Goldbach's conjecture and network centers

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1 Definitions

We note $\mathcal{P} = \{2, 3, 5, \ldots\}$ the set of prime numbers.

In the following, we define a grid in which coordinates sets are :

$$\begin{aligned} X_n &= \{p_k \mid 3 \leq p_k \leq n-3,\ p_k \in \mathcal{P}\} \cup \{0, n\}, \\ Y_n &= \{n - p_k \mid p_k \in X_n\} \cup \{0, n\}. \end{aligned}$$

Fact : $\forall n \geq 6, X_n \neq \varnothing, Y_n \neq \varnothing$.

We define a subset R of points of the euclidean plane $\mathbb{N}^2$:

$$R = \{(x, y) \in \mathbb{N}^2 \mid x \in X_n,\ y \in Y_n\}.$$

We need a function f to number points of R :

$$\begin{array}{rccl} f: & R & \rightarrow & \mathbb{N} \\ & (x, y) & \mapsto & x + y.\text{Card}(X_n) + 1 \end{array}$$

We associate to R the graph (called squared grid-graph) $G = (S, A)$ with :

$$\begin{array}{ll} S & = \{s_i, 1 \leq i \leq (\text{Card}(X_n))^2\}, \\ A & = A_H \cup A_V \end{array}$$

where S is the set of vertices and A is the set of edges (horizontal and vertical) of the graph :

$$\begin{array}{ll} A_H & = \{(s_i, s_j) | (f^{-1}(s_i) = (x_i, y_i)) \wedge (f^{-1}(s_j) = (x_i, y_i + 1))\} \\ A_V & = \{(s_i, s_j) | (f^{-1}(s_i) = (x_i, y_i)) \wedge (f^{-1}(s_j) = (x_i + 1, y_i))\} \end{array}$$

We restrict the squared grid-graph G to the punctured triangular graph $T = (S_{\mathrm{T}}, A_{\mathrm{T}})$:

$$\begin{aligned} S_{\mathrm{T}} &= \{s_i | (f^{-1}(s_i) = (x_i, y_i) \in S) \wedge (x_i \geq y_i)\} \\ A_{\mathrm{T}} &= \{a_i | (a_i = (s_i, s_j)) \wedge (a_i \in A) \wedge (s_i \in S_{\mathrm{T}}) \wedge (s_j \in S_{\mathrm{T}})\} \end{aligned}$$

We must labeled edges of A_{T} with gaps between successive prime numbers. For this, we define the gaps sequence between successives primes (the first gap is the prime number 3 and the last gap is the difference between n and the greatest prime number $\leq n-3$) :

$$\Delta_n = s_k(n) \text{ telle que } \begin{cases} s_1 = 3 \\ \\ s_k = p_{k+1} - p_k \text{ pour } 2 \leq k \leq \mathrm{Card}(X_n) - 2 \\ \\ s_{\mathrm{Card}(X_n)} = n - \max\{p_k | (3 \leq p_k \leq n-3) \wedge (p \in \mathcal{P})\} \end{cases}$$

The labeling function of the graph edges is thus :

$$\begin{array}{rccl} g : & A_{\mathrm{T}} & \to & \mathbb{N}^2 \\ & ((x_i, y_i), (x_i, y_i + 1)) & \mapsto & \delta_i \in \Delta_n \\ & ((x_i, y_i), (x_i + 1, y_i)) & \mapsto & \delta_i \in \Delta_n \end{array}$$

It turns out that the distance between two vertices s and s' of S corresponds to the Manhattan distance (or norm 1 distance) in the euclidean plane :

$$\begin{array}{rccl} d : & S \times S & \to & \mathbb{N}^+ \\ & ((x_i, y_i)(x_j, y_j)) & \mapsto & |x_j - x_i| + |y_j - y_i|. \end{array}$$

The distance between two vertices is thus the length of a path that has a minimum distance leading from s to s' in the graph G_{T}.

Let $s \in S$. The eccentricity of s, denoted by $e(s)$ is defined by :

$$e(s) = \max\{d(s, s'), \forall s' \in S\}.$$

Let E_{T} the set containing T's vertices eccentricities :

$$E_{\mathrm{T}} = \{e(s) | s \in S_{\mathrm{T}}\}.$$

A center of a graph is a vertex of this graph, which has an eccentricity which is equal to the minimum of the eccentricities set of all vertices of this graph.

The s vertex is a center of the graph $G = (S, A)$ if and only if

$$e(s) = m \text{ avec } m = \min\{e(s') | s' \in S\}.$$

In the particular case of the graph T that is of interest for us, a vertex s de S_{T} is a center of T if and only if $e(s) = \min(E_{\mathrm{T}})$.

2 Assertions

Lemma : *The G graph is strongly connected.*

Lemma : *The T graph (punctured triangular) is strongly connected.*

Lemma : *The set of vertices S_{T} and the set of edges A_{T} being two not empty sets, the set of eccentricities of vertices of the graph E_{T} is well defined and not empty. The set of vertices that have an eccentricity that is minimum is well defined and is not empty.*

(**Lemma** : *S (the source) and P (the sink) are the graph vertices that have an eccentricity which is equal to the maximum of the set of eccentricities ; the maximum eccentricity is equal to $2n$.*)

Theorem 1 : *The punctured triangular graph centers correspond to Goldbach components of n. The path that joins the source vertex S (that has $(0,0)$ as coordinates) is composed of a part of the path containing only horizontal edges and whom labels sum is equal to p with p a prime number and a part of the path containing only vertical edges and whom labels sum is equal to q with q also prime.*

Follows from definitions.

Theorem 2 : *The punctured triangular graph contains at least one center.*

It follows from the existence of a minimum and a maximum for every not empty set of integers.

Theorem 3 : *Every even number admits at least one Goldbach decomposition.*

Follows from theorem 1 (that asserts the isomorphism between Goldbach components of n and centers of the graph G_{T}) and theorem 2 (that asserts mandatory existence of at least one center for the graph G_{T}).

3 Illustrations : case $n = 16$

$$\mathrm{X}_{16} = \{3, 5, 7, 11, 13\} \qquad \mathrm{Y}_{16} = \{13, 11, 9, 5, 3\}$$

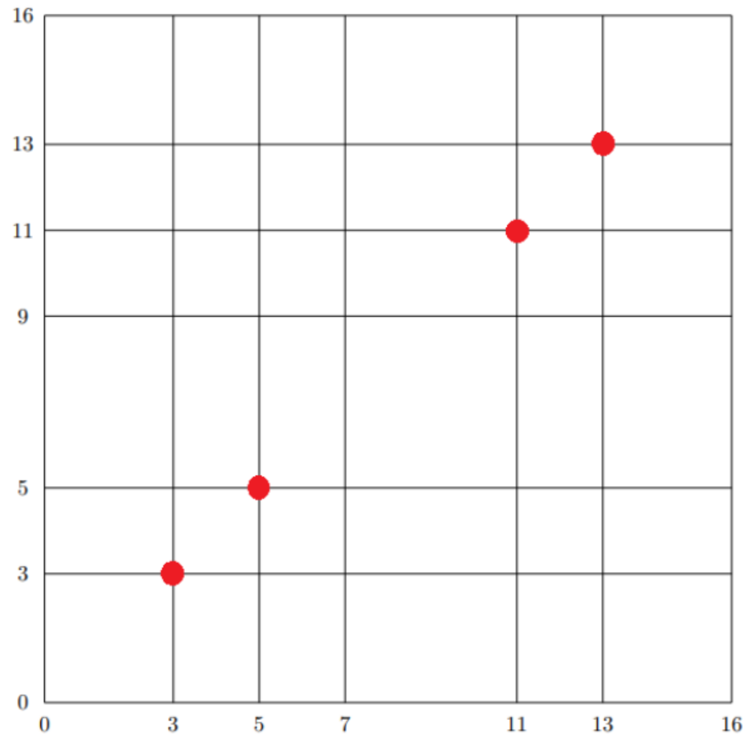

FIGURE 1 : The example of the square for $n = 16$. Points corresponding to Goldbach components of 16 that are $\{3, 5, 11, 13\}$, i.e. points $(3, 3), (5, 5), (11, 11), (13, 13)$ are colored in red on the ascending diagonal.

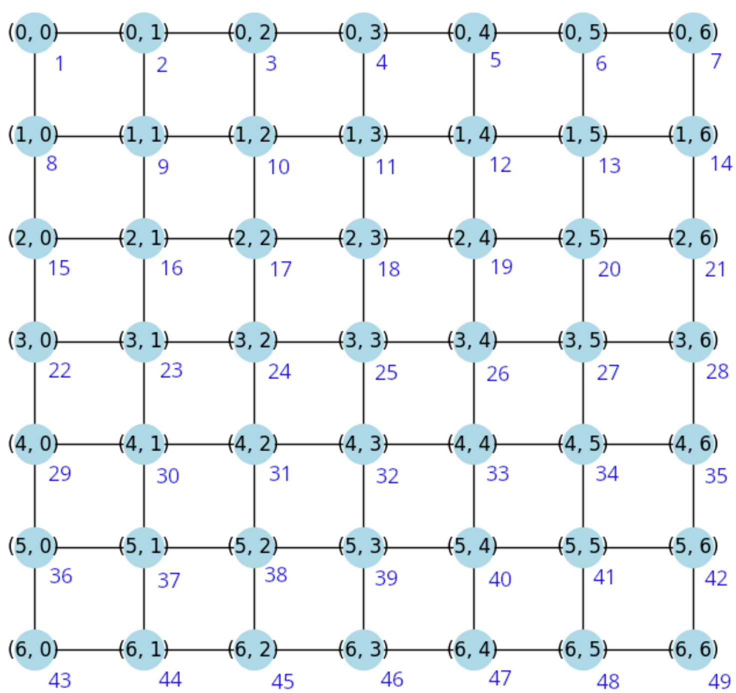

FIGURE 2 : Numbering of graph vertices

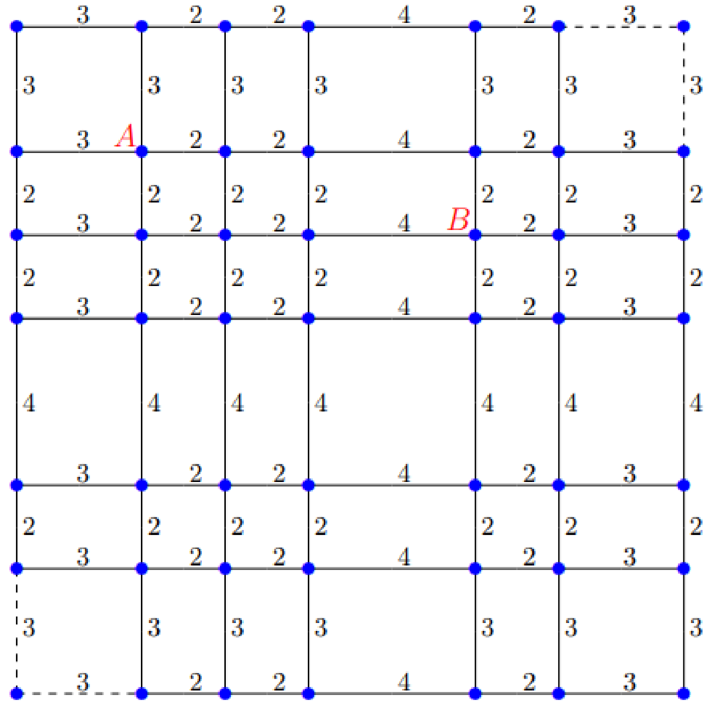

FIGURE 3 : Labeling of vertices forêtes pour $n = 16$. ($d(A, B) = 10$)

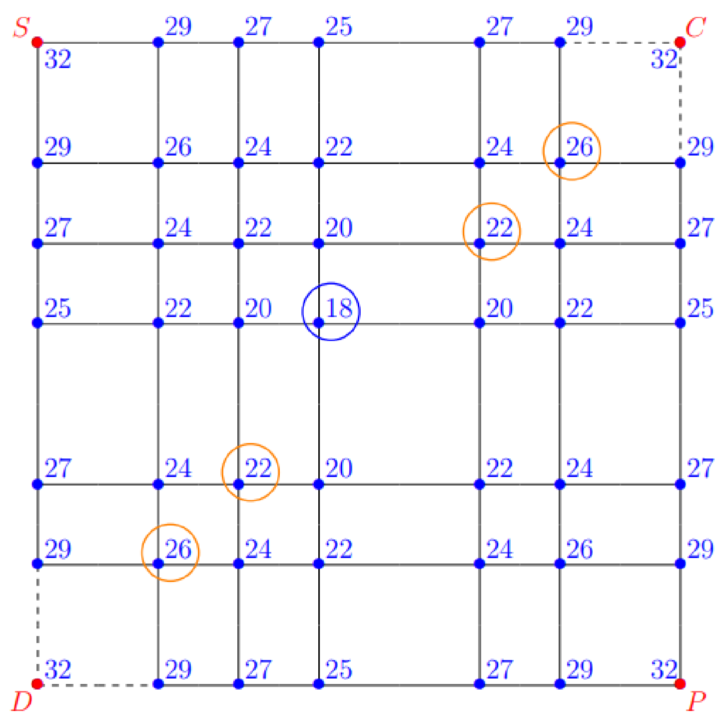

FIGURE 4 : vertices eccentricities.
The eccentricity of the graph center is circled in blue.
Eccentricities of vertices corresponding in the euclidean plane to points having (p_k, p_k) as coordinates with p_k a Goldbach component of n circled in orange.

Note : those vertices *are not* squared graph centers.

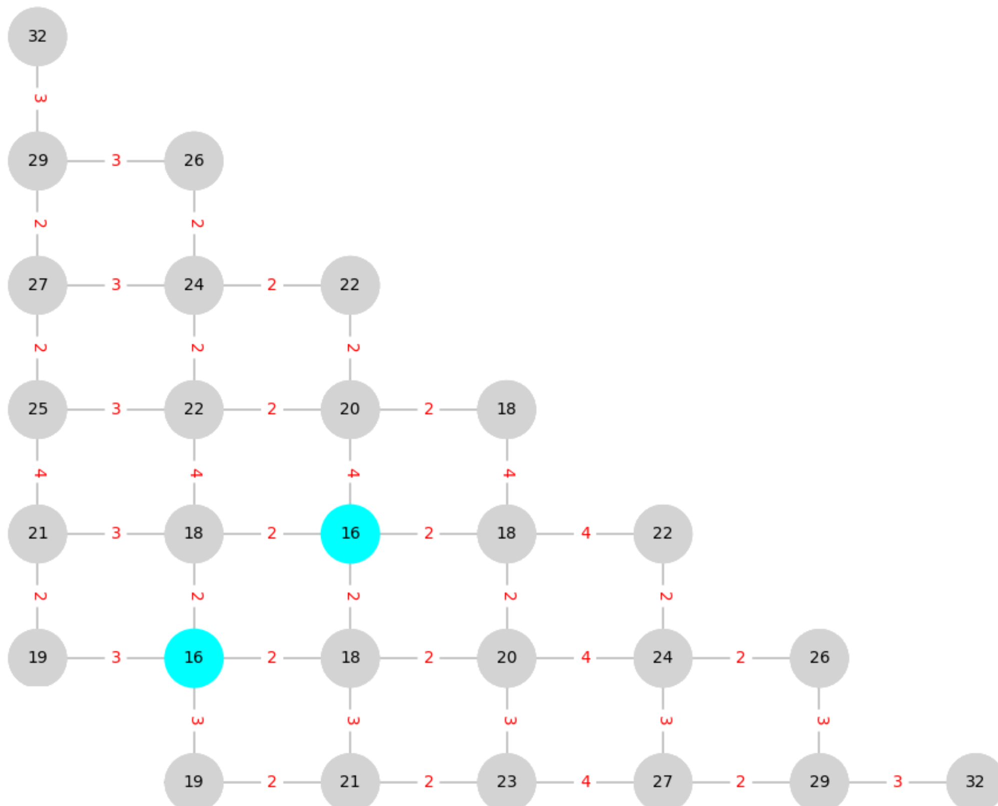

FIGURE 5 : Eccentricities of vertices of the punctured triangular graph ; the centers of the punctured triangular graph are corresponding now to Goldbach components of n.

Remark : Goldbach's conjecture is trivially verified by even numbers that are the double of a prime number. In the modelisation provided here, the trivial Goldbach decomposition $2p_k = p_k + p_k$ with p_k a prime number corresponds to a center of the punctured triangular graph. However, such a center is not at the center of the "euclidean" initial square : it's at the center in terms of edges punctured triangular graph number, i.e. it is always located on the isosceles right triangle hypothenuse that underlies the graph ; the path that leads to the graph center corresponding to the Goldbach trivial decomposition $2p_k = p_k + p_k$ is, as is the whole punctured triangular graph, an isoscele right triangle of the euclidean plane.

The punctured triangular graphs visualizing the graphs centers and so the Goldbach decompositions for integers n between 8 and 102 can be downloaded at this address. The python program computing those centers has been provided by Gemini, following the author instructions to label correctly edges, vertices, and to compute eccentricities, first in squared graphs and then in punctured triangular ones (it can be downloaded at this address). It is provided in appendice 1 below, the graph G_{T} corresponding to the case $n = 98$ is provided in appendice 2.

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Appendice 1 : Program that draws punctured triangular graphs and that visualizes their centers.

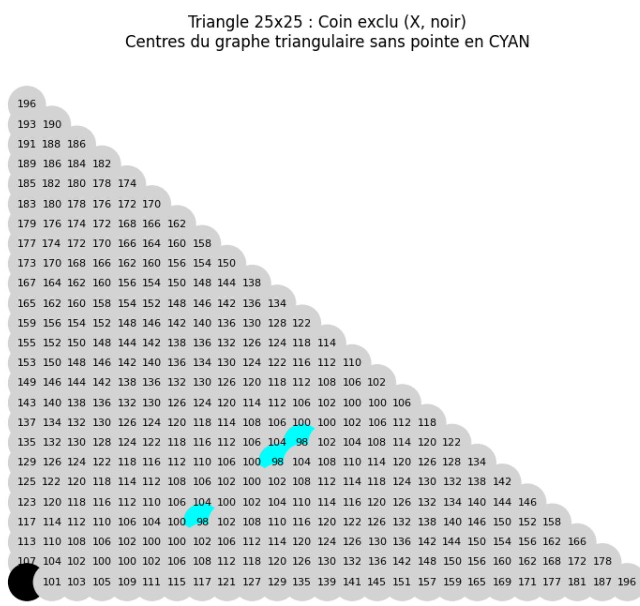
```
import networkx as nx
import matplotlib.pyplot as plt

def premier(atester):
  k = 2
  if atester in [0, 1]: return False
  if atester in [2, 3, 5, 7]: return True
  while True:
    if k * k > atester: return True
    else:
      if atester % k == 0: return False
      else: k = k + 1

for n in range(8,62,2):
  L=[0]
  for k in range(3,n-1,2):
    if premier(k):
      L.append(k)
  L.append(n)
  print('L = ',L)
  ecarts = []
  for indice in range(1,len(L)):
    ecarts.append(L[indice]-L[indice-1])
  print('ecarts = ',ecarts)
  poids_fixes = ecarts
  SIZE = len(poids_fixes) + 1
  G_full = nx.grid_2d_graph(SIZE, SIZE)
  nodes_to_keep = [(i, j) for (i, j) in G_full.nodes() if i >= j]
  G_complet = G_full.subgraph(nodes_to_keep).copy()
  for (u, v) in G_complet.edges():
      (r1, c1), (r2, c2) = u, v
      if r1 == r2: # Horizontale
          G_complet.edges[u, v]['weight'] = poids_fixes[min(c1, c2)]
      else: # Verticale
          G_complet.edges[u, v]['weight'] = poids_fixes[min(r1, r2)]
  node_isole = (SIZE - 1, 0) # Le coin bas-gauche
  G_tri = G_complet.copy()
  if node_isole in G_tri:
      G_tri.remove_node(node_isole) # Supprime le noeud ET ses aretes
  path_lengths = dict(nx.all_pairs_dijkstra_path_length(G_tri, weight='weight'))
  eccs_tri = {node: max(dists.values()) for node, dists in path_lengths.items()}
  min_ecc = min(eccs_tri.values())
  centers_triangle = [n for n, ecc in eccs_tri.items() if ecc == min_ecc]
  plt.figure(figsize=(10, 8))
  pos = {(i, j): (j, -i) for (i, j) in G_complet.nodes()}
  node_labels = {n: f"{eccs_tri[n]}" for n in G_tri.nodes()}
  node_labels[node_isole] = "X" # Marquer le noeud exclu
  node_colors = []
  for node in G_complet.nodes():
      if node == node_isole:
          node_colors.append('black') # Le point exclu est noir
      elif node in centers_triangle:
          node_colors.append('cyan')  # Les centres du "tri" sont cyan
```

```
        else:
            node_colors.append('lightgrey') # Le reste est gris
    nx.draw_networkx_edges(G_complet, pos, alpha=0.5, edge_color='gray')
    edge_labels = nx.get_edge_attributes(G_complet, 'weight')
    nx.draw_networkx_edge_labels(G_complet, pos, edge_labels=edge_labels,
                                                            font_color='red', font_size=8)
    nx.draw_networkx_nodes(G_complet, pos,
                           node_color=node_colors,
                           node_size=800)
    nx.draw_networkx_labels(G_complet, pos, labels=node_labels, font_size=8)
    plt.title(f"Triangle {SIZE}x{SIZE} : Coin exclu (X, noir)
              \n Centres du graphe triangulaire sans pointe en CYAN")
    plt.axis('off')
    plt.show()
    print(f"Le(s) centre(s) du graphe triangulaire (sans pointe) est/sont :
             {centers_triangle}")
    for sommet in centers_triangle:
      print(n,'=',L[sommet[1]],'+',n-L[sommet[1]],'   ',end='')
    nomfic ='triangle'+str(n)
    plt.savefig(nomfic)
    plt.close()
    print('')
```

Appendice 2 : punctured triangular graph for the case $n = 98$. The 3 centers of the graph correspond to the 3 Goldbach decompositions $98 = 19 + 79, 98 = 31 + 67, 98 = 37 + 61$.



Output written to the terminal by the program

L = [0, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 98]
ecarts = [3, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 9]
Le(s) centre(s) du graphe triangulaire (sans pointe) est/sont : [(17, 11), (18, 10), (21, 7)]
98 = 37 + 61 98 = 31 + 67 98 = 19 + 79