denotes the iterated $\vee$-connective and the a_i are propositional variables. F_n is satisfied in every realization with fewer than n elements in which all formulas provable in H are satisfied. For, with every assignment, a_i and a_k are replaced by the same element e in at least one of the summands of F_n, and $e \supset\subset e . \vee b$ yields a designated value for arbitrary b, since the formula $a \supset\subset a . \vee b$ is provable in H. Further, let S_n be the following realization:

$$\text{Elements: } \{1, 2, \ldots, n\}; \text{ designated element: } 1;$$
$$a \vee b = \min(a, b); \quad a \wedge b = \max(a, b);$$
$$a \supset b = 1 \text{ for } a \geq b; \quad a \supset b = b \text{ for } a < b;$$
$$\neg a = n \text{ for } a \neq n, \quad \neg n = 1.$$

Then, for S_n, all formulas of H and the formula F_{n+1}, as well as all F_i with greater subscript, are satisfied, while F_n, as well as all F_i with smaller subscript, are *not* satisfied. In particular, it follows that *no* F_n is provable in H. Besides, the following holds with full generality: a formula of the form $A \vee B$ can only be provable in H if either A or B is provable in H.