On the Intersections of the Trisectors of the Angles of a Triangle.

By

Professor **Frank Morley.**

(From a letter directed to Prof. T. Hayashi.)

Dear Professor Hayashi :—

I have not published the theorem [The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle][1]. It arose from the consideration of cardioids. I noticed, in the Transactions of the American Mathematical Society, vol. 1, p. 115, that certain chains of theorems were true for any number of lines in a plane, when one replaces the intersection of the lines taken two at a time (1) by the centre of a circle touching the lines taken 3 at a time and (2) by the centre of a cardioid touching the lines taken 4 at a time, and so on.

So I was led to think on the cardioids touching 3 lines.

The cardioid is mapped on the unit circle by an equation.

$$x=2t-t^2,$$

x a complex number, t a complex number such that $|t|=1$. The tangent at t is

$$x-3t+3t^2-\bar{x}t^3=0\,,$$

where $\bar{x}$ is the conjugate of x. The 3 tangents from a point x are then such that

$$t_1 t_2 t_3=x/\bar{x}\,.$$

Whence if θ_i are the angles which these tangents make with any fixed line, and ϕ the angle of x itself,

$$3\phi=\theta_1+\theta_2+\theta_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(1) This enunciation of the theorem has been added here by Prof. T. Hayashi.

The image y of any points x in the tangent is given by

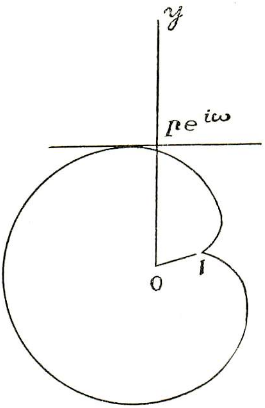
$$y-3t+3t^2-\bar{x}t^3=0\,.$$

Thus the image of the centre $x=0$ is

$$y=3(t-t^2)\,.$$

Hence, if

$$y=2pe^{i\omega}\,,\quad \text{so that}\quad \bar{y}=2pe^{-i\omega}\,,$$



we have

$$4p^2=9(1-t)(1-1/t),$$

$$e^{2i\omega}=-t^3\,,$$

$$t+1/t=-2\cos 2\omega/3\,,$$

and $\qquad p=3\sin\omega/3\ldots.(2)$

This is the line-equation of the cardioid. The equation $p=a\sin\mu\omega$ for any cycloidal curve is given in some of the older books (for instance, in Edwards, Differential Calculus), so that we might begin with equation (2).

If then p_1, p_2, p_3 are perpendiculars from the centre on 3 tangents, and ω_1, ω_2, ω_3 the angles of these perpendiculars, since

$$\sum^{3}\sin\frac{\omega_1}{3}\sin\frac{\omega_2-\omega_3}{3}=0\,,$$

we have

$$\sum^{3}p_1\sin\frac{\omega_2-\omega_3}{3}=0\,.$$

Replacing $\omega_2-\omega_3$ by the angle A_1 of the triangle of tangents, but bearing in mind that in (3) the angles must have a sum congruent to 0, we get for the locus of centres 9 lines, such as

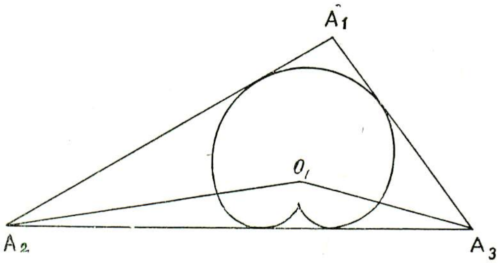
$$p_1\sin\frac{\pi-A_1}{3}+p_2\sin\frac{\pi-A_2}{3}+p_3\sin\frac{-\pi-A_3}{3}=0\,,$$

$$p_1\sin\frac{2\pi-A_1}{3}+p_2\sin\frac{\pi-A_2}{3}+p_3\sin\frac{-2\pi-A_3}{3}=0\,.$$

But from (1) considering those cardioids whose centres are at a great distance (so that the triangle behaves like a point), we see that the 9 lines have only 3 directions, given by

$$3\phi=\theta_1+\theta_2+\theta_3 .$$

They are thus 3 sets of 3 parallel lines, forming equilateral triangles. The centre changes from one line to another when one of the lines is a double tangent.



Consider in particular the cardioids which lie inside the triangle. Let O_1 be the centre of a cardioid with double tangent A_2A_3. We have from (1)

$$\text{angle } A_3A_2O_1=A_2/3 ,$$
$$\text{angle } O_1A_3A_2=A_3/3 ,$$

and we have seen that the 3 lines O_1O_2, O_2O_3, O_3O_1 form an equilateral triangle.

That was the argument. Verification is naturally a much simpler matter. If you think above worth printing I shall be very pleased to have it appear in a Japanese journal.

Further should the matter of the memoir referred to be of interest I shall be glad to send a copy, with a correction, for the use of "direction lines" there is not clear.

With high regards,

sincerely yours.

(Sign)