What do we mean by equal ?

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Vladimir Voevodsky Memorial Conference

September 11, 2018

So in the type theory that Voevodsky is using, equality ($a = b$) makes sense only among objects of the same type, more precisely the syntax allows only to write it only when a and B have been declared as being object of the same type. And it is not as usual to be thought as a proposition but itself[1] is a type. Now this intrigues me especially since I have always been very interested in using transport of structure, which is somewhat at the basis of the univalence axiom. And so I want to understand what we usually mean by equal and now it's related to this formalism. Now it's soon clear that we used the word equal with different meanings. And that's perfectly alright as Humpty Dumpty was saying to Alice, I would say that when you make a word work hard, we pay him extra, being just we should know at each time and in which sense we are using the word. So, if first I use Zermelo-Fraenkel (ZF) set theory, then everything is a set and by axiom you have that X equals Y means that both are the same element ($X = Y \iff \forall x((x \in X \iff x \in Y))$.

Now that's alright, but it's not the usual way ZF is using the sign "equal", but again there is no harm in that because for me Zermelo-Fraenkel is not, or any formal system is not a tool for writing mathematics, it's the tool for analyzing proof. And also a very useful thing is to give a common meaning to what it means to have a proof. If somebody means "I have a proof of something", for me it means that in principle, it can be formalized in Zermelo-Fraenkel set theory. So at least, I know what is meant. And this has not always been the case in to 18th century, in which people were looking at analysis, it was not all clear what a proof was meaning. So that's one meaning of the proof. But as I was telling it's not the usual meaning and it was not the usual meaning both before or after Zermelo-Fraenkel set theory. If I go a two thousands years before Zermelo-Fraenkel, when we say in euclidean plane geometry that two triangles are equal, we don't mean that they are the same triangle. What we mean in fact was not completely clear. It more or less means that we can move one to another by an isometry and everything we care about will not change. In modern terms, we would say that there is an automorphism of the plane mapping one triangle to the other but this is completely anachronistic because first, moving the whole plane or space would not have been seen as very physically possible whereas it's possible to move a piece of paper with a triangle onto another one. And also, the concept of the whole space was not well considered because of some taboo on actual infinity.

But so we have this form long time ago and if I take a more recent example, when you write something like $\pi_n(S^n) = \mathbb{Z}$ or if we write a Künneth formula $H_*(X, F) = H_*(X, F)_*(Y, F)$ for homology with coefficients in a field, we surely don't mean that they are equal in terms of the Zermelo-Fraenkel theory. They are a different kind of objects. So in those cases somewhat as for the triangles, we mean... the weak sense is that there exists an isomorphism between both. But now in all of those cases, existence of isomorphism is a weak statement and as a rule, we mean more. So for instance, if I look at again a triangle in plane geometry, if I look at some triangles (*drawing two triangles on the blackboard*), one has this criterion of equality of triangles that if one has the same angle and the same length of the two sides (*symbolizing those constraints with signs on the blackboard*) then

Référence : https://www.youtube-nocookie.com/embed/WfDcrN5_1wA, septembre 2018.

Transcription : Denise Vella-Chemla, juin 2022.

1. the equality $a = b$

the two triangles are equal, this is a weak statement : you don't just want to know that *there is* an isometry between the triangle, you want to note that this is isometry will carry this point to this point this point to this point and this point to this point and this is a stronger statement. For instance if you look at some triangle with one angle and two equal sides, you can use the criterion of equality of triangles to write $ABC = ACB$. And so you will know that the criterion tells you there is an isometry mapping A to A, B to C and C to B. So you get the symmetry property from the equality criterion that the angle $\widehat{B}$ and $\widehat{C}$ are equal. So that's one aspect that you want to care what isomorphisms you have between objects, because you also care about symmetries between objects. And I said those are weak statements because it's in fact useless to just know existence of some isomorphism : here (*showing the identity* $\pi_n(S^n) = \mathbb{Z}$ *on the blackboard*) you would like to know for instance that the identity map of S^n correspond to one in $\mathbb{Z}$ if you want to apply the statement and here (*showing the Künnteh identity on the blackboard*), you want to know that you have some very explicit isomorphism mapping cohomology classes α and β (*on the right side of the equality*) to some product of the inverse images of α and β by the two factors (*on the left side of the equality*).

So what I want to remember of this is that very often, equality means existence of isomorphisms, but that we also care to understand the isomorphisms between those objects not only the existence. And this corresponds quite well to what you have here. First to speak of isomorphisms of objects, they have to have the same kind of objects : this is the condition that equality between a and b is meaningful only if we have objects of the same type. And the fact that in these cases, for those examples (*sphere and Künneth equality*), equality is a type, here it would be interpreted as it is the set of isomorphisms between a and b. And often you will not just prove equality, but you will exhibit some object of that equality (*writing* $\alpha \ : \ a = b$ *and explaining that*) this double dot means "is of type" meaning you construct an isomorphism and you will use it later, the one you've constructed.

Now, one thing which is already appearing in this formalism... You have this story : you start with some type A, you have two objects of type A $(a, b \ : \ A)$; you can consider the type equality of a and b ; and if you consider $a = a$, in it, you have isomorphism of identity : $id_a \ : \ a = a$. This tells you that you can iterate the process, in this formalism, we can consider two objects of type $a = b$, consider the type of equality between those objects and iterate. And this occurs in various guys quite often, you have the story of homotopy theory, where equality of X and Y corresponds to homotopy equivalence between X and Y. If you have two equivalences, you can consider equivalence between homotopies, if you have two homotopies between homotopies, you can consider homotopies between sames, and so on. So there, you can go further and further. So that's one context, there are other contexts where you have the same story : if you consider categories $\mathcal{C}$ and $\mathcal{D}$, equality essentially corresponds to have an equivalence of categories between them. Now, if you have two equivalences of categories, you can wonder about isomorphism between the equivalences and here, the story stops there but for other categories, it would continue for a while. Also another example if you consider a complex K^* and you consider cohomology classes of this complex (*writing* $H^n(K^*)$), often you want rather to consider cocycles (*writing* c *and* c') and now, an equality between cocycles (*writing* $c = c'$) would be a cochain of one previous dimension such that $da = c - c'$, so something showing that the two have the same cohomology class. And now you can continue. If you have two such a's, their difference is a cocycle and you may want it to be a co-boundary and so on. So this is another setting when you can iterate this notion of equality, and it's often very useful to keep track of it. For instance if I look at some kind of cohomology, if you have cohomology classes h and h', and proof they are equal, more precisely if you prove in two different ways that they are equal, often it will... from this, you get one cohomology class in one lower dimension, the idea being that you look at cocycles c and c' corresponding to your cohomology classes, one proof of equality of the cohomology classes will give you that the difference of the cocycles is d of something (*writing* da), the other prove of equality of the cohomology classes will give that it is d of something else (*writing* da') and now, you have $a-a'$, the difference between the two proofs which is a cohomology class in H^{n-1}.

So I can give one not very important example : consider cohomology of a space with coefficient in $\mathbb{Z}/2$ (*writing* $H^*(X, \mathbb{Z}/2)$), now you know that if we have two cohomology classes, the cup product in cohomology mod 2 will be commutative. And it's true only because of $\mathbb{Z}/2$, otherwise it would have a minus sign (*writing* $\alpha \cup \beta = \beta \cup \alpha$). Now if you consider a class cup itself (*writing* $\alpha \cup \alpha$), this gives you to prove $\alpha \cup \alpha = \alpha \cup \alpha$ it's an obvious thing. And then this proof of commutativity and so from your class α you started with, not only you get a cut product in H^{2n} but the fact that you have the proof of equality of $\alpha \cup \alpha$ which itself give will give you a cohomology class in H^{2n-1}, which is some Steenrod square of α. So it's useful to keep track of those equality between equalities between equalities in a number of different contexts.

Now there is one thing we care about equality and it's the reason we use the same word to describe different situations : if you have some equality $A = B$, you care that everything you do on A, you can do on B in the same way, that's transport of structure, or not having to care which one you use, or univalence axioms. And the usual way of handling this is restricting the language which weneath allow to use, like for instance in Zermelo-Fraenkel theory. Let me give an example. Suppose you have two groups G and G' and you have an isomorphism between them. Then everything you do on G is true in the same way on G' except if you use the full language of Zermelo-Fraenkel, this is clearly false; you can wonder for instance if this particular set is an element of G (*writing* $\{\varnothing\} \in G$*? and* $\{\varnothing\} \in G'$*?)* and you can even isomorphize between the two groups and this is a wrong question so it will not be equivalent for G and G'.

So you should know when it's ok to use a transport of structure and tell that anything on G can be transported to G' and in chapter 4... Oh yes, first I should tell a little more about restricting the language one allows oneself to use. This is something which is very familiar; for instance, consider the notion of integers $\{0, 1, 2, \ldots\}$; there are a number of different definitions : if you like the definition of Frege, you have the empty set, the set reduced to empty set, and so on (*writing* $\varnothing \quad \{\varnothing\} \quad \{\varnothing, \{\varnothing\}\} \ldots$). And in general, $n+1$ is equal to the set of integer smaller or equal to n (*writing* $n+1 = \{i \mid i \leqslant n\}$); so that's one definition. If you like Russel definition, you will tell that it's the class of all sets with the same cardinality as some finite set. And a variant of it is the one used by Bourbaki : you have to use the universal fact...(*Somebody in the audience tells something about what is written on the blackboard : the second one is an idea of Frege too.*). In Bourbaki there is the axiom of universal choice in the formalism selecting an element in every class and then in Bourbaki, an integer is a representative of such a class of equipotent finite sets. Now when you write an article and you use integers, you don't tell which one of the definitions you will be using, but this means that you will not allow yourself to use anything where these definitions would be relevant. For instance if you have an homotopy theorist and then you want to consider the set $\{0, \ldots, n\}$, you will call it maybe Δ_n, you will not call it $n+1$. So restricting to language one use is something but familiar we do all the time, and similarly to tell when one can use transport of structure, Bourbaki made the valiant effort to codify what how one has to restrict the language one was using. But I found it a little ironic that the formalism he was giving was adequate until about the time where this chapter was published, only! So one important thing always is first... I will describe a little what he was doing. And first one has to tell what is a structure on some sets, so you have some sets $E_1 \quad E_\ell$, and you want to define what is a structure of some kind on these sets. Now I simplify a little : Bourbaki was also using some auxiliary sets, like the integers or some ground field which you would be using. So the first thing one has to tell is what kind of object the structure is and this is telling that it's belong to some set built from the basic set by some list of operations; so which one can tell the type of object the set is. But it comes that constructions which Bourbaki is using are very few : there was just product (*writing* $\times$) and and set of all subsets (*writing* $\mathcal{P}$) ; so before telling what a group is, I will first tell that the structure is an element of the set of subsets of E times E times E (*writing* $E \times E \times E$) which I have to think of the structure s as a graph of a map of the composition law from $E \times E$ going to E (*He writes this* $s \in \mathcal{P}(E \times E \times E)$). Now when I have told what the type of the structure is, one thing I get is that if I have a bijection between F_1. *(Erasing a part of what is written vertically* $E_\ell \to F_\ell$*)*... Let me just think to one basic set with another example if you want a structure of a topological space, your s will be in $\mathcal{P}\mathcal{P}(E)$, the set of all open subsets for the topology. Now if you have a bijection from your basic set to another set (*showing* $E^s F$*)* and some s of which you have prescribed the type, this will allow you to go from a structure on E to a structure on F. If you have a subset of E, it gives you a subset of F. If you have a set of subsets, it gives you a set of subsets. But for that, you have of course to prescribe what type of objects you are considering. And now the axioms of the structure should be such that the axioms are true for some s if and only if they are true for some transported s and there is a lot of criterion to know when this is all true and in practice we do this without thinking. We don't put stupid axioms like φ is an element of our basic set (*writing* $\varphi \in E$) or something like that which would not be transportable. No. One basic thing here if you want to make statements which are transportable, one important thing is that you can require equality only between objects of the same type. And this corresponds to also this condition that you should never had an equality between objects of different types (*showing the first definition of* $a = b$ *on the first blackboard*). Here (*showing* $E \to F$, $s \to t$), it would not be transportable.

Now, I was telling that this was adequate almost upto when Bourbaki's book was published, and essentially for two reasons, one is that the type construction allowed are much too restrictive in practice : for instance if you look at some topological space X which you take fixed and consider sheaves on X, so part of the data of the data (or presheaves, what you want) for each open set, you want some set $\mathcal{F}(U)$ and you know what an isomorphism of sheaves is supposed to do from $\mathcal{F}(U)$ to $\mathcal{F}(V)$, if you have $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$, it should make no sense to require equality between s and t ($s = t$) because this, if you have an isomorphism of this sheaf in another one, this kind of condition will not be respected. So this is a basic example where this formalism is unsufficient. And I would say that when you are in this setting the notion of isomorphism follows, is automatic : you don't have to define what isomorphism is; it is just isomorphism of... a bijection from E to F will be an isomorphism of s and t if it maps s to t. So sheaves are an example when things are outside of this formalism and also some constructions we like to do are not allowed into formalism in the sense that we don't want to prescribe a type or it would be artificial. For instance if have over a field two vector spaces (*writing* $k \quad V \quad W \quad V \otimes W$) and you want to consider their tensor product, the only thing you care is that you have a map from V tensor W to the tensor product (*writing* $V \times W \to V \otimes W$) with some universal property. Now you might construct the object first by telling that you take the free group generated by pairs of elements and you would get something of some type over the basic set but that's rather artificial and that what's Bourbaki does to define tensor product. But you just care about universal property with define V tensor W up to unique isomorphism and this is also outside of the formalism which Bourbaki uses.

And one thing I wonder is that if this was a ? and also a psychological barrier to the notion of analytic space with nilpotents or scheme with nilpotents because if you want to consider just complex reduced analytic space X the structure which is enough to give one is a function from X to $\mathbb{C}$

which is analytic (*writing* $f : X \to \mathbb{C}$ *analytic*). So the structure is some set of functions, it has to satisfy a number of axioms and this was fitting with the formalism.

But if you have a nilpotent, you have to give some sheaves with some property and this falls outside of the formalism which was used. But in practice we know, you have enough experience to know what we can tell and what we should not tell so that whatever we do in one situation can be transported to an isomorphic situation, even if there is no actual reference in the literature justifying it in all cases in which we want to do it. Now if we go to categories instead of sets with structure, then the situation is not as good.

Suppose you have two categories ; if you have an equivalence of categories between them, and if you do think in a categorical frame of mind (*writing* $\mathcal{C} \approx \mathcal{D}$), everything you do on $\mathcal{C}$ should be able to be transported to $\mathcal{D}$. And we somewhat know when things we tell can be indeed transported but we don't always respect those rules or we are floppy. For instance if we have some functor... Yes... One basic rule which is not always respected is that it's almost always bad to write equality (*writing* $X = Y$) between objects ; existence of an isomorphism or giving an isomorphism, that's good. But equality between objects is dangerous. So for instance, if you have a functor between categories (*writing vertically* $\mathcal{C} \to \mathcal{D}$) and you want to define the fiber of $\mathcal{C}$ to $\mathcal{D}$ on an object of $D \in \mathcal{D}$, it's a very bad idea to tell that you look at the X such that $F(X)$ equals D (*writing on the left of* $\mathcal{C}$ *the set* $\{X \mid F(X) = D\}$) ; at least it's a bad idea except if this functor has some nice property like being a fibration. The notion which is good is to consider an object of X together with an isomorphism of $F(X)$ with D (*writing* $F(X) \mapsto D$) ; this definition (*showing the first idea*) is not compatible by replacing $\mathcal{C}$ and $\mathcal{D}$ by something equivalent, and this (*showing* $F(X) \mapsto D$) is compatible ; at least, if you replace $\mathcal{C}$ and $\mathcal{D}$ by equivalent things, you will get an equivalent end result. And it's quite important that the statement one makes are invariant by equivalence the same way that it's important when we speak of integers we don't use a specific definition, but something which works equally well for all of them. Because even for very standard categories, there are small variants of definitions which give different categories and it would be a messy feat to tell which exact definition you are using.

So we somewhat know what we should do for categories but here, something new happens is that in the case of structure essentially if we apply some hygienic rules, we know transport of structures will work well. For categories, there are cases where the construction is invariant by this kind of equivalence but where it's not completely obvious on the face of it. So let me give some examples. If you have such categories, you can consider the nerve $\mathcal{NC}$ of $\mathcal{C}$, the nerve $\mathcal{ND}$ of $\mathcal{D}$, and you will get a map of spaces from one to the other and this will be an homotopy equivalence. It's not difficult, but it's not completely obvious on the face of it. For instance if you take here just one object and one automorphism (*showing the side concerning* $\mathcal{C}$ *and* $\mathcal{NC}$) and here you take a few objects with just one isomorphism between two of them (*showing the side concerning* $\mathcal{D}$ *and* $\mathcal{ND}$), here you will get something like a point, and here you get something contractible but still, you might have to prove it. It's not a difficult proof but still in difference with the case of structure, it's not a tautology. And another more useful example : suppose you have some abelian category $\mathcal{A}$, and you want to define the higher category group of $\mathcal{A}$ (*writing* $K_\bullet(\mathcal{A})$), or better you want to define the $K_\bullet$-theory spectrum of $\mathcal{A}$. So somewhat the intuition of what you want to do is somewhat clear. Here in this category you have objects (*writing* A *under* $\mathcal{A}$), you want each object to define... yes if I think to this (*writing* $K_\bullet(\mathcal{A})$) as an infinity category groupoid ahead which I take $2\pi i$, I want to attach to each object a virtual object (*writing* $[A]$ *under* $K_\bullet(\mathcal{A})$), I want a short exact sequence (*writing* $A' \to A \to A''$ *under* A) should give me something and it should be universal in some way.

To touch some intuition, but which I don't think one has formalized, and then they are a number of definitions which I guess always rely on the same intuition, the Q construction of Quillen or the Wegge-Olsen (?) construction of K-theory groups. And in all case it's true that if you have an equivalence of abelian categories, you will get the same K-groups but it requires a proof which is not difficult but which is not completely obvious. And I would say that the situation gets somewhat worse if you are going to higher categories and wondering what definition you want to make exactly.

Now here I should begin by telling that I have only a partial understanding of what I am telling but my take of the univalence axioms is that one wants to have a language which one has a notion of equality which incorporates all those difficulties, and where it is impossible to state things which would contradict transport of structure. I must say I get somewhat worried that it reminds me very much of the novlangue of 1984 of Orwell where the ideal was to have a language where it was impossible to express heretical thoughts. The reason I find it worrying is that there are those examples like definitions of K-theory, groups, nerve of a category, where the thought is not heretical but it's not so obvious it's not heretical and so the language we want to do should make it easy and not difficult to tell of those things. And notice is more something which I would like to be explain things but I wonder for a while and try sometimes to reensure me without the axioms used are strong enough for the intended purpose but maybe I am misunderstanding the intended purpose sometime. So of course I am willing to believe that you can reconstruct or model Zermelo-Fraenkel inside this type theory so that in such sense it is strong enough but it's also not the point if you have to use this beautiful trial to understand what equals means and then throw it away to do what you want to do, that's not the point. And then, I have some concrete questions where I wonder how expressible it is and how much one can prove things which should be obvious or whether some action might maybe missing. So I want to give one example of what I have in mind and maybe I will be told that it's all easy. So suppose you have some type and some element of that type. So the intended picture for me (*writing* $a : A$) is that this (*showing* A) is a topological space representing some homotopy type and here, I have a point of that space (*showing* a). Then in that case the type $a = a$ corresponds to the loop space of A based at this point (*writing* ΩA) and in it, yuo have the identity which is of that type (*writing* $id_a : a = a$). So you see you start from a type with something of that type, and you get again a type with something of that type, it should correspond to the loop space construction, clearly you can iterate this. So you iterate a number of times and you define a $id_a^n : a =_n a$ at level n and which is an object of some type corresponding in some iterate of the loop space. Now suppose that you know, for example, in Zermelo-Fraenkel theory that in $\pi_{n+k}(S^n)$ you have some element (*writing* $\alpha \in \pi_{n+k}(S^n)$) and even possibly that you have it realised as an explicit map from S^{n+k} to S^n. So the question is "from this, can you into type theory (recover) construct something of type (*writing* $[\alpha] : a =_n a \to a =_{n+k} a$) ?", corresponding to the picture that if you have an element of $\pi_{n+k}(S^n)$, this gives you a map from the n full top space to the $n+k$ full top space. So the question is "can this be done ?" and also one would like this to have some universal property, at least that if you have two elements which are distinct (*writing* $\alpha \neq \beta$), there should be instance where the equality between the map you get being equally to contradiction. For me, it is related to some dream that I have also : in Lurie's formalism for higher categories, a thing like this is somewhat put inside the formalism that use a formalism of spaces or simplicial spaces and then this has to come from free. But I still wonder whether one could give or has been given some definition of higher categories, so that statement of this kind can be used to define homotopy group of sphere, possibly compute sometime. In some easy case for instance, if you look at path tree of S^2, this can be done by hand. But in general I have no idea. And I will stop there.