Extensions of Clifford's Chain-Theorem.

By F. Morley.

I propose to state more fully what is implied in the final section of my paper "Metric Geometry of the plane n-line," *Transactions of the American Mathematical Society,* Vol. 1 (1900), p. 115. I refer to this as M. G. First, let us recall Clifford's Theorem, *Works,* p. 51.*

In a plane we take lines, say 1, 2, 3, $\cdots$, n. We complete the figure as follows: We mark the intersections 12, $\cdots$. We mark the circles 123, on 12, 23, 31, $\cdots$. There is a point 1234 on the 4 circles 123, 234, 341, 412. There is a circle 12345 on the 5 points 1234, $\cdots$. There is a point 123456 on the 6 circles 12345 $\cdots$. And so on. For an even number of lines the figure ends with a point—the Clifford point; for an odd number with a circle—the Clifford circle.

Regarding the lines as circles on the point ∞, we have a configuration—the Clifford configuration.

1. *The fundamental curves C^n.* An account of these curves and their osculant theory is given in my paper on Reflexive Geometry,† to which I refer as R. G. I shall state a little differently what is needed here. The aim is to obtain for any given number of lines a curve which plays the part of the circumcircle for three lines.

We take a base circle $|t| = 1$. An equation $x = f(t)$ maps this circle on some curve. The point x is stationary—that is a cusp—of the curve when $dx/dt = 0$.

An especially simple class of curves with $n - 2$ cusps will then be given by

$$(1) \qquad dx/dt = \kappa(t - t_1)(t - t_2) \cdots (t - t_{n-2}).$$

Denote such a curve by C^n.

Thus when $n = 2$, $dx/dt = \kappa$, $x = \kappa t + x_0$, so that C^2 is a circle. C^3 is a cardioid, and so on.

It is convenient here to mean by C^1 a point, for which $dx/dt = 0$; and by C^0 a line. If $dx/dt = \kappa/t$ and $t = e^{\iota\theta}$, then $x = x_0 + \iota\kappa\theta$, which denotes a line. If the equation (1) which gives the cusp-parameters is written

$$(1') \qquad dx/dt + (n-1)\{a_1 + (n-2)a_2 t + \cdots + \bar{a}_1 t^{n-2}\} = 0,$$

* For exceptional cases, see W. B. Carver, *American Journal of Mathematics,* Vol. 42 (1920), pp. 137-167.

† *Transactions of the American Mathematical Society,* Vol. 8 (1907), pp. 14-24.

then the lines of the curve C^n are given by

$$(2) \qquad x - x_0 + na_1 t + \binom{n}{2} a_2 t^2 + \cdots + n\bar{a}_1 t^{n-1} + (\bar{x} - \bar{x}_0)\, t^n = 0.$$

We call x_0 the center of the curve. If conversely we write a self-conjugate equation in t, and regard the end-coefficients as conjugate variables, we have a curve C^n with center 0. R. G. § 3.

The fully polarized form of (2) is

$$(3) \qquad x - x_0 + a_1 s_1 + a_2 s_2 + \cdots + (\bar{x} - \bar{x}_0) s_n = 0,$$

where $s_1, s_2 \cdots s_n$ are the product-sums of $t_1, t_2 \cdots t_n$. This is the osculant line of C^n for the parameters t_i, or points t_i of C^n. If we set $t_1 = t_2 = \cdots = t_m = t$ we have the osculant C^m for the points $t_{m+1} \cdots t_n$. Two osculant C^{n-1}'s have a common osculant C^{n-2} and conversely two C^{n-1}'s with a common osculant C^{n-2} are osculants of a C^n. R. G. § 6.

A curve C^n is defined save as to homologies $x = ay + b$ by its $n - 2$ cusp-parameters. Suppose $n = 2m + 1$. Then there are $2m - 1$ cusp-parameters. There is then a canonizant, m parameters apolar (or harmonic) to the cusp-parameters. These m points give an osculant C^{m+1} which is a repeated point. This point is the Clifford point of the C^{2m+1}. The reason for the existence of this point may be stated thus. An osculant C^m is defined by taking $n - m$ points t_i on the given C^n. These have a polar as to the cusp-parameters. This polar gives the cusp-parameters of C^m. Thus when the polar is arbitrary, the cusp-parameters are arbitrary. And this implies that in (1′) $a_1 = a_2 = \cdots = 0$. For the curve C^{2m+2} with $2m$ cusps the simplest set apolar to the cusp-parameters is a pencil of sets of $m + 1$ points. Such a set has an osculant C^{m+1} which again is a repeated point. The locus of such points is a circle.* This is the Clifford circle of the C^{2m+2}.

To prove this it will suffice to take one case. Suppose we have a C^4, with 2 cusps,

$$x - x_0 + 3a_1 t + 3\rho t^2 + \bar{a} t^3 = 0.$$

The osculant of t_1, t_2 is the circle

$$x - x_0 + a_1 s_1 + \rho s_2 + \bar{a} s_3 = 0.$$

If t_1 and t_2 are apolar or harmonic to

$$a_1 + 2\rho t + \bar{a} t^2 = 0$$

then

$$a_1 + \rho(t_1 + t_2) + \bar{a} t_1 t_2 = 0$$

* Strictly, we should speak of the Clifford circle or arc, for the circle may or may not be complete.

and the circle became the point

$$x - x_0 + a_1(t_1 + t_2) + \rho t_1 t_2.$$

If we eliminate $t_1 + t_2$ from these two equations we have

$$\begin{vmatrix} x - x_0 & a_1 \\ a_1 & a\rho \end{vmatrix} = t_1 t_2 \begin{vmatrix} a_1 & \rho \\ \rho & \bar{a} \end{vmatrix}$$

which is a circle (or the arc of a circle). The argument is general—see M. G. § 4, p. 103. We are here restating that § 4 in geometrical language.

Given a C^{2m+2}, each osculant C^{2m+1} has a Clifford point. The locus of these is the Clifford circle. The reason is that if we have $m + 1$ points t_i apolar to $2m$ points τ_i the polar of t_i has as canonizant the points $t_2 \cdots t_{m+1}$. The algebraic argument is in M. G. § 4.

Given a C^{2m+1} each osculant C^{2m} has a Clifford circle. These are all on the Clifford point. The reason is that if we have m points t_i apolar to $2m - 1$ points, τ_i the polar of t_1 has an apolar set the m points. The algebraic argument is again in M. G. § 4. The gist is that we regard the Clifford chain as a property of a curve C^n and its osculants. The lines come in by taking $n + 1$ points t_i on C^n. Each n of these points has an osculant line, so that we have a set of $n + 1$ osculant lines. These are any lines and they determine the C^n uniquely (M. G. § 1). This C^n is the fundamental covariant, under the group $x = ay + b$ of homologies, of the $n + 1$ lines. The fundamental covariant C^m of any $m + 1$ of the lines is an osculant of C^n—in fact the osculant for the unused t_i. Thus for two lines the C^1 is the intersection. For three lines the 3 intersections are on the C^2 which is the circumcircle. For four lines, the 4 circumcircles are osculants of the C^3, and meet at the cusp (the Clifford point). For five lines the 5 C^3's are osculants of C^4. The five cusps are on its Clifford circle. And so on.

2. *The incenters of an* $(n + 1)$*-line.* A three-line has four inscribed circles. Their centers are the incenters of the three-line. We ask then for the curves C^n which touch $n + 1$ given lines, and more particularly for their centers. These centers we call the *incenters* of the $(n + 1)$-line.

A line is given most simply by the image in it of the base point 0. Let the images be $x_1, x_2 \cdots x_{n+1}$. We have then from (2) $n + 1$ equations

$$-x_0 + a_1 t_i + \cdots + \bar{a}_1 t_i^{n-1} + (\bar{x}_i - x_0) t_i^n = 0.$$

Eliminating $x_0, a_1 \cdots \bar{a}_1$ we have a determinant

$$|\, 1 \; t_i t_i^2 \cdots (\bar{x}_i - \bar{x}_0) t_i^n \,| = 0,$$

and therefore

$$\bar{x}_0 = \sum^{n+1} [\bar{x}_1 t_1^n / (t_1 - t_2) \cdots (t_1 - t_{n+1})]. \tag{4}$$

But the clinant of the line given by x_i is $-x_i/\bar{x}_i$ and is from (2) $-t_i^n$. Thus we have

$$(5) \qquad x_i = \bar{x}_i t_i^n,$$

so that (4) takes the form, the equation of incenters,

$$(4') \qquad \bar{x}_0 = \sum^{n+1} [x_1/(t - t_2) \cdots (t_1 - t_{n+1})].$$

These incenters can be constructed, if we can solve (5) for t_i, that is if we assume that we can divide an angle into n equal parts. This operation is all we need, in addition to Euclid's, in this paper. For any t_i we may substitute ϵt_i where $\epsilon^n = 1$. The set t_i and the set ϵt_i give the same value of x_0. Hence there are n^n incenters, and n^n inscribed C^n's.

Each can be named $1, \epsilon_1, \cdots, \epsilon_n$, where $\epsilon_i^n = 1$, and repetitions are allowed.

3. *The axes of an n-line.* The conjugate of (4) is

$$(5) \qquad x_0 = \sum [x_1 t_2 \cdots t_{n+1}/(t_2 - t_1) \cdots (t_{n+1} - t_1)].$$

Eliminating x_{n+1} from (4′) and (5) we have

$$(6) \qquad x_0 \pm \bar{x}_0 t_1 t_2 \cdots t_n = \sum^{n} [x_1 t_2 \cdots t_n/(t_2 - t_1) \cdots (t_n - t_1)]$$

the sign being $+$ for n odd, $-$ for n even. This is a self-conjugate equation, denoting a line. There are n^{n-1} such lines. We call them the *axes* of the n-lines. They constitute the locus of the centers of inscribed C^n's. The clinant of an axis is a geometric mean of the clinants of the n lines; in other words if a line makes with a base line an angle θ_i (to the modulus π) an axis makes an angle θ where

$$n\theta = \sum \theta_i.$$

Therefore the n^{n-1} axes fall into n sets of parallel lines, inclined successively at angles π/n. In each set are n^{n-2} parallel lines.

Each axis can be named $1, \epsilon_1, \cdots, \epsilon_n$. If in two axes so named we have the same ϵ in the same place, the axes are parallel.

Two curves C^n touching n lines may or may not have their centers on an axis. Then the common lines of two curves C^n fall into sets. To verify this directly we compare with (2) a second equation

$$(2') \qquad x - y_0 + b_1\tau + \cdots + (\bar{x} - \bar{y}_0)\tau^n = 0$$

and we see at once that if these are to be the same equation we must have $\tau^n = t^n$, $\tau = \epsilon t$. For a selected root of unity ϵ, we have on subtraction an equation of degree n in t giving what we call a *tied* set of n common tangents. This equation is

$$y_0 - x_0 + \cdots + (\bar{y}_0 - \bar{x}_0)\, t^n$$

so that

$$(y-x_0)/(\bar{y}_0-\bar{x}_0)=(-)^n t_1\cdots t_n.$$

Accordingly, when the centers of two curves on n lines are on an axis, the n lines are a tied set.

The curves C^n which touch n lines fall then into n^{n-1} discrete systems. The transition from one system to another is when the center falls on two axes. One of the n lines is then a double line of C^n.

It follows that when n is not a prime number the double lines of C^n will fall into sets. For example if $n=4$, the center may be where axes meet at right angles, or where axes meet at $\pi/4$.

If we apply the theory of this section to a triangle abc, we obtain as the locus of centers of inscribed cardioids three sets of three parallel lines, forming equilateral triangles. The vertices of the triangles are the centers of the cardioids which touch a side (say bc) of the given triangle twice. If x_0 be such a center, then the angle x_0bc is a third of the angle abc. For x_0b is an axis of the 3 lines ab and bc twice. Thus if we take the interior trisectors of the angles of a triangle, the points where those adjacent to a side meet form an equilateral triangle.*

4. *The chain for an incenter.* Taking the equation of an incenter, (4′) we interpolate, as in M. G. § 2,

$$\bar{x}=\sum^{n+2}\frac{x_1(t_1-\tau)}{(t_1-t_2)\cdots(t_1-t_{n+2})} \tag{11}$$

this becomes (4′) when $\tau=\epsilon_{n+2}$. It is a circle on selected incenters of any $n+1$ lines out of $n+2$.

There are then for $n+2$ lines n^{n+2} circles, each on $n+2$ incenters. For a second interpolation we write

$$\bar{x}=\sum^{n+3}\frac{x_1(t_1-\tau_1)(t_1-\tau_2)}{(t_1-t_2)\cdots(t_1-t_{n+3})}\,. \tag{12}$$

This becomes (11) when $\tau_2=t_{n+3}$. It is an osculant of the curve

$$\bar{x}=\sum\frac{x_1(t_1-\tau)^2}{(t_1-t_2)\cdots(t_1-t_{n+3})}\,.$$

And the point is that this curve is a C^3. For differentiating we get as cusp-condition

$$\sum\frac{x_1(t_1-\tau)}{(t_1-t_2)\cdots(t_1-t_{n+3})}=0, \tag{13}$$

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* Morley, *Mathematical Association of Japan for Secondary Mathematics*, Vol. 6, Dec. 1924. This theorem, which I obtained in this way long ago, has excited much interest.

whose conjugate is

$$\Sigma \frac{x_1(\tau - t)}{(t_2 - t_1) \cdots (t_{n+3} - t_1)} \times t_1 t_2 \cdots t_n = 0.$$

The equation (13) is then self-conjugate.

Thus the circles attached to $n+2$ out of $n+3$ lines are osculants of a C^3, and therefore meet at its cusp. There are then for $n+3$ lines n^{n+3} cardioids. For $n+4$ lines, the cardioids are osculants of

$$(14) \qquad \bar{x} = \Sigma \frac{x_1(t-\tau)^3}{(t_1 - t_2) \cdots (t_1 - t_{n+4})}.$$

Writing $d\bar{x}/dt$, and forming its conjugate we see again that (14) has 2 cusps, and is a C^4. And so on.

We have then the table:

number of lines	1	2	3	4	5	6	...
axes C^0	1	2	3^2	4^3	5^4	6^5	...
incenters C^1		1	2^2	3^3	4^4	5^5	...
circles C^2			1	2^3	3^4	4^5	...
cardioids C^3				1	2^4	3^5	...
C^4					1	2^5	...
C^5						1	...
:							

The table is read diagonally; each C^n is a first osculant of some C^{n+1} in the next column.

The first column says that a line is its own axis.

The second column says that a two-line has two axes and an intersection.

The third column says that a three-line has 3^2 axes; that it has 2^2 incenters, the intersection of the axes of the two-lines contained in it; and that it has one circumcircle, on the intersection of the two-lines.

For the nth column, the axes are new; the incenters arise from the axes of the preceding column, that is the $n(n-1)^{n-2}$ axes of the component $(n-1)$-lines meet in the $(n-1)^{n-1}$ incenters, there being n axes on a point and $n-1$ points on an axis. The circles arise from the incenters of the preceding column. That is the $n(n-2)^{n-2}$ incenters are on the $(n-2)^{n-1}$ circles, there being n points on each circle, $n-2$ circles on each point. These $n-2$ circles cut at the angle $\pi/(n-2)$. The $(n-3)^{n-1}$ cardioids arise from the $n(n-3)^{n-2}$ circles; the circles are osculants of the cardioids, each C^3 having n osculant C^2's; and each C^2 osculating $n-3$ C^3's. And so on.

The leading diagonal indicates Clifford's chain. We notice that the n-line has a unique C^{n-1}.

The second diagonal indicates the chain discussed by F. H. Loud, *Transactions of the American Mathematical Society.** The ambiguities which enter from (5) are there cleared up by regarding lines as directed. This makes the incenter of a 3-line unique. In general the ambiguities disappear if we recall that two osculants of C^n have a common osculant. Consider the 2^4 C^3's for 5 lines. These for 5/6 lines are osculants of C^4's. The 6×2^4 C^3's are osculants of the 2^5 C^4's; $6C^3$'s on each C^4, $2C^4$'s on each C^3. The C^3 taken from lines 12345 and that taken from 23456 must have a common osculant C^2 (taken from 2345).

The ambiguity is explained in another specific case involving axes by P. S. Wagner in the article following this one.

There is in fact no ambiguity where we name the C^n's by the roots of unity. The naming is carried on from one column to the next.

This completes the object of this paper, but it is convenient to add a canonical equation of the curve C^n.

4. *Canonical equation of a C^n.* The curve C^n is defined save as to homologies by the $n-2$ cusp-parameters. It is proper to give these intrinsically, so far as possible. When n is odd, say $n = 2m+1$, Sylvester pointed out the proper intrinsic or canonical form for an equation, here

$$a_1 + (n-2)\, a_1 t + \cdots + \bar{a}_1 t^{2m-1} = 0$$

namely the equation

$$\sum^{m} A_i (t - \tau_i)^{2m-1} = 0. \tag{15}$$

Here the m numbers τ_i give the canonizant, the unique equation of degree m apolar to the given equation. Accordingly the canonical form for a C^{2m+1} is

$$(1/2m)\,(dx/dt) = \sum^{m} A_i (t-\tau_i)^{2m-1}$$

or

$$x = \sum^{m} A_i (t - \tau_i)^{2m}. \tag{16}$$

The equation (15) must be self-conjugate, that is the same as

$$\sum^{m} \bar{A}_i (t-\tau_i)^{2m-1} / \tau_i{}^{2m-1} = 0.$$

This is secured by

$$\bar{A}_i = A_i T_i{}^{2m-1}. \tag{17}$$

The base-point in this canonical form (8) is the Clifford point of any $2m+2$ lines which are an osculant set of the C^{2m+1}. It is in fact the osculant of the canonizant. The simplest case, $m = 1$, is the cardioid, with

* Vol. 1 (1900), pp. 323-338.

the canonical equation

$$x=(1-t)^2.$$

An osculant point is here

$$x=(1-t_1)(1-t_2)$$

and 3 such points are on the line

$$x=(1-t_i)/(1-t).$$

Four such lines are given by

$$x=\prod^4 \left[(1-t_i)/(1-t)(1-t')\right]$$

and are tangents of the parabola

$$x=\left[\prod^4/(1-t)^2\right].$$

And so in general the C^{2m+1} can be immediately connected with

$$x=\sum^m \left[B_i/(t-\tau_i)^2\right]$$

which is Clifford's m-fold parabola (*loc. cit.*). The case of C^5, with 3 cusps, was analysed, with figures, by Father E. C. Phillips (*American Journal,* Vol. 31, 1909). It may be remarked that if the cusps c_i are given there are four curves, for the Clifford point is given by $\sum\left[1/(x-c_i)^{1/3}\right]=0$, which rationalized is a quartic for which $g_2=0$.

When n is $2m$, we have for C^{2m} a cusp form of degree $2(m-1)$. The lowest apolar forms are of degree m, and therefore as in Sylvester's theory we take

$$\frac{1}{2m-1}\ \frac{dx}{dt}=\sum^m A_i(t-\tau_i)^{2(m-1)}$$

whence the canonical form is

$$x=\sum^m A_i(t-\tau_i)^{2m-1}.$$

This may be regarded as a first osculant of (8), namely

$$x=\sum^m A_i(t_0-\tau_i)(t-\tau_i)^{2m-1}.$$

There is no advantage for small values of m. The circle C^2 is naturally best as $x=t$ and the C^4 as $x=3t-3\mu t^2+t^3$.

The canonical form might have been used throughout, but it would not have been possible to use the references.

It is to be remarked that in the form used the coefficients $a_1, a_2 \cdots$ of the C^{n-1} of an n-line are invariants of the n-line. They form with the parameters t_i a complete set of rational invariants (under homologies). The parameters are by (3) inversely proportional to the clinants of the lines.

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