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K HOMOLOGY AND INDEX THEORY

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This expository note reports on Ext, K homology, and index theory. Proofs and complete details will be given elsewhere. Our point of view is that index theory is based on the equivalence between analytic and topological K homology.

For a topological space X, analytic K homology groups $K_0^a(X)$, $K_1^a(X)$ are defined in the spirit of C* algebras and operator theory. Topological K homology groups $K_0^t(X)$, $K_1^t(X)$ are defined by a topological method using cycles and homologies of cycles. The Dirac operator of a Spin^c manifold gives the link between these two theories, and gives explicit isomorphisms

$$K_0^t(X) \longrightarrow K_0^a(X) \ , \ K_1^t(X) \longrightarrow K_1^a(X)$$

Analytic K homology came from the work of Brown-Douglas-Fillmore on essentially normal operators [23]. A related but slightly different definition of analytic K homology has been given by G. G. Kasparov [37]. In [2] M. F. Atiyah proposed the existence of the analytic theory, and took a basic first step in this direction.

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Topological K homology came from the work of Baum-Fulton-MacPherson [17] [18] on the Riemann-Roch theorem. A quite different topological definition of K homology has been given by G. Segal [44].

The paper is divided into five parts:

(1) Analytic K homology
(2) Topological K homology
(3) Riemann-Roch theorem
(4) Isomorphism of analytic and topological K homology
(5) Index theorems

At the beginning of each part there is a brief introduction to that part. These mini-introductions can be found on pages 4, 16, 28, 34, and 38.

The two definitions of K homology and the isomorphism $K_*^t \to K_*^a$ are developed in 1, 2, 4. Part 3 is devoted to Riemann-Roch. By using K_*^t as defined in Part 2, a Riemann-Roch theorem is stated which extends the Grothendieck-Riemann-Roch theorem to singular varieties. Part 5 shows how various index theorems (e.g. Atiyah-Singer [11], Atiyah-Bott [4], Boutet de Monvel [22], Cheeger [25], Connes [27]) fit into the general framework for index theory provided by the isomorphism $K_*^t \to K_*^a$.

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Contents

Part 1: Analytic K homology

In Part 1, analytic K homology is defined. The motivation for the definition came from two directions: essentially normal operators (Brown-Douglas-Fillmore [23]) and elliptic operators (Atiyah [2]). Both points of view will be made explicit.

§1. Essentially normal operators

Let H, H_0, H_1 be separable Hilbert spaces. Recall that a bounded operator $T : H_0 \to H_1$ is Fredholm if T has a finite dimensional null space, and a closed range of finite codimension. The index of a Fredholm operator T is:

$$\mathrm{Index}(T) = \dim_{\mathbf{C}} \mathrm{Kernel}(T) - \dim_{\mathbf{C}} \mathrm{Cokernel}(T)$$

A bounded operator $A : H \to H$ is essentially normal if the commutator $[A,A^*] = AA^* - A^*A$ is a compact operator. Two essentially normal operators $A : H \to H$ and $B : H \to H$ are unitarily equivalent modulo compact operators if there exists a unitary operator $U : H \to H$ such that:

$$UAU^* - B \text{ is a compact operator.}$$

We then have:

Problem: Classify essentially normal operators up to unitary equivalence modulo compact operators.

The remarkable fact about this problem is that its answer is very simple and only depends on some underlying topological data.

To state the answer, recall that for a bounded operator $T : H \to H$ the essential spectrum of T, denoted $\Sigma(T)$, is defined by:

$$\Sigma(T) = \{\lambda \in \mathbf{C} \mid T - \lambda I \text{ is not Fredholm}\}$$

As usual, I denotes the identity operator. $\Sigma(T)$ is compact, so $\mathbf{C} - \Sigma(T)$ is open. Note that $\mathbf{C} - \Sigma(T)$ comes equipped with an integer-valued function $\lambda \mapsto \mathrm{Index}(T-\lambda I)$. This function has two properties:

(i) $\mathrm{Index}(T-\lambda I)$ is constant on each connected component of $\mathbf{C} - \Sigma(T)$

(ii) For $|\lambda|$ sufficiently large $\mathrm{Index}(T-\lambda I) = 0$

Theorem. [23] Two essentially normal operators $A : H \to H$ and $B : H \to H$ are unitarily equivalent modulo compact operators if and only if

(i) $\Sigma(A) = \Sigma(B)$

(ii) For all $\lambda \in \mathbb{C} - \Sigma(A) = \mathbb{C} - \Sigma(B)$,
$\mathrm{Index}(A-\lambda I) = \mathrm{Index}(B-\lambda I)$

Corollary: Let $A : H \to H$ be an essentially normal operator. Then there exists a normal operator N and a compact operator K with $A = N + K$ if and only if $\mathrm{Index}(A-\lambda I) = 0$ for all $\lambda \in \mathbb{C} - \Sigma(A)$.

The classification is completed by an existence theorem.

Theorem. [23] Let Σ be a compact subset of $\mathbb{C}$. Let $\alpha : \mathbb{C} - \Sigma \to \mathbb{Z}$ be an integer-valued function such that

(i) $\alpha(\lambda)$ is constant on each connected component of $\mathbb{C} - \Sigma$

(ii) $\alpha(\lambda) = 0$ for $|\lambda|$ sufficiently large

Then there exists an essentially normal operator $A : H \to H$ with $\Sigma(A) = \Sigma$ and $\mathrm{Index}(A-\lambda I) = \alpha(\lambda)$ for all $\lambda \in \mathbb{C} - \Sigma$.

§2. The Calkin algebra

For a separable Hilbert space H there is the standard notation:

$L(H)$ = C^* algebra of all bounded operators on H

$K(H)$ = The closed ideal in $L(H)$ consisting of all compact operators on H

$Q(H) = L(H)/K(H)$

The Calkin algebra, $Q(H)$, is a C^* algebra with unit. π denotes the canonical map $L(H) \to Q(H)$.

Let $A : H \to H$ be an essentially normal operator, and set $\Sigma(A) = \Sigma$. Consider the C^* algebra $C(\Sigma)$ of all continuous complex-valued functions on Σ , with

$$\|f\| = \sup_{p\in\Sigma} |f(p)| \qquad f \in C(\Sigma)$$

According to the Stone-Weierstrass approximation theorem, the finite sums

$$(2.1) \qquad a_0 + \sum_{(i,j)} a_{ij} z^i \bar{z}^j \qquad a_0, a_{ij} \in \mathbb{C}$$

form a dense sub-algebra of $C(\Sigma)$. In (2.1) $z : \Sigma \to \mathbb{C}$ is the function which maps a complex number to itself.

Hence the essentially normal operator $A : H \to H$ determines the unital algebra *-homomorphism

$$\tau : C(\Sigma) \to \mathcal{Q}(H)$$

given by

$$\tau(1) = \pi(I)$$

$$\tau(z) = \pi(A)$$

$$\tau(\bar{z}) = \pi(A^*)$$

Note that $A = N + K$ where N is normal and K is compact if and only if there is a commutative diagram of unital algebra *-homomorphisms

$$\begin{tikzcd} & L(H) \arrow[d, "\pi"] \\ C(\Sigma) \arrow[ur] \arrow[r, "\tau"'] & \mathcal{Q}(H) \end{tikzcd}$$

These observations led BDF (BDF = Brown-Douglas-Fillmore) to their solution of the classification problem for essentially normal operators. They also led BDF to define an abelian group $\operatorname{Ext}(X)$ for any compact metrizable space X .

§3. Ext(X)

To define $\operatorname{Ext}(X)$, let $C(X)$ be the C^* algebra of all continuous complex-valued functions on X , with

$$\|f\| = \sup_{p \in X} |f(p)| \qquad f \in C(X)$$

An element of $\operatorname{Ext}(X)$ will be given by a pair (H,τ) where H is a separable Hilbert space and τ is a unital algebra *-homomorphism from $C(X)$ to $\mathcal{Q}(H)$

$$\tau : C(X) \to \mathcal{Q}(H)$$

For such (H,τ) BDF define

(i) Unitary equivalence

(ii) Triviality

(iii) Direct sum

(i) Unitary equivalence. (H_0,τ_0) and (H_1,τ_1) are unitarily equivalent if there exists a unitary operator U from H_0 onto H_1 with commutativity in the diagram

(3.1)
$$\begin{tikzcd} & \mathcal{Q}(H_0) \arrow[d] \\ C(X) \arrow[ur, "\tau_0"] \arrow[r, "\tau_1"'] & \mathcal{Q}(H_1) \end{tikzcd}$$

In (3.1) $\mathcal{Q}(H_0) \to \mathcal{Q}(H_1)$ is the isomorphism of C^* algebras induced by mapping $L(H_0)$ to $L(H_1)$ by $T \to UTU^*$

(ii) Triviality. (H,τ) is trivial if there exists a unital algebra *-homomorphism $\eta : C(X) \to L(H)$ with commutativity in the diagram

$$\begin{tikzcd} & L(H) \arrow[d, "\pi"] \\ C(X) \arrow[ur, "\eta"] \arrow[r, "\tau"'] & \mathcal{Q}(H) \end{tikzcd}$$

(iii) Direct sum. The standard inclusion

$L(H_0) \oplus L(H_1) \subset L(H_0 \oplus H_1)$ induces an inclusion

$\mathcal{Q}(H_0) \oplus \mathcal{Q}(H_1) \subset \mathcal{Q}(H_0 \oplus H_1)$. Given (H_0,τ_0) and (H_1,τ_1) , the direct sum $(H_0,\tau_0) \oplus (H_1,\tau_1)$ is $(H_0 \oplus H_1, \tau_0 \oplus \tau_1)$ where $\tau_0 \oplus \tau_1 : C(X) \to \mathcal{Q}(H_0 \oplus H_1)$ is the composition

$$C(X) \xrightarrow{\tau_0+\tau_1} \mathcal{Q}(H_0) \oplus \mathcal{Q}(H_1) \to \mathcal{Q}(H_0 \oplus H_1)$$

On the collection of all such (H,τ) define an equivalence relation ~ by:

$(H_0,\tau_0) \sim (H_1,\tau_1)$ if there exist trivial (H_0',τ_0') and (H_1',τ_1') with $(H_0,\tau_0) \oplus (H_0',\tau_0')$ unitarily equivalent to $(H_1,\tau_1) \oplus (H_1',\tau_1')$.

$\mathrm{Ext}(X)$ is the set of equivalence classes. Addition in $\mathrm{Ext}(X)$ is given by direct sum. The zero element in $\mathrm{Ext}(X)$ is given by any trivial (H,τ).

Let $f : X \to Y$ be a continuous map. Define a homomorphism of abelian groups $f_* : \mathrm{Ext}(X) \to \mathrm{Ext}(Y)$ as follows. Let $\hat{f} : C(Y) \to C(X)$ be given by

$$\hat{f}(\phi) = \phi \circ f \qquad \phi \in C(Y) \tag{3.2}$$

Then:

$$f_*(H,\tau) = (H,\tau\circ\hat{f}) \tag{3.3}$$

In (3.3) $\tau \circ \hat{f}$ is the composition

$$C(Y) \xrightarrow{\hat{f}} C(X) \xrightarrow{\tau} \mathcal{Q}(H)$$

If $\tau : C(X) \to \mathcal{Q}(H)$ is injective, then by setting $A = \pi^{-1}(\mathrm{Image}\ \tau)$ an exact sequence is obtained

$$0 \to K(H) \to A \to C(X) \to 0$$

Thus $\mathrm{Ext}(X)$ could have been defined as equivalence classes of extensions of $C(X)$ by $K(H)$.

Ext is a covariant functor from the category of compact metrizable spaces to the category of abelian groups.

§4. $K_0^a(X)$

From now on $\mathrm{Ext}(X)$ will be denoted $K_1^a(X)$. This group $K_1^a(X)$ is closely connected to the odd-dimensional homology of X [36]. For example, if S^n is the n-sphere, then

$$K_1^a(S^n) = \begin{cases} 0 & n \quad \text{even} \\ \mathbb{Z} & n \quad \text{odd} \end{cases}$$

To obtain a group $K_0^a(X)$, which will be related to the even-dimensional homology of X, an idea of M. F. Atiyah [2] applies.

An element of $K_0^a(X)$ will be given by a 5-tuple $(H_0,\psi_0,H_1,\psi_1,T)$ such that:

(i) H_0 and H_1 are separable Hilbert spaces.

(ii) $\psi_i : C(X) \to \mathcal{L}(H_i)$ is a unital algebra *-homomorphism. $i = 0,1$

(iii) $T : H_0 \to H_1$ is a bounded Fredholm operator with $T \circ \psi_0(f) - \psi_1(f) \circ T$ compact for each $f \in C(X)$

For these objects $(H_0,\psi_0,H_1,\psi_1,T)$ define:

(1) Isomorphism

(2) Triviality

(3) Direct sum

(1) Isomorphism. $(H_0,\psi_0,H_1,\psi_1,T)$ and $(H_0',\psi_0',H_1',\psi_1',T')$ are isomorphic if there exist unitary operators $U_0 : H_0 \to H_0'$, $U_1 : H_1 \to H_1'$ such that $T' = U_1 T U_0^*$, and for each $f \in C(X)$, $\psi_i'(f) = U_i \psi_i(f) U_i^*$, $i = 0,1$.

(2) Triviality. A trivial object is one of the form (H,ψ,H,ψ,I). Here $H_0 = H_1 = H$, $\psi_0 = \psi_1 = \psi$, and $T = I$ is the identity operator of H.

(3) Direct sum. Given $(H_0,\psi_0,H_1,\psi_1,T)$ and $(H_0',\psi_0',H_1',\psi_1',T')$, the direct sum is evident and is $(H_0 \oplus H_0', \psi_0 \oplus \psi_0', H_1 \oplus H_1', \psi_1 \oplus \psi_1', T \oplus T')$.

On the collection of all these objects $(H_0,\psi_0,H_1,\psi_1,T)$ define an equivalence relation ~ by letting ~ be the equivalence relation generated by

(i) Isomorphism

(ii) Direct sum with a trivial object

(iii) Compact perturbation

(i) Isomorphic objects are equivalent.

(ii) Let $(H_0,\psi_0,H_1,\psi_1,T)$ be an object, and let (H,ψ,H,ψ,I) be a trivial object. Then $(H_0,\psi_0,H_1,\psi_1,T)$ is equivalent to $(H_0 \oplus H, \psi_0 \oplus \psi, H_1 \oplus H, \psi_1 \oplus \psi, T \oplus I)$.

(iii) Compact perturbation. Suppose given two objects $(H_0,\psi_0,H_1,\psi_1,T)$ and $(H_0',\psi_0',H_1',\psi_1',T')$. Assume that

$H_0 = H_1 = H_0' = H_1' = H$ and that $\psi_0 = \psi_1, \psi_0' = \psi_1'$. Thus the two given objects are of the form (H,ψ,H,ψ,T) and (H,ψ',H,ψ',T'). Let V,V' be the partial isometry parts in the polar decompositions of T,T'. Then $(H,\psi,H,\psi,T) \sim (H,\psi',H,\psi',T')$ if $V - V'$ is compact and for each $f \in C(X)$, $\psi(f) - \psi'(f)$ is compact.

Remark. Some care must be taken with compact perturbation. For example, let $(H_0,\psi_0,H_1,\psi_1,T)$ be an object. Assume $H_0 \neq H_1$, $\psi_0 \neq \psi_1$. Let $\psi_0' : C(X) \to L(H_0)$ be a unital algebra $*$-homomorphism with $\psi_0(f) - \psi_0'(f)$ compact for each $f \in C(X)$. Then compact perturbation does not assert that $(H_0,\psi_0,H_1,\psi_1,T)$ is equivalent to $(H_0,\psi_0',H_1,\psi_1,T)$.

$K_0^a(X)$ is the set of equivalence classes. Direct sum makes $K_0^a(X)$ into an abelian group, and the additive inverse of $(H_0,\psi_0,H_1,\psi_1,T)$ is $(H_1,\psi_1,H_0,\psi_0,T^*)$.

For the n-sphere S^n,

$$K_0^a(S^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \quad \text{even} \\ \mathbb{Z} & n \quad \text{odd} \end{cases}$$

For any compact metrizable X, $K_0^a(X)$ surjects onto $\mathbb{Z}$ by $(H_0,\psi_0,H_1,\psi_1,T) \mapsto \text{Index}(T)$.

Denote this by

$$\text{Index} : K_0^a(X) \to \mathbb{Z} \tag{4.1}$$

Let $f : X \to Y$ be a continuous map. As in (3.2) there is $\hat{f} : C(Y) \to C(X)$. Define a homomorphism of abelian groups $f_* : K_0^a(X) \to K_0^a(Y)$ by:

$$f_*(H_0,\psi_0,H_1,\psi_1,T) = (H_0,\psi_0 \circ \hat{f},H_1,\psi_1 \circ \hat{f},T) \tag{4.2}$$

In (4.2) $\psi_i \circ \hat{f}$ is the composition

$$C(Y) \xrightarrow{\hat{f}} C(X) \xrightarrow{\psi_i} L(H_i) \qquad i = 0,1$$

$K_0^a(X)$ and $K_1^a(X)$ are the analytic K homology groups of X.

In the BDF [24] formulation of analytic K homology, $K_0^a(X)$ was defined using Ext and the suspension of the space X. As indicated in [24], the Ext-suspension definition and the definition stated here are equivalent.

§5. Elliptic Operators

One raison d'être for analytic K homology is its very close relationship with elliptic operators. Roughly speaking, an object $(H_0,\psi_0,H_1,\psi_1,T)$ for $K_0^a(X)$ can be thought of as an elliptic operator on the topological space X. An object (H,τ) for $K_1^a(X)$ can be thought of as a self-adjoint elliptic operator on X.

To make this more precise, suppose now that X is a closed C^∞ manifold. (Closed = compact without boundary.) Let E_0,E_1 be two complex C^∞ vector-bundles[1] on X. $C^\infty(E_i)$ denotes the vector-space of C^∞ sections of E_i.
Let $D : C^\infty(E_0) \to C^\infty(E_1)$ be an elliptic pseudo-differential operator. Choose a smooth measure μ on X, and choose C^∞ Hermitian structures on E_0,E_1. Thus each fibre $(E_i)_p$ has been made into a finite-dimensional Hilbert space, and the inner product varies in a C^∞ fashion for $p \in X$. $C^\infty(E_i)$ then has an inner product given by

$$\langle u,v\rangle = \int_X \langle u(p),v(p)\rangle d\mu \qquad u,v \in C^\infty(E_i)$$

Completion with respect to this inner product gives Hilbert spaces $L^2(E_0)$, $L^2(E_1)$. D can be viewed as an unbounded[2] operator from $L^2(E_0)$ to $L^2(E_1)$

$$D : L^2(E_0) \to L^2(E_1)$$

---

1 All manifolds are assumed to be C^∞, Hausdorff, and para-compact. A topological vector bundle on such a manifold can be given a C^∞ structure which is unique up to isomorphism. So on such manifolds there is no essential distinction between topological vector bundles and C^∞ vector bundles.

2 If order $D > 0$, D is unbounded. If order $D \le 0$, D is bounded.

This possibly unbounded operator is closable. Therefore we may take its polar decomposition. Let V be the partial isometry part in the polar decomposition of D . V is then an elliptic pseudo-differential operator of order zero. Hence V is a bounded operator from $L^2(E_0)$ to $L^2(E_1)$.

$$V : L^2(E_0) \to L^2(E_1)$$

Define $\psi_i : C(X) \to \mathcal{L}(L^2(E_i))$ by

$$(\psi_i(f)u)(p) = f(p)\cdot u(p) \qquad p \in X \ , \ u \in L^2(E_i)$$

Then $(L^2(E_0),\psi_0,L^2(E_1),\psi_1,V)$ is an object for $K_0^a(X)$.

The element in $K_0^a(X)$ so obtained depends only on the original elliptic operator $D : C^\infty(E_0) \to C^\infty(E_1)$, and will be denoted by $[D]$ with

(5.1) $$[D] \in K_0^a(X)$$

For a closed C^∞ manifold X , all elements of $K_0^a(X)$ arise in this way from elliptic pseudo-differential operators on X . See §23.

If $D : C^\infty(E_0) \to C^\infty(E_1)$ is an elliptic pseudo-differential operator on the closed C^∞ manifold X , then Kernel(D) and Cokernel(D) are finite dimensional vector-spaces over $\mathbb{C}$. Index(D) is:

$$\mathrm{Index}(D) = \dim_{\mathbb{C}} \mathrm{Kernel}(D) - \dim_{\mathbb{C}} \mathrm{Cokernel}(D)$$

Let $\mathrm{Index} : K_0^a(X) \to \mathbb{Z}$ be as in (4.1), and let $[D] \in K_0^a(X)$ be as in (5.1). Then:

(5.2) $$\mathrm{Index}(D) = \mathrm{Index}[D]$$

§6. Self-adjoint elliptic operators

With X still assumed to be a closed C^∞ manifold, let $A : C^\infty(E) \to C^\infty(E)$ be an elliptic pseudo-differential operator, where E is a C^∞ complex vector-bundle on X . Suppose that a smooth measure μ on X and a C^∞ Hermitian structure $\langle\ ,\ \rangle$

for E can be chosen such that for all $u,v \in C^\infty(E)$,

$$\int_X \langle Au(p),v(p)\rangle d\mu = \int_X \langle u(p),Av(p)\rangle d\mu$$

Hence A is a self-adjoint elliptic operator.

If A is a differential operator, then $L^2(E)$ has an orthonormal basis of eigen-vectors of A . In this case, let $L^2_+(E)$ be the closed subspace of $L^2(E)$ spanned by the eigen-vectors belonging to non-negative eigen-values. If A is only a pseudo-differential operator then apply the spectral theorem and let $L^2_+(E)$ be the spectral subspace corresponding to $[0,\infty)$.

Define $\psi : C(X) \to \mathcal{L}(L^2(E))$ by

$$(\psi(f)u)p = f(p)\cdot u(p) \qquad p \in X \text{ , } u \in L^2(E)$$

For $f \in C(X)$, let $\gamma(f)$ be the compression of $\psi(f)$ to $L^2_+(E)$. Thus $\gamma(f)$ is the composition

$$L^2_+(E) \xrightarrow{\psi(f)} L^2(E) \xrightarrow{P} L^2_+(E) \tag{6.1}$$

In (6.1) P is the Hilbert space projection of $L^2(E)$ onto the closed subspace $L^2_+(E)$.

As in §2 above, π denotes the canonical map

$$\pi : \mathcal{L}(L^2_+(E)) \to \mathcal{Q}(L^2_+(E)) \quad .$$

Define $\tau : C(X) \to \mathcal{Q}(L^2_+(E))$ by

$$\tau(f) = \pi(\gamma(f)) \qquad f \in C(X)$$

Then $(L^2_+(E),\tau)$ is an object for $K^a_1(X)$.

The element in $K^a_1(X)$ so obtained depends only on the original elliptic operator $A : C^\infty(E) \to C^\infty(E)$, and will be denoted by [A], with

$$[A] \in K^a_1(X) \quad . \tag{6.2}$$

For a closed C^∞ manifold X , all elements of $K^a_1(X)$ arise in this way from self-adjoint elliptic operators on X . See §24.

§7. Elliptic complexes

On the C^∞ manifold X, let $E_0, E_1, \ldots, E_r$ be C^∞ complex vector bundles. Let $d_i : C^\infty(E_i) \to C^\infty(E_{i+1})$ be a pseudo-differential operator. The sequence

$$0 \to C^\infty(E_0) \to C^\infty(E_1) \to \ldots \to C^\infty(E_r) \to 0$$

is an <u>elliptic complex</u> if

(i) $d_{i+1} \circ d_i = 0 \qquad i = 0,1,\ldots,r-2$

(ii) The sequence of leading symbols
$0 \to \pi^*E_0 \to \pi^*E_1 \to \ldots \to \pi^*E_r \to 0$
is exact outside the zero section of T^*X

In (ii) T^*X denotes the cotangent bundle of X and π^*E_i is the pull-back of E_i to T^*X by the projection $\pi : T^*X \to X$.

For elliptic complexes see [6], [11].

Let ξ be an elliptic complex on X :

$$\xi = (0 \to C^\infty(E_0) \to C^\infty(E_1) \to \ldots \to C^\infty(E_r) \to 0) .$$

The i-th homology group $H_i(\xi)$ of ξ is:

$$H_0(\xi) = \text{Kernel}(d_0)$$

$$H_r(\xi) = \text{Cokernel}(d_{r-1})$$

$$H_i(\xi) = \text{Kernel}(d_i)/\text{Image}(d_{i-1}) \qquad i = 1,2,\ldots,r-1$$

$H_i(\xi)$ is a vector space over the complex numbers $\mathbb{C}$. If X is a closed C^∞ manifold, then each $H_i(\xi)$ is finite dimensional. So if X is closed, the Euler number $\chi(\xi)$ is defined:

$$\chi(\xi) = \sum_{i=0}^{r} (-1)^i \dim_{\mathbb{C}} H_i(\xi)$$

If $r = 1$, then the elliptic complex ξ is just

$$0 \to C^\infty(E_0) \to C^\infty(E_1) \to 0$$

In this case $d_0 : C^\infty(E_0) \to C^\infty(E_1)$ is an elliptic pseudo-differential operator and

$$\chi(\xi) = \mathrm{Index}(d_0)$$

More generally, assume $r \geq 1$. Let $\mathrm{Index} : K_0^a(X) \to \mathbb{Z}$ be as in (4.1). Then ξ determines an element $[\xi] \in K_0^a(X)$ with

$$\chi(\xi) = \mathrm{Index}[\xi]$$

$[\xi]$ is obtained by "assembling" ξ. To do this, choose a smooth measure on M and for each E_i choose a C^∞ Hermitian structure. $L^2(E_i)$ denotes the Hilbert space of all L^2 sections of E_i. Then $d_i : L^2(E_i) \to L^2(E_{i+1})$ is a (possibly unbounded) closable operator. Let $V_i : L^2(E_i) \to L^2(E_{i+1})$ be the partial isometry part in the polar decomposition of d_i. V_i is a pseudo-differential operator of order zero and is a bounded operator from $L^2(E_i)$ to $L^2(E_{i+1})$. Since $L^2(E_i)$ and $L^2(E_{i+1})$ are Hilbert spaces there is the adjoint operator $V_i^* : L^2(E_{i+1}) \to L^2(E_i)$. Set

$$\mathcal{H}_0 = \bigoplus_{i \text{ even}} L^2(E_i)$$

$$\mathcal{H}_1 = \bigoplus_{i \text{ odd}} L^2(E_i)$$

and define $V : \mathcal{H}_0 \to \mathcal{H}_1$ by

$$V(u_0, u_2, u_4, \ldots) = (V_0 u_0 + V_1^* u_2, V_2 u_2 + V_3^* u_4, V_4 u_4 + V_5^* u_6, \ldots)$$

$$u_i \in L^2(E_i),\ i = 0, 2, 4, \ldots$$

Then V is a Fredholm operator with

$$\mathrm{Index}(V) = \chi(\xi)$$

As in §5 and §6 above, define $\psi_i : C(X) \to \mathcal{L}(L^2(E_i))$ by

$$(\psi_i(f)u)(p) = f(p)\cdot u(p) \qquad p \in X,\ u \in L^2(E_i)$$
$$f \in C(X)$$

Set

$$\phi_0 = \bigoplus_{i \text{ even}} \psi_i$$

$$\phi_1 = \bigoplus_{i \text{ odd}} \psi_i$$

Then $[\xi] \in K_0^a(X)$ is:

$$[\xi] = (H_0, \phi_0, H_1, \phi_1, V)$$

$[\xi]$ depends only on the original elliptic complex ξ .

Two naturally occurring elliptic complexes are the deRham complex of (complex-valued) differential forms on X , and the Dolbeault complex of a holomorphic vector bundle.

For the Dolbeault complex, take X to be a complex-analytic manifold, and let E be a holomorphic vector bundle on X . $\Lambda^q\bar{T}^*$ denotes the C^∞ complex vector bundle on M whose C^∞ sections are the C^∞ q-forms of type $(0,q)$. The $\bar{\partial}$ operator is a first order differential operator

$$\bar{\partial} : C^\infty(E\otimes\Lambda^q\bar{T}^*) \to C^\infty(E\otimes\Lambda^{q+1}\bar{T}^*)$$

The Dolbeault complex of E is:

$$0 \to C^\infty(E) \xrightarrow{\bar{\partial}} C^\infty(E\otimes\bar{T}^*) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} C^\infty(E\otimes\Lambda^n\bar{T}^*) \to 0$$

$n = \dim_{\mathbb{C}} X$. $u \in C^\infty(E)$ has $\bar{\partial}u = 0$ if and only if u is a holomorphic section of E .

In §16 below (See (16.5)) and again in §19 (See (19.7)) the Hirzebruch-Riemann-Roch theorem [35] will be stated. This is a topological formula for the Euler number of the Dolbeault complex.

Part 2: Topological K homology

In Part 2, topological K homology is defined. Preliminary to this, some well-known facts on Spin^c vector bundles and Spin^c manifolds are reviewed. The basic cycles for topological K homology are Spin^c manifolds. Although this has been known for some time [26], it was not previously understood how to give a geometrically defined equivalence relation on these cycles. The geometric equivalence relation defined here (See §11 below.)

yields a definition of K_*^t which makes the theory applicable to index problems.

§8. Spin^c vector bundles

For $n \geq 3$, Spin(n) is the non-trivial double cover of SO(n). Denote the projection of Spin(n) onto SO(n) by $\eta : \text{Spin}(n) \to SO(n)$. Let ϵ be the non-identity element in Kernel(η). As usual S^1 denotes the multiplicative group of complex numbers of modulus one.

$$S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$$

In $\text{Spin}(n) \times S^1$, let Γ be the two element subgroup whose non-identity element is $(\epsilon,-1)$. Spin^c(n) is:

$$\text{Spin}^c(n) = (\text{Spin}(n) \times S^1)/\Gamma$$

For Spin(n), Spin^c(n), see [8]. A Spin (Spin^c) vector bundle can be defined as a vector bundle of $\mathbb{R}$ vector spaces together with a given lifting of the structure group from $GL(n,\mathbb{R})$ to Spin(n) (Spin^c(n)).
M. Hirsch, however, has suggested an elegant equivalent definition [39]. The Hirsch approach will be followed here.

$GL^+(n,\mathbb{R})$ denotes the group of all $n\times n$ invertible matrices $\|a_{ij}\|$ of real numbers with

$$\det \|a_{ij}\| > 0$$

$H^i(Y)$ denotes the i-th cohomology group of the topological space Y with coefficients the integers $\mathbb{Z}$.

If $n \geq 3$, then

$$H^0(GL^+(n,\mathbb{R})) = \mathbb{Z}$$

(8.1) $$H^1(GL^+(n,\mathbb{R})) = 0$$

$$H^2(GL^+(n,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$$

Let E be a vector bundle of $\mathbb{R}$ vector spaces with base Y. For simplicity, assume that Y is connected and that the one-point compactification of Y is a finite simplicial complex.

Assume also:

(8.2) For $p \in Y$, the fibre E_p of E has $\dim_{\mathbb{R}} E_p \geq 3$

Let $F(E)$ be the fibre bundle over Y whose fibre at p is the set of all vector space bases of E_p. Since Y is connected, the topological space $F(E)$ either is connected or has exactly two connected components. If $F(E)$ is connected then the $\mathbb{R}$ vector bundle E is <u>non-orientable</u>. If $F(E)$ has two connected components, then E is <u>orientable</u>. In this case, an <u>orientation</u> for E is a choice of one connected component of $F(E)$. The chosen component is denoted $F^+(E)$. These are the positively oriented bases for the fibres of E.

$F^+(E)$ is a fibre bundle over Y. The fibre $F_p^+(E)$ at a point $p \in Y$ is homeomorphic to $GL^+(n,\mathbb{R})$. By (8.1) and (8.2)

$$H^2(F_p^+(E)) = \mathbb{Z}/2\mathbb{Z}$$

Let $i_p^* : H^2(F^+(E)) \to H^2(F_p^+(E))$ be the cohomology map induced by the inclusion $F_p^+(E) \subset F^+(E)$.

(8.3) <u>Definition</u>. A <u>Spin^c structure</u> for the $\mathbb{R}$ vector bundle E is:

(i) An orientation for E

and (ii) A cohomology class $u \in H^2(F^+(E))$ such that for $p \in Y$, $i_p^*(u) \neq 0$.

A Spin^c vector bundle is a vector bundle with a given Spin^c structure.

If E is an oriented $\mathbb{R}$ vector bundle on Y, some standard algebraic topology gives the exact sequence

$$(8.4) \qquad 0 \to H^2(Y) \xrightarrow{\rho^*} H^2(F^+(E)) \xrightarrow{i_p^*} H^2(F_p^+(E)) \to H^3(Y)$$

where $\rho^* : H^2(Y) \to H^2(F^+(E))$ is the map of cohomology given by the projection $\rho : F^+(E) \to Y$.

Recall that any $\mathbb{R}$ vector bundle E on a topological space Y has Stiefel-Whitney classes $w_1(E), w_2(E), \ldots, w_r(E)$.

$$w_i(E) \in H^i(Y;\mathbb{Z}/2\mathbb{Z}) \qquad r = \dim_{\mathbb{R}}(E_p)$$

For a very clear exposition of Stiefel-Whitney classes see [40]. E is orientable if and only if $w_1(E) = 0$. E can be given a Spin^c structure if and only if $w_1(E) = 0$ and $w_2(E)$ is the mod 2 reduction of some element in $H^2(Y)$. This can be proved by using the exact sequence (8.4). Suppose that the $\mathbb{R}$ vector bundle E can be given a Spin^c structure. The exactness of (8.4) then implies that the set of all Spin^c structures on E inducing one fixed orientation of E can be put into one-to-one correspondence with $H^2(Y)$.

In definition (8.3), it was assumed that Y is connected and that $\dim_{\mathbb{R}} E_p \geq 3$. To remove these simplifying assumptions, first suppose that $\dim_{\mathbb{R}} E_p < 3$. Let Θ^3 be the trivial $\mathbb{R}$ vector bundle on Y with fibre $\mathbb{R}^3$.

$$\Theta^3 = Y \times \mathbb{R}^3$$

Then a Spin^c structure for E is a Spin^c structure for $E \oplus \Theta^3$.

If Y is not connected, a Spin^c structure for E is a Spin^c structure for the restriction of E to each connected component of Y .

By forgetting some structure a complex vector bundle or a Spin vector bundle canonically becomes a Spin^c vector bundle.

$$\begin{array}{ccc} & \text{complex} & \\ & \Downarrow & \\ \text{Spin} & \Rightarrow & \mathrm{Spin}^c \\ & & \Downarrow \\ & & \text{oriented} \end{array}$$

A Spin^c structure for the $\mathbb{R}$ vector bundle E can be thought of as an orientation for E plus a slight extra bit of structure. Spin^c structures behave very much like orientations. For example, an orientation on two out of three $\mathbb{R}$ vector bundles in a short exact sequence determine an orientation on the third vector bundle. Analogous assertions are true for Spin^c structures.

(8.5) <u>Exact Sequence Lemma</u>. Let $0 \to E_0 \to E_1 \to E_2 \to 0$ be an exact sequence of $\mathbb{R}$ vector bundles on Y . If two out of three of E_0, E_1, E_2 have Spin^c structures, then the third has a uniquely determined Spin^c structure.

If E is a Spin^c vector bundle on Y, then the Spin^c structure on E can be reversed. To do this, as in definition (8.3), let the Spin^c structure on E be given by $F^+(E)$ and $u \in H^2(F^+(E))$. Denote the negatively oriented bases by $F^-(E)$. Let $e_1, e_2, \ldots, e_r$ be a negatively oriented basis for E_p. Map $F^-(E)$ to $F^+(E)$ by $\phi : F^-(E) \to F^+(E)$ where

$$\phi(e_1, e_2, e_3, \ldots, e_r) = (-e_1, e_2, e_3, \ldots, e_r)$$

Then the reversed Spin^c structure on E is given by $F^-(E)$ and $\phi^*(u) \in H^2(F^-(E))$.

If E is a Spin^c vector bundle on Y, then E has a first Chern class

$$c_1(E) \in H^2(Y) .$$

To define $c_1(E)$, let $u \in H^2(F^+(E))$ be as in (ii) of definition (8.3). Since $H^2(F_p^+(E)) = \mathbb{Z}/2\mathbb{Z}$, $i_p^*(2u) = 0$. Use the exact sequence (8.4) to define $c_1(E)$ by

$$\rho^*(c_1(E)) = 2u$$

For a Spin^c vector bundle E, the Todd class, denoted $Td(E)$, is

$$Td(E) = e^{c_1(E)/2} \hat{A}(E) \tag{8.6}$$

with $Td(E) \in H^*(Y;\mathbb{Q})$

In (8.6) $\hat{A}(E)$ is the $\hat{A}$ polynomial in the Pontryagin classes of E:

$$\hat{A}(E) = \prod_i \frac{x_i/2}{\sinh(x_i/2)} \tag{8.7}$$

where the j-th Pontryagin class [35] of E is the j-th elementary symmetric function of the x_i^2.

For a complex vector bundle E the Todd class is defined by

$$Td(E) = \prod_i \frac{x_i}{1 - e^{-x_i}} \tag{8.8}$$

where the j-th Chern class [35] of E is the j-th elementary symmetric function of the x_i .

The identity

$$\frac{x}{1 - e^{-x}} = e^{x/2} \frac{x/2}{\sinh(x/2)}$$

implies that for a complex vector bundle E , the Todd class of E as a complex vector bundle equals $Td(E)$ of the underlying Spin^c vector bundle.

§9. Spin^c manifolds

A Spin^c manifold is a C^∞ manifold M with a given Spin^c structure on its tangent bundle TM . M can be compact or non-compact. M may have a boundary.

If M is a Spin^c manifold with boundary, then the boundary (denoted ∂M) is again a Spin^c manifold. To define the Spin^c structure on $T(\partial M)$, let ν be the normal bundle of ∂M in M . On ∂M there is the exact sequence

$$0 \to T(\partial M) \to TM|\partial M \to \nu \to 0$$

and ν can be trivialized by choosing an inward normal vector field on ∂M . This trivialization of ν makes ν a Spin^c vector bundle on ∂M . The exact sequence lemma (8.5) now applies to give $T(\partial M)$ a uniquely determined Spin^c structure.

Various well-known structures on a manifold M make M into a Spin^c manifold.

$$\begin{tikzcd}
 & \text{(complex-analytic)} \arrow[d, Rightarrow] \\
\text{(symplectic)} \arrow[r, Rightarrow] & \text{(almost complex)} \arrow[d, Rightarrow] \\
\text{(contact)} \arrow[r, Rightarrow] & \text{(stably almost complex)} \arrow[d, Rightarrow] \\
\text{(Spin)} \arrow[r, Rightarrow] & (\mathrm{Spin}^c) \arrow[d, Rightarrow] \\
 & \text{(oriented)}
\end{tikzcd}$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

§10. Clutching construction: example 1

Let M be a Spin^c manifold. On M let H be a C^∞ Spin^c vector bundle with even-dimensional fibres. 1 denotes the trivial $\mathbb{R}$ line bundle on M :

$$1 = M \times \mathbb{R} \quad .$$

In a C^∞ way, choose a positive definite symmetric inner product for each fibre of $H \oplus 1$. Let $\hat{M}$ be the unit sphere bundle of $H \oplus 1$.

$$\hat{M} = S(H \oplus 1)$$

The Spin^c structures on TM and H give a Spin^c structure on $T\hat{M}$ and so $\hat{M}$ is a Spin^c manifold. $\hat{M}$ is a sphere bundle over M with even-dimensional spheres as fibres. Denote the projection by $\rho : \hat{M} \to M$.

For $p \in M$, set $n = \dim_{\mathbb{R}}(H_p)$. Since n is even $\mathrm{Spin}^c(n)$ has two ½-Spin representations. The Spin^c structure on H associates to the ½-Spin representations complex vector bundles H_+, H_- on M . Let H_0, H_1 be the pull-backs to H of H_+ and H_- . Clifford multiplication gives a pairing

$$H \times H_+ \to H_-$$

which defines a vector bundle map

$$H_0 \xrightarrow{\sigma} H_1 \quad .$$

σ is an isomorphism off the zero section of H .

$\hat{M} = S(H \oplus 1)$ consists of two copies of the unit ball bundle of H glued together by the identity map of $S(H)$.

$$\hat{M} = B(H) \underset{S(H)}{\cup} B(H)$$

Form a vector bundle $\hat{H}$ on $\hat{M}$ by putting H_0 on one copy of $B(H)$, H_1 on the other copy of $B(H)$ and then attaching together on $S(H)$ by σ .

$$\hat{H} = H_0 \underset{\sigma}{\cup} H_1$$

Starting with M, H this clutching construction has produced $\hat{M}, \hat{H}, \rho$. For each $p \in M$, the restriction of $\hat{H}$ to the even-dimensional sphere $\rho^{-1}(p)$, is the Bott generator vector bundle on $\rho^{-1}(p)$. Compare [8]

This clutching construction will be used in §11 below to define the equivalence relation on the cycles for topological K homology.

§11. Topological K homology

A K cycle on the topological space X is a triple (M,E,ϕ) such that:

(i) M is a compact Spin^c manifold without boundary

(ii) E is a complex vector bundle on M

(iii) ϕ is a continuous map from M to X .

M is not required to be connected, and E may have different fibre dimensions on different connected components of M . Thus for K cycles on X there is the evident disjoint union operation. Denote this by $(M_1,E_1,\phi_1) \cup (M_2,E_2,\phi_2)$. Two K cycles (M,E,ϕ) and (M',E',ϕ') are <u>isomorphic</u> if there exists a diffeomorphism h mapping M onto M' such that h preserves the Spin^c structures, $h^*(E') \cong E$, and there is commutativity in the diagram

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "\phi"'] & & M' \arrow[dl, "\phi'"] \\ & X & \end{tikzcd}$$

Let $\Gamma(X)$ be the collection of all K cycles on X . Define an equivalence relation $\sim$ on $\Gamma(X)$ by letting $\sim$ be the equivalence relation generated by the following three elementary steps.

(i) Bordism

(ii) Direct sum

(iii) Vector bundle modification

(i) <u>Bordism</u>. $(M_0,E_0,\phi_0) \sim (M_1,E_1,\phi_1)$ if there exists a compact Spin^c manifold W with boundary and a complex vector bundle E on W and a continuous map $\phi : W \to X$ such that

$(\partial W, E|\partial W, \phi|\partial W)$ is isomorphic to the disjoint union $(M_0, E_0, \phi_0) \cup (-M_1, E_1, \phi_1)$. Here $-M_1$ denotes M_1 with the Spin^c structure on TM_1 reversed.

(ii) Direct sum. Suppose given (M,E,ϕ) and also given a direct sum decomposition $E = E_1 \oplus E_2$. Then:

$$(M, E_1 \oplus E_2, \phi) \sim (M, E_1, \phi) \cup (M, E_2, \phi)$$

(iii) Vector bundle modification. Suppose given (M,E,ϕ) and also given a C^∞ Spin^c vector bundle H on M with even-dimensional fibres. Use the clutching construction of §10 above to form $\hat{M}, \hat{H}, \rho$. Then:

$$(M,E,\phi) \sim (\hat{M}, \hat{H} \otimes \rho^*(E), \phi \circ \rho)$$

This completes the definition of the equivalence relation $\sim$ on $\Gamma(X)$.

Set $K_*^t(X) = \Gamma(X)/\sim$. Disjoint union makes $K_*^t(X)$ into an abelian group. Note that for a K cycle (M,E,ϕ) on X , the equivalence relation $\sim$ preserves the parity of the dimension of M . In $K_*^t(X)$ let $K_0^t(X)$ $(K_1^t(X))$ be the subgroup given by all (M,E,ϕ) with each connected component of M even-dimensional (odd-dimensional). Then

$$K_*^t(X) = K_0^t(X) \oplus K_1^t(X) \quad .$$

$K_0^t(X)$, $K_1^t(X)$ are the topological K homology groups of X .

If $f : X \to Y$ is a continuous map, there is a homomorphism of abelian groups

$$f_* : K_*^t(X) \to K_*^t(Y)$$

For a K cycle (M,E,ϕ) on X ,

$$f_*(M,E,\phi) = (M,E,f\circ\phi)$$

Let $K_t^0(X)$ denote the Grothendieck group of topological complex vector bundles on X . There is a bilinear pairing (the cap product):

$$K_t^0(X) \otimes K_*^t(X) \to K_*^t(X)$$

Let F be a complex vector bundle on X, and let (M,E,ϕ) be a K cycle on X. The cap product pairing, denoted $F \cap (M,E,\phi)$, is:

$$F \cap (M,E,\phi) = (M,E \otimes \phi^* F,\phi)$$

For a point, a calculation using Bott periodicity gives

$$K_0^t(\cdot) = \mathbb{Z}$$

$$K_1^t(\cdot) = 0$$

Let $\epsilon : X \to \cdot$ be the map of X to a point. Consider $\epsilon_* : K_0^t(X) \to K_0^t(\cdot)$. Since $K_0^t(\cdot) = \mathbb{Z}$, ϵ_* may be viewed as a map from $K_0^t(X)$ to $\mathbb{Z}$.

(11.1) $$\epsilon_* : K_0^t(X) \to \mathbb{Z}$$

This map can be made explicit either by using Bott periodicity or by using characteristic classes. The characteristic class formula for ϵ_* is:

(11.2) $$\epsilon_*(M,E,\phi) = (\mathrm{ch}(E) \cup \mathrm{Td}(TM))[M]$$

In (11.2), $[M]$ is the orientation cycle of M. Also,

(11.3) $$\mathrm{ch}(E) = e^{x_1} + e^{x_2} + \ldots + e^{x_\ell}$$

where the j-th Chern class of E is the j-th elementary symmetric function of the x_i. $\mathrm{Td}(TM)$ is the Todd class of the Spin^c vector bundle TM, as defined by (8.6).

It is not immediately apparent that the right hand side of (11.2) is an integer. This integrality, however, is not difficult to prove as a corollary of Bott periodicity [5] [21].

There is a natural transformation of homology theories: the (homology) Chern character from K_*^t to $H_*(\ ;\mathbb{Q})$. As usual, $H_*(\ ;\mathbb{Q})$ is ordinary homology with coefficients the rational numbers $\mathbb{Q}$. Denote this natural transformation by

$$\mathrm{ch}_. : K_*^t(X) \to H_*(X;\mathbb{Q})$$

To define ch. let (M,E,ϕ) be a K cycle on X. Let $\phi_* : H_*(M,\mathbb{Q}) \to H_*(X;\mathbb{Q})$ be the map of rational homology induced by ϕ. Then $\mathrm{ch}(E) \cup \mathrm{Td}(TM) \cap [M]$ is the Poincaré dual of $\mathrm{ch}(E) \cup \mathrm{Td}(TM)$ and

$$\mathrm{ch.}(M,E,\phi) = \phi_*(\mathrm{ch}(E) \cup \mathrm{Td}(TM) \cap [M])$$

Let $ch : K_t^0 \to H^*(\ ;\mathbb{Q})$ be as in (11.3) ch is the (cohomology) Chern character. ch and ch. preserve cap product; i.e. there is commutativity in the diagram:

$$\begin{tikzcd} K_t^0(X) \otimes K_*^t(X) \arrow[r, "\cap"] \arrow[d, "ch \otimes ch."] & K_*^t(X) \arrow[d, "ch."] \\ H^*(X;\mathbb{Q}) \otimes H_*(X;\mathbb{Q}) \arrow[r, "\cap"] & H_*(X;\mathbb{Q}) \end{tikzcd}$$

If X is a finite simplicial complex, then $K_*^t(X)$ is a finitely generated abelian group [9] and ch. gives an isomorphism

$$K_*^t(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong H_*(X;\mathbb{Q})$$

§12. Clutching construction: example 2

Let X be a finite simplicial complex. Suppose given (V,i,η) such that

(i) V is a (possibly non-compact) Spin^c manifold without boundary

(ii) i is a topological embedding of X into V with the pair $(V,i(X))$ triangulable.

(iii) $\eta = \{0 \to E_0 \to E_1 \to \ldots \to E_r \to 0\}$ is a complex of $\mathbb{C}$ vector bundles on V with η exact on $V - i(X)$

From (V,i,η) a K cycle (M,E,ϕ) on X can be constructed as follows. Denote the map $E_j \to E_{j+1}$ by $\sigma : E_j \to E_{j+1}$. Choose Hermitian structures for the E_j. Each fibre of E_j is then a finite dimensional Hilbert Space, so the map $\sigma : E_j \to E_{j+1}$ has an adjoint $\sigma^* : E_{j+1} \to E_j$. Set $F_0 = \underset{j \text{ even}}{\oplus} E_j$, $F_1 = \underset{j \text{ odd}}{\oplus} E_j$. Map F_0 to F_1 by $\sigma + \sigma^*$.

$$\sigma + \sigma^* : F_0 \to F_1$$

$\sigma + \sigma^*$ is an isomorphism on $V - i(X)$. Regard the embedding $i : X \to V$ as an inclusion $X \subset V$. Include V in $V \times \mathbb{R}$ by setting $V = V \times \{0\}$.

$$X \subset V \subset V \times \mathbb{R}$$

In $V \times \mathbb{R}$ let W be a regular neighborhood of X with smooth boundary. Let $\rho : W \to X$ be the retraction of the regular neighborhood W onto X . Set $M = \partial W$ and $\phi = \rho|M$. The Spin^c structure on TV gives TW a Spin^c structure. $M = \partial W$ so M is a Spin^c manifold. Set:

$$\begin{aligned} M_+ &= M \cap (V\times\mathbb{R}_+) & \mathbb{R}_+ &= \{\lambda\in\mathbb{R}\,|\,\lambda\geq 0\} \\ M_- &= M \cap (V\times\mathbb{R}_-) & \mathbb{R}_- &= \{\lambda\in\mathbb{R}\,|\,\lambda\leq 0\} \\ M_0 &= M \cap (V\times\{0\}) \end{aligned}$$

Then M consists of M_+ and M_- glued together along M_0 .

$$M = M_+ \underset{M_0}{\cup} M_-$$

Map M to $V = V \times \{0\}$ by $(p,\lambda) \to (p,0)$. Form a vector bundle E on M by pulling-back F_0 to M_+ , F_1 to M_- , and then attaching together on M_0 by $\sigma + \sigma^*$.

This completes the construction of the K cycle (M,E,ϕ) .

(12.1) $$(M,E,\phi) \in K_*^t(X)$$

Set $n = \dim(V)$ and define $e(n)$ by:

$$e(n) = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Then $(M,E,\phi) \in K_{e(n)}^t(X)$.

Let Y be a simplicial complex. For an inclusion $X \subset Y$ with the pair (Y,X) triangulable, $K_t^0(Y,Y-X)$ is defined as equivalence classes of vector bundle complexes η on Y such that η is exact on $Y - X$. Compare [8],[18]. The above clutching construction gives a homomorphism of abelian groups

(12.2) $$K_t^0(V,V-X) \to K_{e(n)}^t(X)$$

Using Bott periodicity it can be proved that this map is an isomorphism. (12.2) is Lefschetz duality in topological K theory.

Part 3: Riemann-Roch Theorem

In part 3, the Riemann-Roch problem is stated, and then a Riemann-Roch theorem is given which extends the Grothendieck-Riemann-Roch (=GRR) theorem to singular varieties. For an earlier version of this RR theorem see [18].

K theory has its origins in the RR problem. Following basic work of J. P. Serre [45], Grothendieck gave the first definition and application of K theory in his statement and proof of GRR [20]. Within the context of algebraic geometry, Grothendieck defined the K homology and K cohomology of a projective algebraic variety X. The algebraic geometry K homology, denoted $K_0^{\omega}(X)$, is the Grothendieck group of coherent algebraic sheaves on X. The algebraic geometry K cohomology, denoted $K_{\omega}^0(X)$, is the Grothendieck group of algebraic vector bundles on X.

If the ground field is the complex numbers $\mathbb{C}$, then the underlying topological space of X is a finite simplicial complex and has topological K groups $K_*^t(X)$, $K_t^*(X)$. To extend GRR to singular varieties one has to define natural maps

$$K_{\omega}^0(X) \longrightarrow K_t^0(X)$$

$$K_0^{\omega}(X) \longrightarrow K_0^t(X)$$

The cohomology map is evident. By forgetting some structure, an algebraic vector bundle becomes a topological vector bundle. This gives the cohomology map $K_{\omega}^0(X) \longrightarrow K_t^0(X)$. Using K_*^t as defined in Part 2 above, the homology map $K_0^{\omega}(X) \longrightarrow K_0^t(X)$ is also readily defined.

§13. The Riemann-Roch Problem

Let X be a projective algebraic variety over the complex numbers $\mathbb{C}$. X may have singularities. On X, let F be a coherent algebraic sheaf. $H^i(X,F)$ denotes the i-th cohomology group of X with coefficients in F. For each integer i, $H^i(X,F)$ is a finite dimensional vector space over $\mathbb{C}$. If $i > n = \dim_{\mathbb{C}} X$, then $H^i(X,F) = 0$. Hence the Euler number $\chi(X,F)$ can be defined:

$$\chi(X,\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X,\mathcal{F}) \qquad n = \dim_{\mathbb{C}} X$$

The Riemann-Roch problem is:

RR Problem: Given X and $\mathcal{F}$, compute as topologically as possible, the Euler number $\chi(X,\mathcal{F})$.

If E is an algebraic vector bundle on X, then $\underline{E}$ denotes the sheaf of germs of algebraic sections of E. Consider the RR problem in the special case when X is non-singular and $\mathcal{F} = \underline{E}$. In this case, as in §7, there is the $\bar{\partial}$ complex of E.

$$0 \longrightarrow C^\infty(E) \xrightarrow{\bar{\partial}} C^\infty(E\otimes\bar{T}^*) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} C^\infty(E\otimes\Lambda^n\bar{T}^*) \longrightarrow 0$$

Denote the Euler number of this $\bar{\partial}$ complex by $\mathrm{Index}(\bar{\partial},E)$. According to the Dolbeault theorem [31], the i-th homology group of the $\bar{\partial}$ complex is canonically isomorphic to $H^i(X,\underline{E})$. Therefore:

$$\chi(X,\underline{E}) = \mathrm{Index}(\bar{\partial},E)$$

Thus the general RR problem can be viewed as an interesting index problem on the possibly singular variety X.

§14. K homology in algebraic geometry

X is a complex projective algebraic variety. $K_0^\omega(X)$ is the Grothendieck group of coherent algebraic sheaves on X. $K_\omega^0(X)$ is the Grothendieck group of algebraic vector bundles on X.

Tensor product of vector bundles makes $K_\omega^0(X)$ into a ring. $K_0^\omega(X)$ is a module over this ring

(14.1) $$K_\omega^0(X) \otimes_{\mathbb{Z}} K_0^\omega(X) \longrightarrow K_0^\omega(X)$$

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be coherent algebraic sheaves on X. $\mathcal{O}_X$ denotes the structure sheaf of X, i.e. the sheaf of germs of algebraic functions on X. Then $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$ is again a coherent algebraic sheaf on X. The pairing (14.1), denoted $E \cap \mathcal{F}$, is:

(14.2) $$E \cap \mathcal{F} = \underline{E} \otimes_{\mathcal{O}_X} \mathcal{F}$$

In (14.2) $\underline{E}$ is the sheaf of germs of algebraic sections of E.

If Y is another projective variety, and $f : X \longrightarrow Y$ is a morphism of algebraic varieties, then there are induced maps

$$f^* : K_\omega^0(Y) \longrightarrow K_\omega^0(X) \tag{14.3}$$

$$f_* : K_0^\omega(X) \longrightarrow K_0^\omega(Y) \tag{14.4}$$

f^* is given by vector bundle pull back. f_* is defined by

$$f_*F = \sum_{i=0}^{n} (-1)^i (R^i f) F \qquad n = \dim_{\mathbb{C}} X$$

$R^i f$ is the i-th derived functor of direct image [20]. In particular, define $\chi : K_0^\omega(X) \longrightarrow \mathbb{Z}$ by

$$\chi(F) = \chi(X,F) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X,F) \qquad n = \dim_{\mathbb{C}} X$$

Let $\epsilon : X \longrightarrow \cdot$ be the map of X to a point. A coherent algebraic sheaf on a point is just a finite dimensional vector space over $\mathbb{C}$. To such a vector space assign its dimension. This gives an isomorphism

$$K_0^\omega(\cdot) = \mathbb{Z}$$

By definition, the map $\epsilon_* : K_0^\omega(X) \longrightarrow K_0^\omega(\cdot)$ is $\chi : K_0^\omega(X) \longrightarrow \mathbb{Z}$.

§15. Riemann-Roch theorem

As above, X is a complex projective algebraic variety which may be singular. The underlying topological space of X is a finite simplicial complex and so has topological K groups $K_t^0(X)$, $K_0^t(X)$. $K_t^0(X)$ is the Grothendieck group of topological complex vector bundles on X. $K_0^t(X)$ is the group of K cycles on X as defined in §11 above.

To solve the RR problem, define two natural maps

$$\alpha^\cdot : K_\omega^0(X) \longrightarrow K_t^0(X)$$

$$\alpha_\cdot : K_0^\omega(X) \longrightarrow K_0^t(X)$$

$\alpha^\cdot$ is evident. By forgetting some structure an algebraic vector bundle on X becomes a topological vector bundle on X. This forgetting operation gives $\alpha^\cdot : K_\omega^0(X) \longrightarrow K_t^0(X)$. $\alpha^\cdot$ is a natural tranformation of contravariant functors.

For $\alpha_\cdot : K_0^\omega(X) \longrightarrow K_0^t(X)$, let (M,E,ϕ) be a triple such that:

(i) M is a non-singular projective algebraic variety

(ii) E is an algebraic vector bundle on M

(iii) ϕ is a morphism of algebraic varieties mapping M to X . $\phi : M \longrightarrow X$

Such a triple will be called an algebraic K cycle on X .

Given an algebraic K cycle (M,E,ϕ) on X , consider (as in (14.4)) $\phi_* : K_0^\omega(M) \longrightarrow K_0^\omega(X)$.
$\underline{E}$ denotes the sheaf of germs of algebraic sections of E .
Thus $\phi_*(\underline{E}) \in K_0^\omega(X)$. By forgetting some structure an algebraic K cycle (M,E,ϕ) becomes a topological K cycle and determines an element $(M,E,\phi) \in K_0^t(X)$. $\alpha. : K_0^\omega(X) \longrightarrow K_0^t(X)$ is given by

(15.1) $$\alpha.\ \phi_*(\underline{E}) = (M,E,\phi)$$

Two observations are needed. First, the $\phi_*(\underline{E})$ obtained from algebraic K cycles on X generate $K_0^\omega(X)$. If X is non-singular this is not difficult, and follows from the well-known lemma that any coherent algebraic sheaf on a non-singular variety can be resolved by locally free sheaves [20]. For singular X , this is a corollary of Hironaka's resolution of singularities theorem [34].

Second, suppose that (M,E,ϕ) and (M',E',ϕ') are two algebraic K cycles on X such that in $K_0^\omega(X)$

$$\phi_*(\underline{E}) = \phi'_*(\underline{E}')$$

It is not obvious that in $K_0^t(X)$ one then has

$$(M,E,\phi) = (M',E',\phi')$$

Although not obvious this is true. In fact, this assertion is the heart of the RR matter.

(15.2) Proposition. Let (M,E,ϕ) and (M',E',ϕ') be two algebraic K cycles on X such that in $K_0^\omega(X)$, $\phi_*(\underline{E}) = \phi'_*(\underline{E}')$. Then in $K_0^t(X)$, $(M,E,\phi) = (M',E',\phi')$.

Once this proposition has been proved, it is then immediate that for a morphism of algebraic varieties $f : X \longrightarrow Y$ there is commutativity in the diagram:

$$\begin{tikzcd} K_0^\omega(X) \arrow[r, "f_*"] \arrow[d, "\alpha."'] & K_0^\omega(Y) \arrow[d, "\alpha."] \\ K_0^t(X) \arrow[r, "f_*"'] & K_0^t(Y) \end{tikzcd}$$

This is summarized by

(15.3) <u>RR Theorem</u>. $\alpha. : K_0^\omega(X) \longrightarrow K_0^t(X)$ is a natural transformation of covariant functors.

§16. Properties of the RR map

The RR map $\alpha. : K_0^\omega(X) \longrightarrow K_0^t(X)$ has several very useful properties.

<u>Cap Product</u>. $\alpha^{\bullet}$ and $\alpha.$ preserve cap product; there is commutativity in the diagram

$$\begin{tikzcd} K_\omega^0(X) \otimes K_0^\omega(X) \arrow[r] \arrow[d, "\alpha^{\bullet}\otimes\alpha."'] & K_0^\omega(X) \arrow[d, "\alpha."] \\ K_t^0(X) \otimes K_0^t(X) \arrow[r] & K_0^t(X) \end{tikzcd}$$

<u>Embedding</u>. Given $\mathcal{F}$ on X, let $i : X \longrightarrow V$ be an algebraic embedding of X into a non-singular projective variety V. On V let

(16.1) $0 \longrightarrow \underline{E}_0 \longrightarrow \underline{E}_1 \longrightarrow \ldots \longrightarrow \underline{E}_r \longrightarrow i_*\mathcal{F} \longrightarrow 0$

be a resolution of the direct image sheaf $i_*\mathcal{F}$ by locally free sheaves. Let η be the complex of vector bundles on V obtained by deleting $i_*\mathcal{F}$ from (16.1).

$$\eta = \{0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \ldots \longrightarrow E_r \longrightarrow 0\}$$

η is exact on $V - i(X)$. The clutching construction of §12 above now applies to (V,i,η) to give a K cycle (M,E,ϕ) on X. Then:

(16.2) The element in $K_0^t(X)$ obtained by applying the clutching construction of §12 to (V,i,η) is $\alpha.\mathcal{F}$.

<u>Solution of the RR problem</u>. Let $\epsilon : X \longrightarrow \cdot$ be the map of X to a point. It follows from the RR theorem that there is commutativity in the diagram

$$\begin{tikzcd} K_0^{\omega}(X) \arrow[d, "\alpha."'] \arrow[dr, "\chi"] & \\ K_0^t(X) \arrow[r, "\epsilon_*"'] & \mathbb{Z} \end{tikzcd}$$

$\epsilon_* : K_0^t(X) \longrightarrow \mathbb{Z}$ is as in (11.1). In principle at least, this solves the RR problem because it replaces the algebraic geometry problem of computing $\chi(X,\mathcal{F})$ by the topological problem of computing $\epsilon_*\alpha.\mathcal{F}$. This topological problem can be solved either by using Bott periodicity or by applying the Chern character to obtain a characteristic class formula.

<u>Chern Character</u>. $\alpha^{\bullet}$ and $\alpha.$ can be composed with the Chern character.

$$K^0_{\omega}(X) \xrightarrow{\alpha^{\bullet}} K^0_t(X) \xrightarrow{ch} H^*(X;\mathbb{Q})$$

$$K_0^{\omega}(X) \xrightarrow{\alpha.} K_0^t(X) \xrightarrow{ch.} H_*(X;\mathbb{Q})$$

Let $\epsilon_* : H_*(X;\mathbb{Q}) \longrightarrow H_*(\cdot;\mathbb{Q}) = \mathbb{Q}$ be the map of rational homology induced by mapping X to a point.
Let $td(X) = ch.\alpha.(\mathcal{O}_X)$.

$$td(X) \in H_*(X;\mathbb{Q})$$

If E is any algebraic vector bundle on X , the RR theorem implies

$$(16.3) \qquad \chi(X,\underline{E}) = \epsilon_*(ch(E) \cap td(X))$$

If X is non-singular, then $td(X)$ is the Poincaré dual of $Td(TX)$.

(16.4) For non-singular X , $td(X) = Td(TX) \cap [X]$

Hence for non-singular X , (16.3) can be rewritten

$$(16.5) \qquad \chi(X,\underline{E}) = (ch(E) \cup Td(TX))[X]$$

(16.5) is the Hirzebruch-Riemann-Roch formula [35].

In (16.4) and (16.5) , $Td(TX)$ is the Todd class of the complex vector bundle TX as defined by (8.8) .

<u>Resolution of singularities</u>. If X is singular , let $\pi : \tilde{X} \longrightarrow X$ be a resolution of singularities of X in the sense of Hironaka [34].

$\pi_* : H_*(\tilde{X};\mathbb{Q}) \longrightarrow H_*(X;\mathbb{Q})$ is the map of rational homology induced by π. $\pi_*(td(\tilde{X}))$ is usually not equal to $td(X)$. However, $\pi_*(td(\tilde{X}))$ is intrinsic to X; i.e. $\pi_*(td(\tilde{X}))$ does not depend on the choice of the resolution of singularities. The difference $td(X) - \pi_* td(\tilde{X})$ is made up of contributions from the singularities of X [14],[17].

Part 4: Isomorphism of analytic and topological K homology

Part 4 describes an isomorphism of homology theories $\mu : K_*^t(X) \longrightarrow K_*^a(X)$. Suppose given a K cycle (M,E,ϕ) on the topological space X. Let D be the Dirac operator of the Spin^c manifold M. Form the tensor product of D with the identity map of E to obtain an elliptic first order differential operator $D\otimes I_E$ on M. As in Part 1, $D\otimes I_E$ determines an element (denoted $E\cap[D]$) in $K_*^a(M)$.

If M is even dimensional, then $E\cap[D] \in K_0^a(M)$. For M odd dimensional $D\otimes I_E$ is self-adjoint, so in this case $E\cap[D] \in K_1^a(M)$. If $\phi_* : K_*^a(M) \longrightarrow K_*^a(X)$ is the map of analytic K homology induced by $\phi : M \longrightarrow X$, then $\mu : K_*^t(X) \longrightarrow K_*^a(X)$ is defined by:

$$\mu(M,E,\phi) = \phi_*(E\cap[D])$$

§17. Dirac Operator

Let M be a C^∞ manifold. T^* denotes the complexified cotangent bundle of M.

$$T^* = \mathbb{C} \otimes_{\mathbb{R}} T^*_{\mathbb{R}}(M)$$

For E a C^∞ complex vector bundle on M, $C^\infty(E)$ is the $\mathbb{C}$ vector space of all C^∞ sections of E. d is the de Rham operator

$$d : C^\infty(\Lambda^j T^*) \longrightarrow C^\infty(\Lambda^{j+1} T^*)$$

Given a C^∞ complex vector bundle E on M, it is always possible to choose a connection for E. This is a $\mathbb{C}$ linear map

$$\nabla : C^\infty(E) \longrightarrow C^\infty(T^*\otimes E)$$

such that for any C^∞ function $f : M \longrightarrow \mathbb{C}$ and any $s \in C^\infty(E)$,

$$\nabla(f\cdot s) = df \otimes s + f\cdot\nabla s$$

Now let M be an even dimensional closed Spin^c manifold with $n = \dim_{\mathbb{R}}(M)$. The Spin^c structure on the tangent bundle of M asociates to the two 1/2 - Spin representations of $\mathrm{Spin}^c(n)$ C^∞ complex vector bundles F_+, F_- on M. Clifford multiplication gives a bilinear pairing

(17.1) $$T^* \otimes F_+ \xrightarrow{\gamma} F_-$$

Choose a connection ∇ for F_+

(17.2) $$C^\infty(F_+) \xrightarrow{\nabla} C^\infty(T^*\otimes F_+)$$

The Dirac operator of M is the composition

$$C^\infty(F_+) \xrightarrow{\nabla} C^\infty(T^*\otimes F_+) \xrightarrow{\gamma} C^\infty(F_-)$$

Denote this operator [3] by

$$D : C^\infty(F_+) \longrightarrow C^\infty(F_-)$$

More generally, if E is a C^∞ complex vector bundle on M, define

$$D\otimes I_E : C^\infty(F_+\otimes E) \longrightarrow C^\infty(F_-\otimes E)$$

as follows. Tensor (17.1) with the identity map of E to obtain

(17.3) $$T^*\otimes F_+\otimes E \xrightarrow{\gamma\otimes I_E} F_-\otimes E$$

Choose a connection ∇ for $F_+\otimes E$. Then $D\otimes I_E$ is the composition

$$C^\infty(F_+\otimes E) \xrightarrow{\nabla} C^\infty(T^*\otimes F_+\otimes E) \xrightarrow{\gamma\otimes I_E} C^\infty(F_-\otimes E)$$

As in §5 above, $D\otimes I_E$ determines an element $[D\otimes I_E] \in K_0^a(M)$. Set $[D\otimes I_E] = E\cap[D]$. (In a sense which can be made precise, $E\cap[D]$ is the cap product of E and $[D]$.)

---

[3]The Dirac operator of a Spin^c manifold is not unique. For example, different choices of connection ∇ for F_+ will give slightly different operators. There is, however, no difficulty in verifying that the element $[D] \in K_0^a(M)$ is well defined and depends only on the given Spin^c structure for TM.

Let M be an odd dimensional closed Spin^c manifold with $n = \dim_{\mathbb{R}}(M)$. The Spin^c structure on the tangent bundle of M associates to the Spin representation of $\mathrm{Spin}^c(n)$ a C^∞ complex vector bundle F on M. Clifford multiplication gives a bilinear pairing

$$T^* \otimes F \xrightarrow{\gamma} F$$

Choose a connection ∇ for F. The Dirac operator D of M is the composition

$$C^\infty(F) \xrightarrow{\nabla} C^\infty(T^* \otimes F) \xrightarrow{\gamma} C^\infty(F)$$

More generally, if E is a C^∞ complex vector bundle on M, choose a connection ∇ for $F \otimes E$. The operator $D \otimes I_E : C^\infty(F \otimes E) \longrightarrow C^\infty(F \otimes E)$ is the composition

$$C^\infty(F \otimes E) \xrightarrow{\nabla} C^\infty(T^* \otimes F \otimes E) \xrightarrow{\gamma \otimes I_E} C^\infty(F \otimes E)$$

$D \otimes I_E$ is a self-adjoint first-order elliptic differential operator on M. As in §6 above, $D \otimes I_E$ determines an element $[D \otimes I_E] \in K_1^a(M)$.

Set $[D \otimes I_E] = E \cap [D]$.

§18. The isomorphism $K_*^t(X) \longrightarrow K_*^a(X)$

Let (M, E, ϕ) be a K cycle on the topological space X. $\phi_* : K_*^a(M) \longrightarrow K_*^a(X)$ is the map of analytic K homology induced by $\phi : M \longrightarrow X$. With $E \cap [D]$ as in §17 above, define $\mu : K_*^t(X) \longrightarrow K_*^a(X)$ by

$$\mu(M, E, \phi) = \phi_*(E \cap [D]) \tag{18.1}$$

It is not immediately clear that μ is well-defined. This basic point is asserted by:

(18.2) <u>Proposition</u>. Let (M, E, ϕ) and (M', E', ϕ') be two K cycles on X such that in $K_*^t(X)$

$$(M, E, \phi) = (M', E', \phi')$$

Then in $K^a_*(X)$,

$$\phi_*(E\cap[D]) = \phi'_*(E'\cap[D'])$$

(18.3) Remarks. Naturality of μ is a corollary of the proposition; if $f : X \longrightarrow Y$ is a continuous map then there is commutativity in the diagram

$$\begin{array}{ccc} K^t_*(X) & \xrightarrow{f_*} & K^t_*(Y) \\ \mu\downarrow & & \downarrow\mu \\ K^a_*(X) & \xrightarrow[f_*]{} & K^a_*(Y) \end{array} \tag{18.3}$$

Mapping X to a point gives a commutative diagram

$$\begin{array}{ccc} K^t_0(X) & & \\ \mu\downarrow & \searrow^{\epsilon_*} & \\ K^a_0(X) & \xrightarrow[\text{Index}]{} & \mathbb{Z} \end{array} \tag{18.4}$$

For Index : $K^a_0(X) \longrightarrow \mathbb{Z}$ and $\epsilon_* : K^t_0(X) \longrightarrow \mathbb{Z}$

See (4.1), (11.1), (11.2).

So far there is a very strong formal similarity to the Riemann-Roch map $\alpha.$ of Part 3. But $\alpha.$ is usually not an isomorphism. The contrary is true for μ , and the main result of this note is:

(18.5) Theorem. If X is a finite simplicial complex, then $\mu : K^t_*(X) \longrightarrow K^a_*(X)$ is an isomorphism of abelian groups.

Outline of proof. Groups $K^t_*(X,A)$ and $K^a_*(X,A)$ are defined for any simplicial pair (X,A) . Thus K^t_* and K^a_* become homology theories on the category of simplicial pairs. $\mu : K^t_* \longrightarrow K^a_*$ becomes a natural tranformation of homology theories.

To prove that μ is an isomorphism, it then suffices to check that μ is an isomorphism for the n-sphere S^n.

The theorem gives insight into both the analytic and the topological theories. All basic features of the two theories (cap product, slant product, external product, Chern character) are preserved by μ.

Part 5. Index Theorems

An index theorem is a formula for the index of a naturally occurring Fredholm operator in terms of topological information associated to the operator. A premier index theorem is the Atiyah-Singer index formula [11]. Proceeding in the direction indicated by Atiyah-Singer several other index theorems have been proved: e.g. Atiyah [3], Atiyah-Bott [1] [4], Atiyah-Singer [13], Boutet de Monvel [22], Cheeger [25], Connes [27], Connes-Moscovici [29] [30], Lazarov [38], and Miscenko-Fomenko [41].

The isomorphism $\mu : K_*^t(X) \longrightarrow K_*^a(X)$ provides a unified framework for these diverse index theorems. Given an index problem on a space X, the analytical data of the problem determines an element in $K_*^a(X)$. To solve the index problem it suffices to explicitly construct, from associated topological information, the corresponding element in $K_*^t(X)$. This is precisely what the Atiyah-Singer theorem does. See §23.

Other index theorems also do this.

One example is an index theorem for self-adjoint elliptic operators which is stated in §24 below. The Boutet de Monvel formula [22] for the index of Toeplitz operators on a strictly pseudo-convex domain is obtained as a corollary.

A second striking example is the Connes index theorem[27][28] for foliated manifolds. At this conference, Connes observed that his theorem appears to fit quite naturally into the general framework provided by the isomorphism of topological and analytic K homology. See §26 and also the article by Connes in these Proceedings.

For the other index theorems cited above some questions remain open, but in each case at least a partial understanding has already been achieved.

§19. Even dimensional Spin^c manifolds: Index of the Dirac operator

For any finite simplicial complex X, there is the commutative diagram

$$\begin{tikzcd} K_0^t(X) \arrow[d, "\mu"'] \arrow[dr, "\epsilon_*"] & \\ K_0^a(X) \arrow[r, "\text{Index}"'] & \mathbb{Z} \end{tikzcd}$$

Consider the special case when X is a point

(19.1)
$$\begin{tikzcd} K_0^t(\cdot) \arrow[d, "\mu"'] \arrow[dr, "\epsilon_*"] & \\ K_0^a(\cdot) \arrow[r, "\text{Index}"'] & \mathbb{Z} \end{tikzcd}$$

Since there is only one way to map to a point, an object for $K_0^t(\cdot)$ is a pair (M,E) where M is a closed even dimensional Spin^c manifold and E is a complex vector bundle on M. As in Part 4, $D \otimes I_E$ denotes the tensor product of the Dirac operator of M with the identity map of E. According to (11.2),

(19.2) $$\epsilon_*(M,E) = (\mathrm{ch}(E) \cup \mathrm{Td}(TM))[M]$$

by the definition of μ:

(19.3) $$\mathrm{Index}(\mu(M,E)) = \mathrm{Index}(D \otimes I_E)$$

(19.2),(19.3), and the commutativity of (19.1), then give:

(19.4) Theorem. $\mathrm{Index}(D \otimes I_E) = (\mathrm{ch}(E) \cup \mathrm{Td}(TM))[M]$

(19.4) is, of course, a special case of the Atiyah-Singer theorem. (19.4) easily implies that the Atiyah-Singer formula is valid for any elliptic operator on a closed even dimensional Spin^c manifold. From this point it is then not difficult to prove the full Atiyah-Singer theorem. These observations can be isolated out from the general discussion of K homology and lead to a very pleasant proof (based on Bott periodicity) of the

Atiyah-Singer theorem [15][16] .

One special case of (19.4) occurs when M is the underlying Spin^c manifold of a complex analytic manifold, and E is a holomorphic vector bundle on M . Let ξ be the Dolbeault complex of E . Set $\chi(\xi) = \mathrm{Index}(\bar{\partial}, E)$. As in §7 above, assemble ξ to obtain $[\xi] \in K_0^a(M)$. It is quite straightforward to show that in $K_0^a(M)$,

(19.5) $$[\xi] = [D \otimes I_E]$$

Therefore

(19.6) $$\mathrm{Index}(\bar{\partial}, E) = \mathrm{Index}(D \otimes I_E)$$

Combining (19.6) and (19.4) gives

(19.7) $$\mathrm{Index}(\bar{\partial}, E) = (\mathrm{ch}(E) \cup \mathrm{Td}(TM))[M]$$

(19.7) is the Hirzebruch-Riemann-Roch theorem [35].

§20. Odd dimensional Spin^c manifolds: Toeplitz operators

On a closed odd dimensional Spin^c manifold the interesting index problem is not for the Dirac operator itself (which is self-adjoint and so has index zero) but rather for Toeplitz operators on the positive space of the Dirac operator. To construct these Toeplitz operators, let $D : C^\infty(F) \longrightarrow C^\infty(F)$ be the Dirac operator of M . Choose a Hermitian structure for F and a smooth measure for M , and view D as an unbounded self-adjoint operator on the Hilbert space of L^2 sections of F .

$$D : L^2(F) \longrightarrow L^2(F)$$

Then $L^2(F)$ has an orthonormal basis consisting of eigen-vectors of D . Let $L^2_+(F)$ $(L^2_-(F))$ be the closed subspace of $L^2(F)$ spanned by the eigen-vectors belonging to non-negative (negative) eigen-values.

$$L^2(F) = L^2_+(F) \oplus L^2_-(F)$$

Given a continuous function $f : M \longrightarrow \mathbb{C}$ define the multiplication operator $\psi(f)$ on $L^2(F)$ by

$$(20.1) \qquad (\psi(f)u)(p) = f(p)\cdot u(p) \qquad u \in L^2(F), \qquad p \in M$$

The Toeplitz operator T_f is the compression of $\psi(f)$ to $L_+^2(F)$. Thus if $P : L^2(F) \longrightarrow L_+^2(F)$ is the Hilbert space projection, then T_f is the composition

$$L_+^2(F) \xrightarrow{\psi(f)} L^2(F) \xrightarrow{P} L_+^2(F)$$

More generally, let β be a continuous map from M to the group of $n \times n$ invertible matrices

$$(20.2) \qquad \beta : M \longrightarrow GL(n,\mathbb{C})$$

For $p \in M$, set $\beta(p) = \| \beta_{ij}(p) \|$. Then the $n \times n$ matrix of operators $\| T_{\beta_{ij}} \|$ is an operator on

$$\mathbb{C}^n \otimes_{\mathbb{C}} L_+^2(F) = L_+^2(F) \oplus L_+^2(F) \oplus \dots \oplus L_+^2(F)$$

Set $T_\beta = \| T_{\beta_{ij}} \|$. T_β is a bounded Fredholm operator on $\mathbb{C}^n \otimes_{\mathbb{C}} L_+^2(F)$. The topological formula for $\mathrm{Index}(T_\beta)$ is:

(20.3) <u>Theorem</u>. $\mathrm{Index}(T_\beta) = (\mathrm{ch}(\beta) \cup \mathrm{Td}(TM))[M]$

In (20.3) $\mathrm{Td}(TM)$ is the Todd class of the Spin^c vector bundle TM, and $\mathrm{ch}(\beta)$ is defined as follows. Recall that the rational cohomology of $GL(n,\mathbb{C})$ is an exterior algebra on well-known [19] standard generators:

$$(20.4) \qquad \begin{aligned} &H^*(GL(n,\mathbb{C});\mathbb{Q}) = \Lambda(e_1, e_3, \dots, e_{2n-1}) \\ &e_i \in H^i(GL(n,\mathbb{C});\mathbb{Q}) \qquad i = 1,3,\dots,2n-1 \end{aligned}$$

With β as in (20.2), let β^* be the map of rational cohomology induced by β, and define $\mathrm{ch}(\beta) \in H^*(M,\mathbb{Q})$ by:

$$(20.5) \qquad \mathrm{ch}(\beta) = \beta^*\left(\frac{e_1}{0!} - \frac{e_3}{1!} + \frac{e_5}{2!} - \frac{e_7}{3!} + \dots + (-1)^{n-1}\frac{e_{2n-1}}{(n-1)!}\right)$$

The formula (20.3) for $\mathrm{Index}(T_\beta)$ can be proved directly from Bott periodicity [5],[21], or as a corollary of the Atiyah-Singer theorem. To apply Atiyah-Singer, first assume that β is a C^∞ map from M to $GL(n,\mathbb{C})$. There is no loss of generality in this assumption because any continuous map from M to $GL(n,\mathbb{C})$ can be approximated by C^∞ maps. Let $S_\beta : \mathbb{C}^n \otimes_{\mathbb{C}} L^2(F) \longrightarrow \mathbb{C}^n \otimes_{\mathbb{C}} L^2(F)$ be defined by:

$$S_\beta(u) = \begin{cases} T_\beta(u) & \text{if } u \in \mathbb{C}^n \otimes_{\mathbb{C}} L^2_+(F) \\ u & \text{if } u \in \mathbb{C}^n \otimes_{\mathbb{C}} L^2_-(F) \end{cases}$$

S_β is an elliptic pseudo-differential operator of order zero. Clearly $\text{Index}(S_\beta) = \text{Index}(T_\beta)$. Applying Atiyah-Singer to S_β gives (20.3)

(20.6) Remark. As in §5 above S_β determines $[S_\beta] \in K_0^a(M)$. For any finite simplicial complex X , one can define the cap product

$$K_a^1(X) \otimes K_1^a(X) \longrightarrow K_0^a(X)$$

In $K_0^a(M)$, $[S_\beta]$ is the cap product of $\beta \in K_t^1(M) = K_a^1(M)$ and $[D] \in K_1^a(M)$

$$[S_\beta] = \beta \cap [D] \tag{20.7}$$

(20.3) is obtained by applying the Chern character to (20.7) and then mapping M to a point. This explains the close resemblance of (20.3) and (19.4) .

§21. Index formula of Boutet de Monvel

Let Ω be a compact strictly pseudo-convex domain. $L^2(\partial\Omega)$ is the space of L^2 functions on the boundary $\partial\Omega$. $H^2(\partial\Omega)$ is the Hardy space, i.e., the space of L^2 functions on $\partial\Omega$ which admit a holomorphic extension to Ω . The Szegö projection $Q : L^2(\partial\Omega) \longrightarrow H^2(\partial\Omega)$ is the Hilbert space projection of $L^2(\partial\Omega)$ onto the closed subspace $H^2(\partial\Omega)$.

Let $f : \partial\Omega \longrightarrow \mathbb{C}$ be a continuous function. M_f is the multiplication operator of f on $L^2(\partial\Omega)$:

$$(M_f u)(p) = f(p)\cdot u(p) \qquad p \in \partial\Omega$$

The Toeplitz operator $\eta(f)$ is the compression of M_f to $H^2(\partial\Omega)$. Thus $\eta(f)$ is the composition

$$H^2(\partial\Omega) \xrightarrow{M_f} L^2(\partial\Omega) \xrightarrow{Q} H^2(\partial\Omega)$$

As in §2 above let

$$\pi : \mathcal{L}(H^2(\partial\Omega)) \longrightarrow \mathcal{Q}(H^2(\partial\Omega))$$

be the canonical map. Define $\tau : C(\partial\Omega) \longrightarrow \mathcal{Q}(H^2(\partial\Omega))$ by

(21.1) $$\tau(f) = \pi(\eta(f)) \qquad f \in C(\partial\Omega)$$

Then $(H^2(\partial\Omega),\tau)$ is an object for $K_1^a(\partial\Omega)$.

(21.2) $$(H^2(\partial\Omega),\tau) \in K_1^a(\partial\Omega)$$

More generally, let β be a continuous map from $\partial\Omega$ to the general linear group

$$\beta : \partial\Omega \longrightarrow GL(n,\mathbb{C})$$

For $p \in \partial\Omega$, set $\beta(p) = \|\beta_{ij}(p)\|$. The $n \times n$ matrix of operators $\|\eta(\beta_{ij})\|$ is an operator on

$$\mathbb{C}^n \otimes_{\mathbb{C}} H^2(\partial\Omega) = H^2(\partial\Omega) \oplus H^2(\partial\Omega) \oplus \dots \oplus H^2(\partial\Omega)$$

Set $V_\beta = \|\eta(\beta_{ij})\|$. Then V_β is a bounded Fredholm operator on $\mathbb{C}^n \otimes_{\mathbb{C}} H^2(\partial\Omega)$. The Boutet de Monvel formula [22] for the index of V_β is:

(21.3) <u>Theorem.</u> $\operatorname{Index}(V_\beta) = (ch(\beta) \cup Td(T\partial\Omega))[\partial\Omega]$

To prove (21.3), set $M = \partial\Omega$ and as in §20 above let T_β be the Toeplitz operator constructed by using the Dirac operator D of $M = \partial\Omega$. Then (21.3) will be implied by (20.3) if it can be shown that

(21.4) $$\operatorname{Index}(V_\beta) = \operatorname{Index}(T_\beta)$$

With $(H^2(\partial\Omega),\tau) \in K_1^a(\partial\Omega)$ as in (21.2) and $[D] \in K_1^a(\partial\Omega)$ as in §6, (21.4) is a corollary of

(21.5) <u>Proposition</u>. In $K_1^a(\partial\Omega)$, $(H^2(\partial\Omega),\tau) = [D]$.

Proposition (21.5) follows from basic properties of the isomorphism $\mu : K_*^t \longrightarrow K_*^a$. A strictly pseudo-convex domain Ω has the property:

(21.6) Let ξ be the $\bar{\partial}$ complex of Ω. Then $H_i(\xi)$ is finite dimensonal for $i > 0$.

One might conjecture that Boutet de Monvel's formula (21.3) is valid for Toeplitz operators on the Hardy space of any domain satisfying (21.6).

§22. Clutching construction: example 3

Let X be a closed C^∞ manifold. X is not required to be orientable. T^* is the cotangent bundle of X

$$T^* = T^*_{\mathbb{R}}(X) .$$

π denotes the projection of T^* onto X.

$$\pi : T^* \longrightarrow X$$

$T^* - \{0\}$ denotes T^* with the zero section deleted.

Suppose given (E_0,E_1,σ) such that:

(i) E_0,E_1 are complex vector bundles on X

(ii) σ is a vector bundle isomorphism
$\sigma : \pi^*E_0|T^*-\{0\} \longrightarrow \pi^*E_1|T^*-\{0\}$

Starting with (E_0,E_1,σ) form a K cycle $(\tilde{X},E_\sigma,\phi)$ on X as follows. Let 1 be the trivial $\mathbb{R}$ line bundle on X

$$1 = X \times \mathbb{R} .$$

Choose a Riemannian metric for X, and let $\tilde{X}$ be the unit sphere bundle of $T^* \oplus 1$.

$$\tilde{X} = S(T^*\oplus 1)$$

$\tilde{X}$ is a Spin^c mainfold. The Spin^c structure for $T\tilde{X}$ comes from the Spin^c structure on $TX \oplus T^*X$. Note that the Riemannian metric for X determines an isomorphism of $\mathbb{R}$ vector bundles $T^*X \cong TX$. Hence on X the $\mathbb{R}$ vector bundle $TX \oplus T^*X$ is isomorphic to $TX \oplus TX = \mathbb{C} \otimes_{\mathbb{R}} TX$ which is a complex vector bundle and so has a Spin^c structure.

$\tilde{X} = S(T^*\oplus 1)$ is a sphere bundle over X.
Let $\phi : \tilde{X} \longrightarrow X$ be the projection. $\tilde{X}$ consists of two copies of the unit ball bundle of T^* glued together by the identity map of the unit sphere bundle.

$$\tilde{X} = B(T^*) \underset{S(T^*)}{\cup} B(T^*)$$

Let E be the vector bundle on $\tilde{X}$ obtained by putting E_0 on one copy of $B(T^*)$, E_1 on the other copy of $B(T^*)$ and then attaching together on $S(T^*)$ by σ

$$E_\sigma = E_0 \underset{\sigma}{\cup} E_1$$

Starting with (E_0,E_1,σ) this clutching construction has produced a K cycle $(\tilde{X},E_\sigma,\phi)$ on X. X is even dimensional since $\dim(\tilde{X}) = 2\dim(X)$. Hence $(\tilde{X},E_\sigma,\phi)$ is an object for $K_0^t(X)$.

$$(\tilde{X},E_\sigma,\phi) \in K_0^t(X)$$

This clutching construction will be used in the statement of the Atiyah-Singer index theorem.

§23. Atiyah-Singer index theroem

For a closed C^∞ manifold X, $K^*(T^*)$ denotes the K cohomology with compact supports of the cotangent bundle $T^* = T^*_{\mathbb{R}}(X)$. Thus an object for $K^0(T^*)$ is a triple (E_0,E_1,σ) such that:

(i) E_0,E_1 are complex vector bundles on X

(ii) σ is a vector bundle isomorphism
$\sigma : \pi^*E_0|T^*-\{0\} \longrightarrow \pi^*E_1|T^*-\{0\}$

Recall [11] that given such a triple an analytic construction produces a pseudo-differential operator $P(\sigma) : C^\infty(E_0) \longrightarrow C^\infty(E_1)$ such that:

$$\text{Symbol}(P(\sigma)) = \sigma$$

As in §5, $P(\sigma)$ determines an element $[P(\sigma)] \in K_0^a(X)$. $(E_0,E_1,\sigma) \longmapsto [P(\sigma)]$ gives an isomorphism of abelian groups

$$P : K^0(T^*) \longrightarrow K_0^a(X)$$

On the other hand, given such a triple (E_0,E_1,σ) the clutching construction of §22 above applies to produce a K cycle $(\tilde{X},E_\sigma,\phi)$ on X . Set $(\tilde{X},E_\sigma,\phi) = c(E_0,E_1,\sigma)$. Then $(E_0,E_1,\sigma) \longmapsto c(E_0,E_1,\sigma)$ gives an isomorphism of abelian groups

$$c : K^0(T^*) \longrightarrow K_0^t(X)$$

$\mu : K_0^t(X) \longrightarrow K_0^a(X)$ is the isomorphism of Part 4.

(23.1) <u>Theorem</u> (Atiyah-Singer [11]). Let X be a closed C^∞ manifold. Then there is commutativity in the diagram

$$\begin{tikzcd} & K^0(T^*) \arrow[dl, "c"'] \arrow[dr, "P"] & \\ K_0^t(X) \arrow[rr, "\mu"'] & & K_0^a(X) \end{tikzcd}$$

(23.2) <u>Corollary</u>. Let $d : C^\infty(E_0) \longrightarrow C^\infty(E_1)$ be an elliptic pseudo-differential operator on X . Set $\sigma = \text{Symbol}(d)$ and $c(E_0,E_1,\sigma) = (\tilde{X},E_\sigma,\phi)$. Then in $K_0^a(X)$:

(23.3) $$\mu(\tilde{X},E_\sigma,\phi) = [d]$$

and

(23.4) $$\text{Index}(d) = (\text{ch}(E_\sigma) \cup \text{Td}(T\tilde{X}))[\tilde{X}]$$

(23.5) <u>Remark</u>. As in corollary (23.2) let $d : C^\infty(E_0) \longrightarrow C^\infty(E_1))$ be an elliptic pseudo-differential operator on X , and set $\sigma = \text{Symbol}(d)$. Let E be a C^∞ vector bundle on X . Form the tensor product of d with the identity map of E to obtain an elliptic pseudo-differential operator

$d\otimes I_E : C^\infty(E_0\otimes E) \longrightarrow C^\infty(E_1\otimes E)$. As in §11 there is the cap product pairing

$$K^0_t(X) \otimes K^t_*(X) \longrightarrow K^t_*(X)$$

Then in $K^a_0(X)$,

$$\mu(E\cap(X,E_\sigma,\phi)) = [d\otimes I_E] \tag{23.6}$$

and

$$\mathrm{Index}(d\otimes I_E) = (\phi^* ch(E)\cup ch(E_\sigma)\cup Td(T\tilde{X}))[\tilde{X}] \tag{23.7}$$

(23.8) Remark. (23.4) is the Atiyah-Singer formula as stated in [42]. To obtain the Atiyah-Singer formula as stated in [11] , combine the diagram of Theorem (23.1) with (18.4) to obtain a commutative diagram

(23.8)
$$\begin{tikzcd} & K^0(T^*) \arrow[dl, "c"'] \arrow[dr, "P"] & \\ K^t_0(X) \arrow[rr, "\mu"'] \arrow[dr, "\epsilon_*"'] & & K^a_0(X) \arrow[dl, "\mathrm{Index}"] \\ & \mathbb{Z} & \end{tikzcd}$$

Set $i_t = \epsilon_*\circ c$ and $i_a = \mathrm{Index}\circ P$. Each is a map of abelian groups $K^0(T^*) \longrightarrow \mathbb{Z}$.
The Atiyah-Singer theorem as formulated in [11] is:

$$i_t = i_a \tag{23.9}$$

§24. Index theorem for self-adjoint elliptic operators

For a closed C^∞ manifold X , $K^*(T^*)$ denotes the K cohomology with compact supports of the cotangent bundle $T^* = T^*_{\mathbb{R}}(X)$. Choose a Riemannian metric for X and let $S(T^*)$ be the unit sphere bundle of T^* . π is the projection $S(T^*) \longrightarrow X$. In [10] (See [10] Proposition (3.1), page 80) it was shown that $K^1(T^*)$ is in one-to-one correspondence with the stable homotopy classes of self-adjoint symbols on X . So an object for $K^1(T^*)$ is a pair (E,σ) such that:

(i) E is a complex vector bundle on X with a given Hermitian structure

(ii) $\sigma : \pi^*(E) \longrightarrow \pi^*(E)$ is a self-adjoint automorphism of $\pi^*(E)$

Starting with such an (E,σ) an analytic construction [11] produces a self-adjoint zero-order elliptic pseudo-differential operator $A : C^\infty(E) \longrightarrow C^\infty(E)$ such that:

$$\mathrm{Symbol}(A) = \sigma$$

As in §6, A determines $[A] \in K_1^a(X)$. Set $[A] = P(E,\sigma)$. Then

$$P : K^1(T^*) \longrightarrow K_1^a(X)$$

is an isomorphism of abelian groups.

On the other hand, given a self-adjoint symbol (E,σ) note that σ gives a direct sum decomposition

(24.1) $$\pi^*(E) = E_+ \oplus E_-$$

where E_+ (E_-) is spanned by the eigen-vectors belonging to the positive (negative) eigen-values of $\sigma : \pi^*(E) \longrightarrow \pi^*(E)$.

Set $\hat{X} = S(T^*)$. $\hat{X}$ is odd-dimensional since $\dim(\hat{X}) = 2(\dim X) - 1$. $\hat{X}$ is a Spin^c manifold with the Spin^c structure on $T\hat{X}$ coming from the Spin^c structure on $TX \oplus T^*X \cong \mathbb{C} \otimes_{\mathbb{R}} TX$. Thus the triple $(\hat{X},E_+,\pi)$ is a K cycle on X .

(24.2) $$(\hat{X},E_+,\pi) \in K_1^t(X)$$

Set $c(E,\sigma) = (\hat{X},E_+,\pi)$. Then

$$c : K^1(T^*) \longrightarrow K_1^t(X)$$

is an isomorphism of abelian groups.

(24.3) <u>Theorem</u>. Let X be a closed C^∞ manifold. Then there is commutativity in the diagram

$$\begin{tikzcd} & K^1(T^*) \arrow[dl, "c"'] \arrow[dr, "P"] & \\ K_1^t(X) \arrow[rr, "\mu"'] & & K_1^a(X) \end{tikzcd}$$

(24.4) Corollary. Let $A : C^\infty(E) \longrightarrow C^\infty(E)$ be an elliptic self-adjoint pseudo-differential operator on X.
Set $\sigma = \text{Symbol}(A)$ and $c(E,\sigma) = (\hat{X},E_+,\pi)$. Then in $K_1^a(X)$

$$\mu(\hat{X},E_+,\pi) = [A] \tag{24.5}$$

(24.6) Remark. With A as in corollary (24.4), Toeplitz operators can be constructed on the positive space of A. To do this, let $[A] = (L_+^2(E),\tau)$ be as in §6. Suppose that $\beta : X \longrightarrow GL(n,\mathbb{C})$ is a continuous map from X to the general linear group. For $p \in X$ set $\beta(p) = \| \beta_{ij}(p)\|$ and consider the $n \times n$ matrix

$$T_\beta = \| \gamma(\beta_{ij})\| \tag{24.7}$$

where $\gamma : C(X) \longrightarrow L(L_+^2(E))$ is as in (6.1). T_β is a bounded Fredholm operator on

$$\mathbb{C}^n \otimes_{\mathbb{C}} L_+^2(E) = L_+^2(E) \oplus L_+^2(E) \oplus \ldots \oplus L_+^2(E) .$$

Recall that $\text{ch}(\beta)$ is defined by (20.5). As in Corollary (24.4) set $\sigma = \text{Symbol}(A)$ and $c(E,\sigma) = (\hat{X},E_+,\pi)$. Then:

(24.8) Corollary. $\text{Index}(T_\beta) = (\pi^*\text{ch}(\beta) \cup \text{ch}(E_+) \cup \text{Td}(T\hat{X}))[\hat{X}]$

(24.9) Remark. As in §6, $L^2(E) = L_+^2(E) \oplus L_-^2(E)$. Suppose that $\beta : X \longrightarrow GL(n,C)$ is a C^∞ map. With T_β as in (24.7) define an elliptic pseudo-differential operator S_β by

$$S_\beta(u) = \begin{cases} T_\beta(u) & \text{if } u \in \mathbb{C}^n \otimes_{\mathbb{C}} L_+^2(E) \\ u & \text{if } u \in \mathbb{C}^n \otimes_{\mathbb{C}} L_-^2(E) \end{cases} \tag{24.10}$$

According to §5, S_β determines $[S_\beta] \in K_0^a(X)$. Using the cap product pairing

$$K_a^1(X) \otimes K_1^a(X) \longrightarrow K_0^a(X) \tag{24.11}$$

one then has, that in $K_0^a(X)$:

$$\beta \cap [A] = [S_\beta] \tag{24.12}$$

(24.12) explains the close resemblance of (24.8) and (23.7).

Theorem (24.3) was not explicitly stated by Atiyah-Singer. To some extent, however, it is implicit in their work. For example, Corollary (24.8) can be proved by applying the Atiyah-Singer index formula to the elliptic pseudo-differential operator S_β of (24.10).

§25. General index problem

The isomorphism $\mu : K_*^t(X) \longrightarrow K_*^a(X)$ of Part 4 leads to:

General index problem. Let X be a finite simplicial complex. Given $v \in K_*^a(X)$, explicitly compute the unique $\tilde{v} \in K_*^t(X)$ with $\mu(\tilde{v}) = v$.

The Atiyah-Singer theorem can be viewed as solving this problem for a very significant special case. If X is a closed C^∞ manifold and $d : C^\infty(E_0) \longrightarrow C^\infty(E_1)$ is an elliptic pseudo-differential operator on X, then the Atiyah-Singer theorem asserts (See (23.3)) that in $K_0^a(X)$,

$$\mu(\tilde{X}, E_\sigma, \phi) = [d]$$

Similarly the index theorem for self-adjoint elliptic operators (24.3) also solves the general index problem in an interesting special case. Here $A : C^\infty(E) \longrightarrow C^\infty(E)$ is a self-adjoint elliptic pseudo-differential operator on the closed C^∞ manifold X. Then according to (24.5), in $K_1^a(X)$:

$$\mu(\hat{X}, E_+, \pi) = [A]$$

For the Boutet de Monvel formula, let $M = \partial\Omega$ be the boundary of a strictly pseudo-convex domain Ω. With $(H^2(\partial\Omega), \tau)$ as in §21, the crucial point in proving the Boutet de Monvel formula is stated by Proposition (21.5):

(25.1) $$\text{In } K_1^a(M),\ (H^2(\partial\Omega), \tau) = [D].$$

$[D] \in K_1^a(M)$ is given by the Dirac operator of M. By the definition of μ, $[D] = \mu(M, 1, I_M)$ where $1 = M \times \mathbf{C}$ is the trivial complex line bundle on M and I_M is the identity map of M. So Proposition (21.5) can be rewritten

(25.2) $$\text{In } K_1^a(M),\ \mu(M, 1, I_M) = (H^2(\partial\Omega), \tau)$$

which solves the general index problem for $v = (H^2(\partial\Omega), \tau)$.

Other index problems also appear to be special cases of the general index problem. Two examples are the Atiyah-Bott theorem [1],[4] for elliptic differential operators on manifolds with boundary and the signature operator for singular spaces [25] studied by J. Cheeger.

If X is a compact C^∞ manifold with boundary, then an elliptic differential operator $d : C^\infty(E_0) \longrightarrow C^\infty(E_1)$ on X (with no boundary condition) determines an element in the group $K_0^a(X, \partial X)$. By imposing an elliptic boundary condition on d , Atiyah-Bott [1] [4] obtain an element in $K_0^a(X)$. The symbol of the operator d together with the symbol of the elliptic boundary condition then gives an explicit construction of the corresponding element in $K_0^t(X)$.

For certain spaces X , which are not manifolds, J. Cheeger has defined an operator [25] whose index is the Goresky-MacPherson signature [33] of X . This operator gives an object for $K_0^a(X)$ and so determines an element $v \in K_0^a(X)$. At the present time it is not known how to explicitly construct $\tilde{v} \in K_0^t(X)$ with $\mu(\tilde{v}) = v$. But with $ch. : K_*^t(X) \longrightarrow H_*(X;\mathbb{Q})$ as in §11, it is known that $ch.(\tilde{v})$ is the Goresky-MacPherson L class [33] of X.

For some special cases, R. MacPherson has observed that recent results of P. Deligne do give an explicit solution to the general index problem for the Cheeger operator.

§26. Possible further developments

Index theory for singular spaces. Let X be a finite simplicial complex. Embed X in $\mathbb{R}^n$ with n even. Let W be a regular neighborhood of X in $\mathbb{R}^n$ with ∂W a C^∞ sub-manifold of $\mathbb{R}^n$. The group $K^0(W, \partial W)$ is independent of the choice of embedding, and the clutching construction of §12 gives an isomorphism $c : K^0(W, \partial W) \longrightarrow K_0^t(X)$. There should then be a commutative diagram

$$\begin{tikzcd} & K^0(W,\partial W) \arrow[dl, "c"'] \arrow[dr, dashed, "P"] & \\ K_0^t(X) \arrow[rr, "\mu"'] \arrow[dr, "\epsilon_*"'] & & K_0^a(X) \arrow[dl, "\text{Index}"] \\ & \mathbb{Z} & \end{tikzcd} \tag{26.1}$$

If X is a C^∞ manifold, then $K^0(T^*) \cong K^0(W,\partial W)$. So for X a C^∞ manifold (26.1) is equivalent to (23.8). If X is not a C^∞ manifold, then all arrows are known in (26.1) except the dashed arrow. Defining the dashed arrow involves constructing interesting operators on X. One example of such an operator is the signature operator studied by J. Cheeger [25].

Equivariant theory. The isomorphism $\mu : K_*^t(X) \longrightarrow K_*^a(X)$ should be stated and proved in the category of G-spaces, i.e., spaces with a given action of a topological group G.

Other K theories. The isomorphism $\mu : K_*^t(X) \longrightarrow K_*^a(X)$ should be stated and proved for other K theories. For example, a K cycle for $KO_*^t(X)$ is a triple (M,E,ϕ) such that:

(i) M is a compact Spin manifold without boundary

(ii) E is an $\mathbb{R}$ vector bundle on M

(iii) ϕ is a continuous map from M to X

This theory is periodic of order eight.

A topological definition of $KO_* \otimes \mathbb{Z}[\frac{1}{2}]$, which is periodic of order four, has been given by P. Haskell. The elliptic operator which relates this theory to analysis is the Hirzebruch signature operator [35].

Heat-equation and refined theory. The heat-equation proof [7] of the Atiyah-Singer theorem does not fit in with the isomorphism $\mu : K_*^t(X) \longrightarrow K_*^a(X)$. There should be a more refined K theory which does relate to the heat equation proof.

Bivariant theory. W. Fulton and R. MacPherson have introduced bivariant theory [32]. The isomorphism $K_*^t(X) \longrightarrow K_*^a(X)$ should extend to the bivariant case. Thus given two finite simplicial complexes X, Y and a continuous map $f : X \longrightarrow Y$ there should be a topologically defined group $K_*^t(X \xrightarrow{f} Y)$, an analytically defined group $K_*^a(X \xrightarrow{f} Y)$, and an isomorphism

$$K_*^t(X \xrightarrow{f} Y) \longrightarrow K_*^a(X \xrightarrow{f} Y)$$

Both the topological and the analytic bivariant groups are only partly understood at the present time. If $X = X_1 \times Y$ and

$f : X \longrightarrow Y$ is the projection $X_1 \times Y \longrightarrow Y$, then the analytic bivariant group $K_1^a(X_1 \times Y \longrightarrow Y)$ is $\mathrm{Ext}(X_1, Y)$ [43].

Bivariant theory is closely related to the Atiyah-Singer index theorem for families of elliptic operators [12]. Suppose that $f : X \longrightarrow Y$ is a fibration with X as total space, Y as base space, and a closed C^∞ manifold M as fibre. Then an elliptic operator along the fibres (i.e. a family of elliptic operators parametrized by Y) gives an object for the analytic bivariant group $K_0^a(X \xrightarrow{f} Y)$. Similarly, a self-adjoint elliptic operator along the fibres (i.e. a family of self-adjoint elliptic operators parametrized by Y) gives an object for the analytic bivariant group $K_1^a(X \xrightarrow{f} Y)$.

Index theory for foliated manifolds. At this conference A. Connes observed that there are close connections between his index theorem for foliated manifolds and the point of view on index theory presented in this note. Let V be a closed C^∞ manifold with a given foliation F. Then the K cohomology (with compact supports) of the "space of leaves" $\mathcal{B}$ can be defined analytically or topologically. The analytical definition uses the C^* algebra $C^*(V,F)$ defined by Connes in [27]. The topological definition uses appropriate cycles (M,E,ϕ) as in the definition of K_*^t. Connes has defined a map

$$\mu : K_t^*(\mathcal{B}) \longrightarrow K_a^*(\mathcal{B})$$

There is good evidence for the conjecture that μ is an isomorphism.

Suppose given a pseudo-differential operator d on V with d elliptic along the leaves of the foliation F. The analytic index $i_a(d)$ of d is an element of $K_a^*(\mathcal{B})$, and the topological index $i_t(d)$ of d is an element of $K_t^*(\mathcal{B})$. A reformulated version of the Connes index theorem [27] would assert that these two elements correspond under μ

$$\mu(i_t(d)) = i_a(d)$$

This proposed reformulation of the Connes index theorem is more precise than the statement in [27] and allows one to remove the undesirable hypothesis that the foliation F admit a holonomy invariant transverse measure. See the lecture of Connes in these Proceedings. A detailed report on the K theory of foliations and the relation to the Connes index theorem will be given elsewhere.

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