

# Goldbach's conjecture and network centers

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Goldbach's conjecture asserts that every even integer  $n \geq 4$  is the sum of two prime numbers. In the following,  $n$  will designate an even number  $\geq 6$ .

## 1 Definitions

We note  $\mathcal{P} = \{2, 3, 5, \dots\}$  the set of prime numbers.

In the following, we define a grid in which coordinates sets are :

$$\begin{aligned} X_n &= \{p_k \mid 3 \leq p_k \leq n - 3, p_k \in \mathcal{P}\} \cup \{0, n\}, \\ Y_n &= \{n - p_k \mid p_k \in X_n\} \cup \{0, n\}. \end{aligned}$$

*Fact* :  $\forall n \geq 6, X_n \neq \emptyset, Y_n \neq \emptyset$ .

We define a subset  $R$  of points of the euclidean plane  $\mathbb{N}^2$  :

$$R = \{(x, y) \in \mathbb{N}^2 \mid x \in X_n, y \in Y_n\}.$$

We need a function  $f$  to number points of  $R$  :

$$\begin{aligned} f : R &\rightarrow \mathbb{N} \\ (x, y) &\mapsto x + y \cdot \text{Card}(X_n) + 1 \end{aligned}$$

We associate to  $R$  the graph (called squared grid-graph)  $G = (S, A)$  with :

$$\begin{aligned} S &= \{s_i, 1 \leq i \leq (\text{Card}(X_n))^2\}, \\ A &= A_H \cup A_V \end{aligned}$$

where  $S$  is the set of vertices and  $A$  is the set of edges (horizontal and vertical) of the graph :

$$\begin{aligned} A_H &= \{(s_i, s_j) \mid (f^{-1}(s_i) = (x_i, y_i)) \wedge (f^{-1}(s_j) = (x_i, y_i + 1))\} \\ A_V &= \{(s_i, s_j) \mid (f^{-1}(s_i) = (x_i, y_i)) \wedge (f^{-1}(s_j) = (x_i + 1, y_i))\} \end{aligned}$$

We restrict the squared grid-graph  $G$  to the punctured triangular graph  $T = (S_T, A_T)$  :

$$\begin{aligned} S_T &= \{s_i \mid (f^{-1}(s_i) = (x_i, y_i) \in S) \wedge (x_i \geq y_i)\} \setminus \{(\text{Card}(X_n) - 1, 0)\}, \\ A_T &= \{a_i \mid (a_i = (s_i, s_j)) \wedge (a_i \in A) \wedge (s_i \in S_T) \wedge (s_j \in S_T)\} \end{aligned}$$

We must labeled edges of  $A_T$  with gaps between successive prime numbers. For this, we define the gaps sequence between successive primes (the first gap is the prime number 3 and the last gap is the difference between  $n$  and the greatest prime number  $\leq n - 3$ ) :

$$\Delta_n = s_k(n) \text{ telle que } \begin{cases} s_1 = 3 \\ s_k = p_{k+1} - p_k \text{ for } 2 \leq k \leq \text{Card}(X_n) - 2 \\ s_{\text{Card}(X_n)} = n - \max\{p_k | (3 \leq p_k \leq n - 3) \wedge (p_k \in \mathcal{P})\} \end{cases}$$

The labeling function of the graph edges is thus :

$$g : \begin{array}{ll} A_T & \rightarrow \mathbb{N}^2 \\ ((x_i, y_i), (x_i, y_i + 1)) & \mapsto \delta_i \in \Delta_n \\ ((x_i, y_i), (x_i + 1, y_i)) & \mapsto \delta_i \in \Delta_n \end{array}$$

It turns out that the distance between two vertices  $s$  and  $s'$  of  $S$  corresponds to the Manhattan distance (or norm 1 distance) in the euclidean plane :

$$d : \begin{array}{ll} S \times S & \rightarrow \mathbb{N}^+ \\ ((x_i, y_i), (x_j, y_j)) & \mapsto |x_j - x_i| + |y_j - y_i|. \end{array}$$

The distance between two vertices is thus the length of a path that has a minimum distance leading from  $s$  to  $s'$  in the graph  $T$ .

Let  $s \in S$ . The eccentricity of  $s$ , denoted by  $e(s)$  is defined by :

$$e(s) = \max\{d(s, s'), \forall s' \in S\}.$$

Let  $E_T$  the set containing  $T$ 's vertices eccentricities :

$$E_T = \{e(s) | s \in S_T\}.$$

A center of a graph is a vertex of this graph, which has an eccentricity which is equal to the minimum of the eccentricities set of all vertices of this graph.

The  $s$  vertex is a center of the graph  $G = (S, A)$  if and only if

$$e(s) = m \text{ avec } m = \min\{e(s') | s' \in S\}.$$

In the particular case of the graph  $T$  that is of interest for us, a vertex  $s$  of  $S_T$  is a center of  $T$  if and only if  $e(s) = \min(E_T)$ .

## 2 Assertions

**Lemma :** *The  $G$  (squared) graph is strongly connected.*

**Lemma :** *The  $T$  (punctured triangular) graph is strongly connected.*

**Lemma :** *The set of vertices  $S_T$  and the set of edges  $A_T$  being two not empty sets, the set of eccentricities of vertices of the graph  $E_T$  is well defined and not empty. The set of vertices that have an eccentricity that is minimum is well defined and is not empty.*

**(Lemma :**  *$S$  (the source) and  $P$  (the sink) are the graph vertices that have an eccentricity which is equal to the maximum of the set of eccentricities ; this maximum eccentricity is equal to  $2n$ .)*

**Theorem 1 :** *The punctured triangular graph centers correspond to Goldbach components of  $n$ . The path that joins the source vertex  $S$  (that has  $(0,0)$  as coordinates) is composed of a part of the path containing only horizontal edges and whom labels sum is equal to  $p$  with  $p$  a prime number, and a part of the path containing only vertical edges and whom labels sum is equal to  $q$  with  $q$  also prime.*

Follows from definitions.

**Theorem 2 :** *The punctured triangular graph contains at least one center.*

It follows from the existence of a minimum and a maximum for every not empty set of integers.

**Theorem 3 :** *Every even number admits at least one Goldbach decomposition.*

Follows from theorem 1 (that asserts the isomorphism between Goldbach components of  $n$  and centers of the graph  $T$ ) and theorem 2 (that asserts mandatory existence of at least one center for the graph  $T$ ).

### 3 Illustrations : case $n = 16$

$$X_{16} = \{0, 3, 5, 7, 11, 13, 16\}$$

$$Y_{16} = \{16, 13, 11, 9, 5, 3, 0\}$$

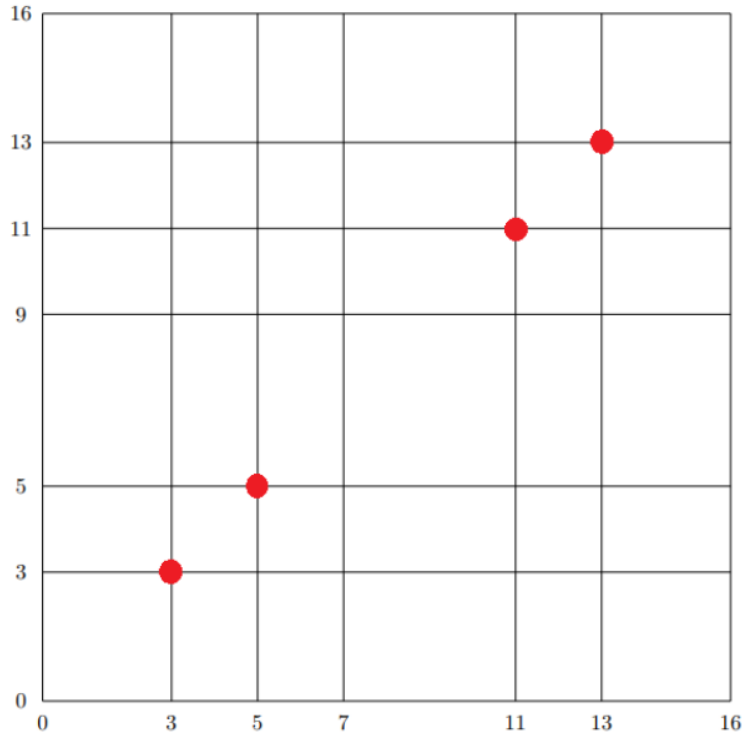


FIGURE 1 : The example of the square for  $n = 16$ . Points corresponding to Goldbach components of 16 that are  $\{3, 5, 11, 13\}$ , i.e. points  $(3, 3), (5, 5), (11, 11), (13, 13)$  are colored in red on the ascending diagonal.

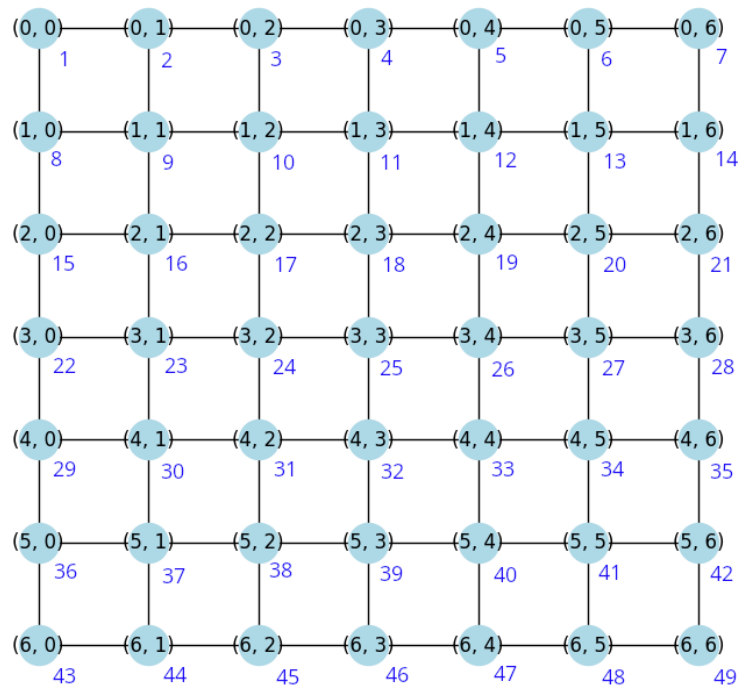


FIGURE 2 : Numbering of graph vertices





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## Appendice 1 : Program that draws punctured triangular graphs and that visualizes their centers.

```
import networkx as nx
import matplotlib.pyplot as plt

def premier(atester):
    k = 2
    if atester in [0, 1]: return False
    if atester in [2, 3, 5, 7]: return True
    while True:
        if k * k > atester: return True
        else:
            if atester % k == 0: return False
            else: k = k + 1

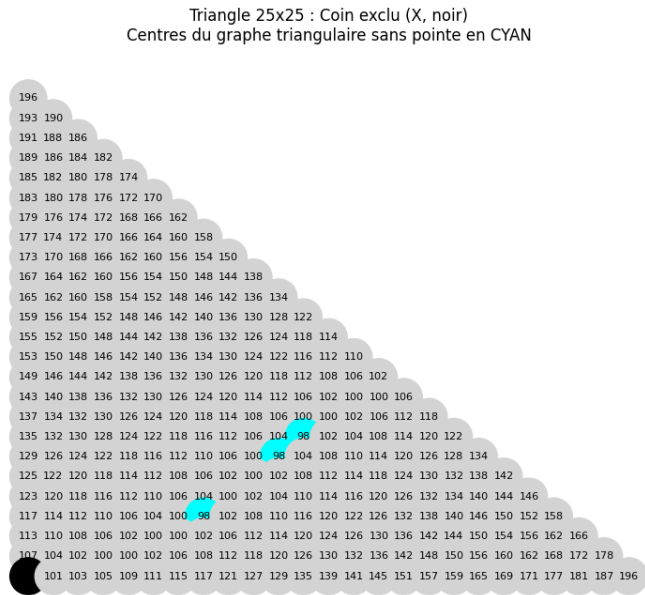
for n in range(8,62,2):
    L=[0]
    for k in range(3,n-1,2):
        if premier(k):
            L.append(k)
    L.append(n)
    print('L = ',L)
    ecarts = []
    for indice in range(1,len(L)):
        ecarts.append(L[indice]-L[indice-1])
    print('ecarts = ',ecarts)
    poids_fixes = ecarts
    SIZE = len(poids_fixes) + 1
    G_full = nx.grid_2d_graph(SIZE, SIZE)
    nodes_to_keep = [(i, j) for (i, j) in G_full.nodes() if i >= j]
    G_complet = G_full.subgraph(nodes_to_keep).copy()
    for (u, v) in G_complet.edges():
        (r1, c1), (r2, c2) = u, v
        if r1 == r2: # Horizontale
            G_complet.edges[u, v]['weight'] = poids_fixes[min(c1, c2)]
        else: # Verticale
            G_complet.edges[u, v]['weight'] = poids_fixes[min(r1, r2)]
    node_isele = (SIZE - 1, 0) # Le coin bas-gauche
    G_tri = G_complet.copy()
    if node_isele in G_tri:
        G_tri.remove_node(node_isele) # Supprime le noeud ET ses aretes
    path_lengths = dict(nx.all_pairs_dijkstra_path_length(G_tri, weight='weight'))
    eccs_tri = {node: max(dists.values()) for node, dists in path_lengths.items()}
    min_ecc = min(eccs_tri.values())
    centers_triangle = [n for n, ecc in eccs_tri.items() if ecc == min_ecc]
    plt.figure(figsize=(10, 8))
    pos = {(i, j): (j, -i) for (i, j) in G_complet.nodes()}
    node_labels = {n: f"{eccs_tri[n]}" for n in G_tri.nodes()}
    node_labels[node_isele] = "X" # Marquer le noeud exclu
    node_colors = []
    for node in G_complet.nodes():
        if node == node_isele:
            node_colors.append('black') # Le point exclu est noir
        elif node in centers_triangle:
            node_colors.append('cyan') # Les centres du "tri" sont cyan
```

```

else :
    node_colors.append('lightgrey') # Le reste est gris
nx.draw_networkx_edges(G_complet, pos, alpha=0.5, edge_color='gray')
edge_labels = nx.get_edge_attributes(G_complet, 'weight')
nx.draw_networkx_edge_labels(G_complet, pos, edge_labels=edge_labels,
                             font_color='red', font_size=8)
nx.draw_networkx_nodes(G_complet, pos,
                       node_color=node_colors,
                       node_size=800)
nx.draw_networkx_labels(G_complet, pos, labels=node_labels, font_size=8)
plt.title(f"Triangle {SIZE}x{SIZE} : Coin exclu (X, noir)
        \n Centres du graphe triangulaire sans pointe en CYAN")
plt.axis('off')
plt.show()
print(f"Le(s) centre(s) du graphe triangulaire (sans pointe) est/sont :
      {centers_triangle}")
for sommet in centers_triangle:
    print(n, '=', L[sommet[1]], '+', n-L[sommet[1]], ', ', end='')
nomfic = 'triangle'+str(n)
plt.savefig(nomfic)
plt.close()
print('')

```

**Appendice 2 : punctured triangular graph for the case  $n = 98$ . The 3 centers of the graph correspond to the 3 Goldbach decompositions  $98 = 19 + 79, 98 = 31 + 67, 98 = 37 + 61$ .**



*Output written to the terminal by the program*

```

L = [0, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 98]
ecarts = [3, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 9]
Le(s) centre(s) du graphe triangulaire (sans pointe) est/sont : [(17, 11), (18, 10), (21, 7)]
98 = 37 + 61 98 = 31 + 67 98 = 19 + 79

```