Creativity in music and math

Pierre Boulez and Alain Connes

Introduction : Good evening everyone, welcome to the heart of IRCAM in the projection space, for this original meeting between a mathematician, Alain Connes, and a composer, Pierre Boulez. So, this meeting belongs to the Agora festival, which questions the relationship between invention and constraint, finally between intuition and logic. And it seemed very important to us to place this nodal meeting point this evening, this attempt to meet between two worlds which coexist and which, maybe, have things to say to each other.

So I just wanted to point out that obviously there will be deduction in artistic operation as well as intuition in mathematical operation. And it's an unfathomable relationship and quite complex. Gérard Assayag, director of the Joint Research Unit CNRS-Ircam, will lead, if necessary, this debate, in any case, will serve as a catalyst. And I also wanted to say that this debate is part of the Mathematics and Music Conference, an international conference taking place at this time at Agora.

Maybe this conference will decree the irreducibility between artistic invention and mathematical invention. But irreducible is a term which was questioned by mathematicians. So we stay in the mathematical field. By way of launching, I only wanted to make one quote, like we often do in France to start or to finish, a quote of the most intuitive, and maybe of the most deductive of all minds, Leibniz, who said and who certainly spoke to composers as much as to scientists :

> *"The perfect world is the simplest world in hypotheses and the richer in phenomena.".*

I give the floor to Gérard Assayag.

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Gérard Assayag : Thank you Franck. We will start with a short presentation by Pierre Boulez and then engage in a dialogue in partly, but in partly only, improvised.

Pierre Boulez : Okay, so a little text at the beginning, to launch a little debate, because it is not at all a definitive and dogmatic text. It is a text, on the contrary, rather skeptical, I would say. If, to account of a work, we talk about mathematical music, it is not an alloy very cordial. These two words, so close to each other, indicate a work barbative, dry, inexpressive, boring.

It does not come from the heart, does not return to the heart, to quote, once in addition, this great model, but comes out of the brain and doesn't even go to another brain. So it's already a kind of rehabilitation of thinking, of musical reflection than one way of directly bringing the two words mathematic, music, and to add the third word contact, discrete word, unpretentious, but a sign of a will that could not be more determined. Good. Obviously, this is not the first time that this rapprochement has been attempted.

From the quadrivium in the Middle Ages, to the work of Rameau and d'Alembert and even the mystical constructions of Scriabin. We even have much written. And yet, there is still some sort of border, said, between musical creativity and the structure of the language explained or the less scientifically approached. When a musician, a composer, close to the computer tool, who wishes to use electronic equipment, many misunderstandings can arise, which are difficult to overcome. Desiring before all the tools that allow him to work step by step, he attaches himself to an immediate return. He expects to be made proposals, let him to be given examples. From there, he can imitate these examples or try to transgress them by modifying the parameters which one proposes to him. But he may as well not go further and abandon this tool that he has just touched. The second pitfall is to transcribe too literally rather, diagrams supplied to it by the mathematical tool or arithmetic.

That which in a case makes sense is no longer relevant and does not make sense in the literal transcription. As much the first approach that I point out is based on immediate perception and does not care to codify for a launch pledge, as much the second approach worries very little, if at all, about the perception, and relies much more on the notion of schema that can be applied regardless of any parameter. Taking into account only the perception, we cannot organize a language, the objects that we found are not strong enough for that. If you don't take perception into account, language can only be constituted in a properly hazardous manner, the parameters not having the same value in the template and in the transcript.

This is where the aesthetic criterion appears to choice or reject the so proposed solutions ? Facing the picture was total possibilities. Intuition becomes like an indispensable short circuit. This is how among all possible universes of intervals, durations, dynamics, etc., intuition is going to choose the one that will serve the composer when the solution will acquire all its necessity. The more we will be able to master this universe of possibilities, the more the intuition will have been used as an absolute criterion in this instant of the choice more or less approached a certain truth that we need at a given time.

Besides, whether we think music with or without an interpreter, music combined between electronics and instrument or purely electronic music, it remains to find the gesture and the form. We are no longer dealing with objects, but with textures which, by continuously changing or breaking, will occupy a space-time. What mathematical model will give us the possibility of finding this gesture which will justify all the other categories ?

From this point of view, I found the quote from Malarmy placed at the head of this symposium : "A dice will never abolish the hazard.". To sum up my attitude as a composer, I would say that I did not wait the whole from a systematic organization of a few parameters that these are. I suppose that the invention, if it is carried out, can only be done if it admits the accident, the unexpected that questions to what we thought establish.

As far as I can tell, scientific intuition goes through the same phases. And on this uncertain ground, it is able to confront with musical intuition. It is a very fragile profession of faith, of course, that I propose, but I owe, I believe, more to this fragility than to the security of dogmas. I believe it to be full of promise.

Gérard Assayag : Your conclusion illustrates a tension which, I think, crosses your work, which is the tension between system and freedom. And in a recent interview with the magazine Musik Blätter, returning to *The* *hammer without master*, you clearly indicated how this work had marked its time by a combination of very finished constructivism, even a little rigid, stemming from the school of Vienna, but with an ornamental freedom and a certain freshness that we could call the French spirit let's say. So the idea was to work with constructivism, but so as to be free there. It's one thing which is not obvious and I say to myself that it may be a problem that also the mathematician meets. What do you think, Alain Connes ?

Alain Connes : Let's say that I've thought about these two things a bit. There are aspects of which we speak relatively little in mathematics, which are precisely creativity and the role of aesthetics. And I think I'll deliver some thoughts I had on that, but just like a starting point, I think it will match what you have told. So, in fact, a priori, when we talk about creativity in mathematics, the mathematician is a little skeptical because most of the task of the mathematician is problem solving. And it's basically a task of discovery. That is to say, the mathematician is looking for truths, that preexist his presence, before he begins to search. And what is quite extraordinary, precisely, you were talking about this relationship with mathematics, which is quite extraordinary, is to see that mathematics evolution that took place in the XX$^{\text{th}}$ century, in fact, already allows the close relationship between music and mathematics. Why ? Because, in fact, the role of mathematics which, at the beginning, was a role that one could roughly summed up as part of physics, has become, over time, in a thematic called modern mathematics of XX$^{\text{th}}$ century in fact, it has become a kind of substitute for philosophy at the level of concept creation. And what's quite remarkable, in fact, is that so far, this transition can almost be traced back to Galois. And which is quite remarkable, is that a bit like in music, it has generated at the start considerable resistance which continues to manifest itself sporadically. But I'll quote you... Does it bother if I do a quote in English because it's a text that is in English at the beginning. But it is a very recent text by a well-known mathematician who shovel Vladimir Arnold, and who talks about mathematics, and who talks about the teaching of mathematics, and who speaks about modern mathematics. Don't worry, I'm French, so I will defend the French point of view after. But I still have to expose this point of view. So he says :

> "*Mathematics is a part of physics, physics is an experimental science, a part of natural science, mathematics is the part of*

> *physics where experiments are cheap.* (laughs). *In the middle of the twentieth century, it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science, and of course in total ignorance of any other science. They first began teaching their ugly scholastic pseudo-mathematics to their students.*"

It continues, it continues, and his text is very funny, it is full of spikes, etc. And then, he says :

> "*The ugly building built by under-educated mathematicians who were exhausted by their inferiority complex and who were unable to make themselves familiar with physics, reminds one's...*"

Well, then after, he speaks of an axiomatic of odd numbers, etc. and then he says so, finally, that he interviewed French math students for example mathematics students, he asked them "2 + 3 ?". And a french primary school pupil replies "3 + 2 because addition is commutative". And then he explains :

> "*Judging by my teaching experience in France, the university students'idea of mathematics, I feel sorry for them because they are very intelligent but deformed kids, is as poor as that of this pupil.".*"

The student who answered "2 + 3 = 3 + 2.". And then he give examples.

But in fact when we deepen this text by Arnold a little, we realize if he wants, what he criticizes is mathematics. What they criticize is if you want all the examples he takes where he says the modern mathematicians don't know how to do that, etc., it's mathematics XIX$^{\text{th}}$ century and the mathematics, he gives examples of curves plot in the plane or things like that, it's math that now are completely digested and the computer does a lot better than a mathematician, he does it in a quarter of a second. And what he did not digested, what he doesn't explain is the wonderful phenomenon which occurred in the mathematics of the XX$^{\text{th}}$ century and which precisely, allow ... I'll read you a little text from Grothendieck. And what Grothendieck says is :

> "*The progressive clarification, precisely, of the notions of definitions, statements, demonstrations, mathematical theories including everything we could, if we only did mathematics as being a part of physics, we could ignore them completely. And to say that it is fantasies of axiomaticians, was in this respect very beneficial and made us become aware of the power of childlike simplicity however. That is to say that mathematical concepts, in fact, should not to fear. In general, they have a childish version and this childish version is much closer to their reality than the extremely versions elaborate," therefore of a childish simplicity, however, which we have to formulate with perfect precision the very people who might seem indivisible by virtue of a sufficiently rigorous use of language more or less current pledge. If there is one thing that fascinated me about mathematics since my childhood, it is precisely this power to define in words and to express perfectly the essence of such mathematical things which, at first glance, appear in such an elusive form or so mysterious that they seem beyond words.*"

And that, if you will, is an extremely important thing because most people, when you talk to them about math, they think about arithmetic, they think of numbers.

Okay, they may be thinking about geometry, but they don't realize that modern mathematics, that is to say the mathematics of the XX$^{\text{th}}$ century, they have just succeeded in perfecting the current language by concepts which are extremely precise, but which have potential of applications that goes far beyond physics.

So, when you think about music and if you want, for well situating things in relation to mathematics, I'm going to read you a little text that I wrote a long time ago and where I was talking about the link in between and I said,

> *"It is crucial to me for a child to be exposed to music very early. I think exposing a child to music at the age of 5 or 6 makes it possible to balance a little the preponderance his sense of sight intellect and this incredible, purely visual wealth, that a child acquires very early and which, therefore in fact is related to geometry.*".

It is linked to geometry as long as it fits into space through a mental image. If you want, there is the same phenomenon in mathematics than in relation to a musician. When a non mathematician sees a mathematician working in the metro. What does he see ? He sees a page full of formulas. They have no meaning. When a non-musician sees a musician working in the metro and reading a partition, it is exactly the same, it feels like ... it's the same ! Now, there is an essential part of the work of mathematicians which is precisely to create mental images. But when I talk about mental images, it has to do with geometry. We see a geometric figure, we see, it fits into space. But what's really amazing is that so far, in the functioning of the mathematician, there is not only the geometric image, there is algebra. And there is nothing visual about algebra, but on the other hand, algebra has a temporality, that is to say that the algebra fits into the time.

When you do a calculation, when you expose a demonstration, that takes place over time. It's just like the musician who, after having understood a musical work, having it completely zipped in his mind, something that has nothing to do with it, spreads it out. For the mathematician, it's the same. When he does an algebraic calculation, it takes place over time, but it's something which is very close to language, which has this diabolical precision of language and in a way, if you will, there is a pretty incredible collusion between algebraic computation, that part of mathematics that has to do with the language, which takes place over time and certain musical works.

And that, I can't help thinking about it. In other words, for me, there are some relatively short musical works that say something. And I even had this impression, you will laugh but I even had that impression when we saw these rooms that go repeatedly listen to Beethoven's sonatas of years and years. It reminded me, if you will, people who are there and who are trying to understand. And we repeat the same to them. They know there is something, and this thing is not transferable by another way than through music. We can't transform into something else than what is transmitted, but it transmits something.

So, we cannot say that it is not a language. Likewise, we can't say that maths don't have a language aspect. They have a language aspect which is extremely important.

But the gist of what I said, if you like, is that this aspect launched pledge of mathematics has become much more flourishing. He became much more expressive. It has become much broader than, precisely, the mathematics of the XIX$^{\text{th}}$ century. And when we stay with mathematics of the XIX$^{\text{th}}$ century, of course, you can say "Oh, these mathematics have a connection with music because there is arithmetic, there is log 3 on log 2, which is the well-tempered keyboard, etc."

But it will not go beyond. In fact, mathematical language, so far has crossed many other frontiers and in a way, now, we can hope that, precisely, there is a possibility of reconciliation which is much bigger because of that. People still must accept mathmodern materials. And people still must have absorbed all this elaboration, which is not at all obvious.

Gérard Assayag : So this algebra / geometry duality is one of your workhorses. It's extremely interesting because it brings to the heart of the math-music problem, because it's a duality that we constantly meet in musical research, namely metaphorically, we discussed it, either technically, and I could possibly give examples. So especially in musical analysis, that is to say that when we look at a score, there is an expression which I heard and which I like well, it is the partition seen plane, when we try to understand this mechanism, that is to say it keep as a whole but we have the right to do what we want. We can jump from one point to another and relate a point to another freely and it's an obviously geometric vision.

Alain Connes : Of course.

Gérard Assayag : Or there is another way of approaching it, which is from the point of view of these generating mechanisms. And here we have a point much more local, because we look at the mechanics. Is that, it's something you feel, this tension when you, when you watch music, not when you create it, but when you watch an existing music ?

Pierre Boulez : When I watch music, I start first by trying to understand the form, because it is what directs you, simin the evaluation. We did one experiment here out of three levels of understanding of music. We first gave, say, a Mozart's sonata or a movement, or a half-movement (the first half of the movement), and we asked someone who is absolutely not a musician, someone who has no musical culture, we asked him what did he think ? He gave a very vague description of it and not at all relevant if I can say. Then the medium level. He had listened several times sonatas by Mozart, so he could find a form, at least a contrast between themes. It's already a much more precise and the cultured, then, described exactly what happened. Then we spent a Schophausen's piano work, a fragment of course. Well, the three responses were very similar because everyone created their own theater of the shape and they would point to the passages that had particularly struck them, that is, there was no conception of form.

But there was a conception of events and events that were still not linked by a form but separate events, which had struck, either because they were very strong, or because they were played by a specialized instrument, etc., etc. So you can see that it's very difficult to approach even a form, because a form is really, disounds, what... how the person looks at it. And there, when one is a musician, obviously, we are trying to have a view, let's say more objective, and not only subjective.

This is how I see music. So when we see the detail, we indeed see how the discourse is constructed and if it is constructed more horizontally than vertically or more vertically than horizontally, or if it is built by breakage, or if it is built by continuity, etc., etc. There is a lot of ways of looking at the very perception of music and I'm convinced that there are many people who also make a kind of... who... since they cannot understand the musical form, who make themselves a certain narration, especially when they have listened to a work several times, they make a personal narration and it is this narration that they follow. That's why people, the general public, if they don't make an effort, settles so well in a work because they always listen to it by the same way and therefore that they has before theim the same images, the same stereotypes, the same shots, I would say, rather than the same pictures. And that's how they absorb music, they don't absorb it by a kind of description of continuity, they accepts it as a whole, followed by a whole, followed by a whole.

Gérard Assayag : But the expert-analyst, the composer who watches another composer, doesn't he have this freedom when he is looking at a score, to finally see it as a space where he can walk around at will which is not realistic insofar as the score was not generated from that way, by acting simultaneously on all parties ?

Pierre Boulez : Yes, certainly, when you analyze ... Me, what interested me in analysis, it's even false analysis, but which generates something.

I remember once when Stockhausen showed me an analysis of the Webern quartet, but he looked at the density of meetings. What has nothing to do with Webern, which was just a four-voice counterpoint, and therefore a four-way counterpoint, especially if it's a cannon, things are offset from each other. So if there is an incorrect individual phrasing, things are obviously not always of constant intensity. But for him, what interested him at the time was the phenomenon of intensity.

How can a four-voice cannon give intensities of this order, statistically speaking. I find it more interesting than analyzing even simply how the composer designed it. What is interesting in an analysis, it's not when you want to redo what the composer has done, it is to see by what process he arrived at such a result. And so, even if the analysis is false, is completely false, the analysis is much more interesting because it is productive.

Alain Connes : Okay, there is still a rather frank difference, precisely, there, we are talking about Works. So we see... So if we look at a particular aspect of mathematics, which is a demonstration, we can say the next thing which is a little bit similar, it is that if you want there are two ways to look at a demonstration. There is a check line by line. And that, I think, is a bit like someone playing a song of music which he has not yet digested and who is obliged to have the score before the eyes.

So we can do that. We can check a demonstration line by line. But there is a second step which is extremely important. Because in fact, a mathematician knows he only understands a demonstration when he is able in his brain to zip it in half a second. That is to say that he will not have the successive ingredients of the demonstration, but he will have immediately the entire demonstration.

Pierre Boulez : Can I open a parenthesis ?

Alain Connes : Of course.

Pierre Boulez : In music, that very much depends on a very different point of view , according to the fact you are a performer, or if you are a composer. If you are a composer, you have plenty of time to navigate and you move from one point to another and you try to consolidate your analysis by comparison from one point to another, what are the differences, what are the similarities, etc. If you're a performer, this way, let's say, of amassing the knowledge is a consequence ... is a kind of unconscious thing.

When you are at point D, for example, you know that you have already played point A, and its successions, and you know you're going to meet the point N and its successions. But you don't know exactly. But you know, the closer it gets, the more you are aware of what will follow. And the more it goes away, the more you are aware that it goes away and therefore that the form has reached a point of the present, that is to say that we constantly have these three dimensions in the head, present, of course, where you are and the past who brought you there, the future which will lead you to ...

Alain Connes : Of course, of course. But what I mean is that precisely, this kind of linearity of the work, there is something that is extremely striking for the mathematician, that is to say that if a mathematician tries to understand a demonstration, there is this process which is to try to read it linearly. There is another process which is much more efficient, which is to look at the theorem statement and start by looking for a demonstration yourself.

And when we've done that, what happens is that reading the demonstration, at that time, we will say : "But that's nothing. That's nothing". And we are going to say : "There, there is something going on". And it's only like that, it's only from this mechanism that we really understand what is going on... So that, I don't know if there is something analogous to that in a musical work. That is to say, does a musical work answer a question, etc. And we can say when the work takes place "Ah !". Well, I sometimes had that impression at the end of certain pieces where there was a kind of moment when there was a moexplanatory a posteriori or vice versa. I mean by the time we see that there is a theme that will then unfold, etc. But in mathematics, it is something extremely strong.

That is to say, there is a huge difference, precisely, between mathmatician who vaguely understands the statement and then begins to check the demonstration step by step, etc. And the mathematician who is going to have an act which is not at all passive, but will start to think for himself and after, only after, go and watch the demonstration.

Gérard Assayag : Does that have anything to do with compression, this zipping you are talking about ?

Alain Connes : Absolutely, of course, of course. That is to say, the mathematician works by levels of abstraction, by hierarchical levels of abstraction, that is to say that in fact, it means that he cannot progress, as the concepts are very complicated, and with this notions of zipping, he is able to make them occupy a space which is almost zero and afterwards, he will be able to manipulate them abstractly without knowing what the zipped contains, simply by having an intuitive idea of "what this motion signify. Of course for that, language is extremely important, that's why, well, there are very creative mathematicians like Grothendieck, etc. who gave 36 new names like *schema*. Schemas have a very precise mathematical sense, etc. And it's only with this zipping mechanism that we can progress through hierarchical levels of understanding.

Pierre Boulez : For music, it is mainly memory that plays a role. I see, for example, it's very striking, when I was mainly in orchestral charge, I did introductory sessions, but explanations on musicians'Works and I always noticed that there was always a need for examples. By that, when you play the work, the example immediately comes to mind.

And there, the memory works so as to magnetize the perception in a direction or in another.

Alain Connes : Okay, yes, so I think there is something which is very analogous in this case, because, well, there are some mathematicians like Grothendieck who work a little bit backwards, that is to say that they start from the general case and then they... but most of the mathematicians work differently, that is, if they are given a good example and have been explained something concretely on an example, a general phenomenon, precisely, they are perfectly capable to immediately generalize and to have the general case and I guess that in musique, finally, we can see in Beethoven's music or things like that, we can see that there is a generative system that allows from things relatively simple to generate quantities of things which are deduced from it and this, in mathematics, is a fairly general phenomenon. So there is this side of almost automatic generation that occurs and that plays a very important role, very, very important.

Gérard Assayag : So to come back to this duality concerning algebra versus geometry, you mention, so it's very important, that on the algebraic side, you put time, there is a begetting, and so a generation. There is a combinatorial of symbols. There are production rules. These are things that we use a lot in music. The musicians were interested a lot, for example, in formal grammars or production rules to find interesting sequences, or not elsewhere, of notes. But as soon as it produces sequences, we agree, but sequences are they sufficient to define time ? It's a question that I'm going to ask both to the mathematician and to the musician.

Alain Connes : Of course, I will answer because I mean : my first mathematical work consisted exactly in that, that is to say if you want, and what is quite incredible, is that, precisely, we realize that this called non-commutativity, what does that mean ? It means that when you write a word, it's not all of the letters of the word that matters, but it's also the order in which it is written. Okay, well, we can give 36 examples. And what is absolutely unbelievable, what is absolutely incredible, is that precisely, you realize when you do math, you realize that when you look at non-commutative geometry, that is to say the algebra precisely, in which one does not dare to say that $abab$ is equal to a^2b^2, well, time is spawned in a natural way. This is much stronger than saying that algebra takes place over time.

In fact, and that comes from the quantum, that is to say the quantum taught us that precisely, when we were doing mechanical calculations, in a quantum paradigm, we couldn't, that's what Heisenberg found, we could not swap quantities like position and time, etc. We could no longer calculate too simply when we are interested in microscopic systems, which is absolutely amazing. And the philosophical potential has not been sufficiently exploited at all, therefore. The fact is that when we take an algebra of a certain quality, which we call an operator algebra, which is non-commutative, well, it generates its own time. It has a group of automorphisms which is parameterized by a parameter t but that is really the time in the physical examples which turns over time. So this is amazing and it comes exactly because you cannot swap a and b. So when you write a word, the order of the letters is important, while when Descartes, etc., when people of that time were doing calculations, they were doing calculations commutatively, that is, by swapping the letters.

Gérard Assayag : If I understand correctly, it is the algebra that evolves itself and which transforms itself, which therefore generates a series

Alain Connes : It creates a passage of time. So that had already been tipped since Hamilton had written utterly prophetic sentences, precisely, and where he was talking about the relationship between algebra and time.

So what struck me earlier was that you were explaining yourself that, precisely, in the work of an interpreter, there is always this present. And then there is the past and the future, etc. So we can see, I will say it roughly speaking, it is a deep, microscopic analysis of time. It's an understanding of time which goes further and further into finesse. But what is quite amazing, is that at the algebraic level, there are exactly the same thing that happens and does it rightly, not only, well sure, an algebraic calculation is done in a linear way, with ordered terms in time, that's nothing.

But what's amazing is the reverse. It's the fact that even if we were doing math outside of time, etc., well time would be there and would be present. It would be generated naturally.

Gérard Assayag : You mentioned another point earlier which was without saying it, I will say the technical term, you will excuse me, the Curry-Howard correspondance, i.e. the fact that a proof, we can also look at it as a program, as a calculation.

Alain Connes : Yes, if you want, yes, of course.

Gérard Assayag : It brings up a question we asked ourselves here during the very first Mathematics and Music congress which was organized in 1999 at the request of the European Mathematical Society with M. Bourguignon.

We decided to put this under the umbrella of the question “Is there a correspondence between what musicians call musical logic, which is always an organizational logic, and what the math just call logic, mathematical logic or formal logic, mathematical logic ?

And we obviously had not decided this question, we had just managed to say the following : there is a lot of logic in the organization of music. There are many formal terms that we generate. There are even things that look like axioms, that is to say starting hypotheses that we give ourselves to generate a material. But there are two things that are not present ; in music, there is no notion of truth : we do not seek that these terms we aggregate, which will eventually form a partition, establish a certain value of truth, that is not the problem. It’s not the problem of logic. The problem of musical logic is not the problem of mathematical logic. Do you agree with me ?

Pierre Boulez : I certainly do not agree with that. I said it discreetly but I think so.

Gérard Assayag : We can say it and I think it’s easy to establish. Here there is no truth value, so already, it removes a whole computational aspect because often that’s what we’re looking for. And then there is another problem much deeper, which is as follows : in pure logic, when we unroll a demonstration, I can use a term A for my demonstration. And I have every right to reuse it afterwards, but nothing happens. It doesn’t cost me anything. I use it, I can use it a thousand times if I want, if I longed for. When you consider a musical sequence, an element of language musical as a little bit like a demonstration and that we look at the terms that we aggregate, notes, chords, etc., well, the fact to have exhibited a musical object is not at all innocent. And the second time that we expose it, it doesn’t have the same value at all as the first time we had exposed him. So already, already, we are no longer in this hypothesis. *(Laughs.)* I see, I think I see you coming.

Alain Connes : No, no, in fact, if you want, that means that you don’t know a certain part of mathematical development, which is what is called linear logic. In logic, in linear logic, especially listen to Jean-Yves Girard from Marseille, when we used it once, we can no longer use.

So I mean, don’t believe that the mathematicians are missing of imagination. They used this logic. It has already appeared to them. But in fact, if you will, well, just bounce a little bit on what you say about what happens in music at the logic level, what I would say, it is that there is indeed for the mathematician a role of the aesthetics, when he watches a demonstration. That is to say a mathematician is able to tell by watching a demonstration how likely it is to be true. He is able, by looking at a formula even obtained by a computer, to tell the chances that it has to be true. So there is a role of aesthetics. But if you want for me, you shouldn't believe at all that quality, well, is a quality necessary for a mathematical statement to be true, to be correct, a demonstration of being correct. But the concept, which is much more interesting and much more difficult to obtain and which is much closer to music, is the notion of meaning, that is, if you want a mathematical statement, you could brick a computer that would make you 36 mathematical statements at the shovel and that would all be correct because he would have made them by making correct demonstrations. It would be easy. However, if you looked all of these statements, most of them would be completely uninteresting because they wouldn't make sense.

What does meaning mean ? The notion of sense is something that is ... which does not respond to logic, because the statement in question is correct. But there is for the mathematician a notion of a statement which is wonderful, which has a meaning. And I think that here, we have a connection with music. Because you told me a musical piece doesn't have to be correct, of course, but it must have meaning. If it doesn't make sense then, well, well, I'll say, we could do anything. We could invent 36 music skins. And there, I think that we touch on an essential point because the notion of correct is a necessary condition. It is a necessary condition for the mathematician, of course. But a mathematician could spend his life doing what Arnold said about odd number axioms or things like that. And that means he would have wasted his time. he would have wasted his time because he would not have found the truth that makes sense. He would not have revealed a part of this mathematical reality, but precisely, things that make sense. And this is an extremely difficult thing to define in mathematics. And I think it's also difficult to define that in music, in a certain way.

Pierre Boulez : Yes, it is very difficult because during history, we see people who have less, especially in the XVIII$^{\text{th}}$ century, vocabulary and in one case, the work is very beautiful and in the other case, the work is going to be completely uninteresting. That is, the same grammar can serve not for the purposes at least, but can serve very different purposes.

Alain Connes : Yes, so, I mean, it just means that when we stick to the level of structure, logic, etc., we don't touch the essential problem and the essential problem for mathematics, so far, really, of course, there is the problem of truth, there is the problem that we can talk long, wide and cross. But there is a problem much more difficult, much more important, which is to see precisely in what sense what we found reveals a little corner of mathematical reality. And that, that means to make sense. Exactly.

Gérard Assayag : The problem you raise is the problem that meet automatic theorem provers, programs that demonstrate theorems, they can demonstrate correct theorems, but they don't know how to say that a theorem is interesting. And so, they can demonstrate billions of internal things without sens. So it's interesting because it can join a problem that we know here, which is computer assisted composition, where we have computer programs that composers use to calculate interesting materials or structures.

But they could calculate billions that would not be of interest. It is ultimately the composer who decides. So could you help us ? How could you, composer, help us to converge in a finer, more interesting way, towards results that are not only correct from the point of view of calculation, but likely to interest the musician ?

Pierre Boulez : The first thing I can answer is a very silly answer, it's because I like it, simply because what you give me, what you offer me, I like it.

Gérard Assayag : This is how we work.

Pierre Boulez : Yes, but the whole reasoning of music is based on that, of course, we're not going to say that stupidly : what pleases to me, so I choose it. You may have a terrible taste, the kitsch, and so to say, I like it too, of course. But what's interesting is that when you have so many possibilities, you can't listen, if you have a thousand possibilities, after a hundred you will be tired or you will have absolutely no judgment. That's what is dangerous in music. The more you listen to the different solutions, the less you have reactions, let's say, to choose things. And so, at some point, two things are needed.

First, narrow the scope of the choice and second, decide : "yes, that, why do I choose it ? Because it looks better to me, for this reason, and this reason". But deep down, you're trying to justify yourself. But the main thing is only ... it's not only, but it's mainly intuition and intuition, well, it exists and it's a gift that you have, even if you are very gifted, you have it one day, you don't have it the next day.

That is to say it is very variable and sometimes, you are very sharp, others times less sharp, because you are more seduced by the... And there are also a difficult question in music that is how to join the abstract structure if we can say, and the concrete object, because the concrete object which is very interesting, is maybe in a completely inept structure. And on the contrary, a very intelligent structure can have objects which are completely uninteresting facts. And so, it's this combination that is not easy either more to organize, which makes the work acquire great validity. But that, that has always been the case. I mean, if you look at the history of music, you have for example two very distinct personalities like Berlioz and Schumann, I take these two examples on purpose. In Berlioz's work, there is a sense of instrumentation which is absolutely remarkable even when he was very young. But the sense of harmony, that is to say of harmonic language, was very primitive per se. So we explain it's because he played the guitar when he was young and therefore the guitar simplified his vocabulary. It wouldn't enchant guitarists, if we say that. But while Schumann on the contrary had a very... much more refined harmonic language.

But his instrumental language was really, let's say, without a lot of meaning, without many colors, even, quite simply.

And so, it's very rare to have people in the same musicians who are also gifted, for the different components. So when you have someone like Wagner, obviously you have it, you have everything.

But Wagner who, let's say, never talked about a system, he always talked about new music, music of the future, etc. But he never codified his language. Not at all even. But he took the language as he found it, and under the influence, in particular of Liszt, he diverted the language of the function on which this language lived, and therefore ultimately, he invented this language very ambiguous where all relationships are possible. In more classic language, let's even say Beethoven, not to mention Mozart, you have chords that made revolving chords, so to speak that helped modulation, helping to go a little bit to a neighboring country, but in Wagner, you are... sometimes you have no idea where you are because he uses only ambiguous things.

This ambiguity became generalized gradually and led to Schönberg, who has again created a dogma.

And this dogma was interesting in a certain way, because it did indeed he organized musical language in another way. But this dogma, this dogma, ignored vertical phenomena, and, or barely took into account vertical phenomena, and this is the weakness of the twelve-tone language of Schönberg. Is that one dimension prevails over the others or over the other, specifically, that is, the horizontal domain prevails over the domain vertical and in Bach, that was typical, the vertical and horizontal domains rates will be completely controlled.

And there, the vertical domain, you perceive it immediately ; the horizontal domain, counterpoint, you perceive it when you have studied the score is the difference. You do not perceive the music of the same way if it is written in one way or another. And that, there is nothing to do, we will never change that, because it is a phenomenon of perception.

Alain Connes : Yes, what I wanted to say is at the general level of structure. It's, well, finally, if you want, we can roughly summarize a little bit of the mathematician's work saying that from time to time there is a mathematician who ... finds a big phenomenon. An example of that is for instance when Riemann finds the relation between prime numbers and zeros of a certain function. Okay ? And it's a find, that is to say that it is something that afterwards, we will be able to verify until a certain level with a computer, etc.

But it will give mathematicians a century later, two centuries after a kind of objective. And the reason is that we know that this phenomenon is deep enough and mysterious enough to be sure that all the concepts that will be invented, discovered during this research, that is, to try to find a demonstration of this fact, will have meaning, will have a lot of meaning. So what ?

Precisely, where I think that there is a connection that is possible, if you want with music, is that we can say in fact that there are two aspects in the work of the mathematician. That is to say, of course, there is an incredibly rational aspect which consists, once we have an idea of a demonstration, to try to verify that it is correct, of course. That is pure rationalism. But there is an aspect which is much more interesting and which has to do with intuition. And this aspect that has to do with intuition is that there is a period in which the mathematician must absolutely not say to himself "Is what I say correct ? etc. Have I checked all the little details ? etc." And in which, precisely, he must allow himself to dream ? He must afford to see much further and in that period, which is basically, set a little bit like a poetic impulse. It is something that is not transferable in words. That is to say that if a mathematician is in this period, he is unable to explain it to people he is going to meet who will say "yes, well, but then ?".

And he is unable to write it. Because if he writes it, it's like he's said to catch something that will disappear from the moment it goes write it down. But the question I ask myself is to what extent, precisely, this intuition which is terribly present, which is something extremely strong, can be translated in another way. Can it express itself in a musical form, can it express itself otherwise. Because it comes from something that is very deep, which is inside. And if you want, there is a text by Grothendieck that I will read to you if I have time and who speaks precisely of the dream in mathematics and who says to what point, precisely, the dream is not admitted in mathematics. It is not admitted. Why ? Because when a mathematician writes an article, he will not write about dreams he has fantasized, etc. He will write demonstrations. And so there is an invisible part of the mathematician's work which is never visible.

The visible part is going to be a rigorous, written demonstration, etc. And it is going to be a whole... something that is completely hidden and that is all this invisible part and that consisted of these... all these days, etc. in which there was a dream, which was present in intuition, which was present in mind and not yet realized. Well, this, it makes me think if you want the music to work, it feels as if we are at this level of intuition, of something that is not yet realized, etc. but that we managed to transmit, on the other hand. We managed to transmit it in musical form and from the moment when, precisely, there had been something real behind it, there was a real inspiration, etc., there, that makes sense and finally, you get through music to convey someone. So what ? The funny thing is that it finally happened to me to have an outside contribution through a musical work for a problem that I was asking myself and that this musical contribution is more important than if I had read a mathematical text.

I used to listen to relatively short musical works, but that had a meaning, and it was a meaning that fit in with some kind of intuition that I had at one time, but could not translate otherwise, I couldn't translate it into words. I couldn't say "Good, well, etc.". But on the other hand, there was for example, I don't know, a Prelude, which corresponded exactly to this intuition. I did not know why. So there, there is something, in my opinion, if you will, in the notion of meaning and all that.

Pierre Boulez : No, I say that the transcription of a musical intuition, from mathematics to music, is very, very uncomfortable. It is very, very uncomfortable because the choices are not the same. The culture is not the same and the choices are not the same. I was saying just now, I take the case of a composer who did it, Xenakis for not to name it, which used a lot of glissandos, curves, so we saw superb, magnificent curves, etc. But what do we hear, we hear an extremely poor material.

Alain Connes : That was not what I was talking about at all. If you want, there are two very, very different things. There is the fact of using mathematics, well, I remember listening, in fact, to a conference of Xenakis, a very, very long time ago, at a given moment when I was asking to myself if I was going to do math or if I was going to be interested in music ? Things like that.

And he disgusted me, really, because he had come to the Sorbonne, he had made a presentation and in his presentation, he had surrounded the painting in which he had some general formulas by mathematical formulas, and these mathematical formulas had nothing to do with what he was talking about. So, they were there only as a psychological tool for, how to say, scaring people who didn't know math and for, so, imposing something on them like that. So it was not that at all I was talking about.

What I was talking about was a problem that is completely open to my opinion, which is that there are certain mathematical notions, certain mathematical intuitions which are not transmissible by words at the moment.

Pierre Boulez : Yes, but what I wanted to say is not just as a criticism. But let's say a glissando, which follows a curve or another, it's an extremely primitive material, it's a limoth. What interests us in a continuity like that is the notion of cutoff, i.e. the interval, because the interval really defines the way you perceive things. And so when we target, for example, we had seen, even a curve that inspires you a kind of gesture... But what gesture, it should be transmitted not by a direct gesture like that, but you have to transmute it, practically, with intervals that will really give it sense. And that's why I say it's the transposition, or trans-figuration of that, and it's really less primitive than we think.

Alain Connes : Okay, but what I had in mind, for example, you mentioned, about Wagner, the ambiguity between the tones, etc. And then, precisely, there is a mathematical idea which is relatively simple to explain, which is due to Galois and which is not yet, how to say, captured mathematically. And this is precisely the idea of ambiguity. And so, what I have in mind, this is the next thing, is that precisely, like math can capture concepts at levels of conceptualization which are very high... For example, what Galois did, what he understood, is that in fact, people before him, were looking for symmetries, and he, he managed to understand that in fact the first thing to do was to break full symmetry between the roots. And after, once we broke completely symmetry, we managed to find the interior structure by other processes. But what I have in mind is that this idea, well, you go and read 36 math texts around this idea. There is none of these texts which completely exhaust him. There are none. That is to say during that you write it in rational terms, etc., you can't exhaust it. And I am persuaded that there are surely certain musical structures which would arrive at transmit part of the content of this idea, in a complementary way, in the rational way of saying it. That's what I have in mind, not at all that we can use mathematics to guide certain things... It is something which is much more, which is much more at a conceptual level, and to the fact, precisely, that there are mathematical concepts much more elaborate, much more complicated and much more ticks, how to say ? And at the same time much more childish than one could believe and that, precisely, we cannot perceive them completely when we only use linear, rational language, etc. And that polyphonic music etc. can help considerably, if only by the polyphony, that is to say the written language is a unique linear language. There is only one, only one narrator. And polyphony, precisely, well, well, we know that. And in my opinion, precisely, that should allow us to go tobeyond certain things that we are only able to do at the moment.

Gérard Assayag : The question you are asking is really the one concerning the source of creativity. In other words, if I transform it a little, "Are there very deep, pre-verbal levels of representation, almost conceptual, but we're not really going to say that since they are still not verbalized, but which could then, by the time they hatch and where they appear, transform in various ways into mathematics, into language."

Alain Connes : It's exactly that. But what I mean, is that I always come back to Grothendieck, but he shows well how precisely, the process of creativity is a process of back to childhood. It is in this sense that it's a process which is to try to get rid of all dogmas, everything what was imposed on us, etc. And to return to a perception completely childish. But precisely, well, after precisely, having been able to make it universal and to transmit it. So that's obviously at the heart of music, but it's also, it's similar within mathematics.

Pierre Boulez : But is it possible to be so childish ? I was going to say infantile, excuse me, for being so childish, having done all the same experiences that have marked you ?

Alain Connes : Exactly ...

Pierre Boulez : Isn't it artificial ?

Alain Connes : I don't think it's artificial. I do not think this let it be artificial : the example of Grothendieck, which is an extreme example, mentally striking because at one point, precisely, it has, to return to the CNRS because he had left and he had made a request to the CNRS and his text was called *Children's drawings*. So you read this, it's a child, you can say, it's infantile, etc. But in fact, it was connected to one of the deepest math problems which is what's called understanding of the Galois group of the algebraic closure of Q, etc. And it is very often the case, in fact, that when people become professionals, they surround themselves more and more with a protective layer which precisely prevents them from returning to this state. And on the contrary, I think that what is absolutely essential, precisely, is to allow the dream, to allow, to try to go beyond the prohibition of the dream, etc., and to return to that source. And I think when we go back to the source, for example, of the notion of ambiguity, which is a notion that exists and that could be manifested in quite a few areas, well then it will have effectively various forms, it will take various forms. And we will not arrive never to sum it up to an expression.

There will never be a single expression that will sum it up and it will remain a constant source of inspiration. And this is the case for Galois theory, that is to say that it is a theory which is not exhausted and it is not exhausted at the sense where it stays... that's when people really understand it, that is to say that someone could read a book on Galois theory and understand nothing about it just because he would not have understood the initial idea. And it's an idea, precisely, which is a childish idea, which is the idea of ambiguity.

But this idea, when understood, sets things in motion.

It's a real idea, it puts things in motion, and I think, it's very similar to music. Because you get the impression, if you want my impression, me, on creativity in music, it is not an impression, it's more than an impression compared to classical music, romantic music, that is music that is emotional. But my impression was more, compared to the mathematician, that there was a kind of emotional battery that charges, regardless of the instrumental expression and then, once it's charged enough, there is a job which is extremely difficult, which is to make individual emotion universal, transform it, and to make it universal. And it's a process that can seem extremely different from the mathematical process. But schematizing it, it is the same because what does the mathematician do ? What is the role of mathematician's intuition ? The role of his intuition, he is exactly like a hunter. He says "There is something there !". He feels it very, very deeply. But after that thing, he has to go to get it and there is a a reality which is extremely cruel, etc., and which prevents him from going to seek it.

So afterwards, he has a real job, and this job, I think it's the same. It's very similar to the work of having a personal emotion, trying to make it universal. So there is a parallel, of course, these are different things but the role of intuition is the absolutely driving role at startup and it's the same in both, I think.

Pierre Boulez : Yes, I also think so, most certainly, but, in more than that, I would say that there are two constraints : first, the object does not exist, whatever you imagine, so it remains to be built, and secondly, what we have, in music that is instrumental, for example, we have to take into account what is transmission. And this transmission will hurt if for example, the idea is brilliant but the realization is insufficient. And until this difference between the objects you use, for example, the notes. When you have, for example, a very remarkable object, I think, quite simply, because everyone knows that, to the sound of a tom-tom. A tom-tom sound is much more interesting than a violon sound, just like that, but what does it do ? This sound is so interesting that it gets out of context automatically, so you have to restrict it on the contrary, to use it in a very very measured way so that it has its place.

While you have a F#, a G, or whatever, it is neutral, and therefore you can use it for your discovery, that is to say that there are objects that are ready for discovery, and objects that are not ready for discovery, which monopolize ...

Alain Connes : It's a bit like a Chinese character that makes sense in itself as opposed to a letter of the alphabet that has no meaning in itself.

Pierre Boulez : And it is not convenient to have to use both.

Gérard Assayag : We could continue this amazing discussion for a very long time but we have to make the antenna, so that the Festival and the Symposium continue. I think we had two very nice ending words and I would just like to mention a conclusion about emotion, I remember reading in one of your works, the one with Changeux, and which is that so that one day the machines can imagine goals, and therefore become more interesting, they should suffer. We have a great program as computer scientists, to make sure that machines can suffer, too. Thank you Alain Connes, thank you Pierre Boulez.

Meeting between two major figures of musical creation and contemporary mathematics, Pierre Boulez and Alain Connes.

What is the place of intuition in mathematical reasoning and in artistic activity ? Is there an aesthetic dimension in mathematical activity ? The concept elegance of a mathematical demonstration or a theoretical construction in music does it play a role in creativity ?

This dialogue around invention in the two disciplines is led by Gérard Assayag, director of the CNRS / Ircam Sciences and technologies of music and sound laboratory.

Introduction : Frank Madlener, director of IRCAM.

As part of the third international conference Mathematics and Computation in Music (MCM 2011), Agora 2011.

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Excerpts from a round table listenable here
https://www.youtube.com/watch?v=d08lHGeIlJY

Alain Connes : Yes then first, if you want, there is a specificity of mathematics compared to the other disciplines and I think that I must speak about it first : it is that, for a mathematician, by far, the most valuable thing and I'll explain why, it's time. And the way the CNRS helped me, if you will, when I was at the École Normale, in fact, I refused to pass the aggregation because I knew I wanted to do research, and I didn't want to go back at all in a spirit of cramming, which had been the spirit of preparation for entering the École Normale, etc., it was a blessed time, it was the beginning of the 70s ; so in fact, I was immediately taken on as an intern at the CNRS and I had, if you want, five extraordinary years in which I had all the time it took to reflect, to work, etc., on, precisely, the works which afterwards earned me the Fields medal.

And in 75, I went to do my cooperation in an underdeveloped country, which was in Kingston, English Canada. I had gotten some help if you want from friends, who had sent me there to a university, which was a very, very long time also profitable in terms of work and fortunately, because it was supposedly my military service, but well, it was an extraordinary way to get through. And at that time, I made a mistake, I made a mistake which was that I learned that I was offered a teaching position in Paris and I let myself be tempted, being far I said to myself : "oh, I've done enough research, etc.", I accepted this position. When I came back to France, therefore, I left the CNRS, I resigned from the CNRS all immediately, I returned to France, and when I returned to France, I started my work as a teacher, and I realized at that time what my time had become. Before, when I worked at the CNRS, my time was a time that was continuous, which was continuous, I could think all day. Of course, when we think about a math problem, we don't need..., we don't even need paper, pencil, we have..., we can go for a walk, we think about this problem. What we need to know is that for three hours, for five hours, we will not be interrupted. When I was in university, when I worked at university, I knew that I had, for example, an hour and a half to reflect. I started, I started to think, etc., my brain heated, and started to be available to the problem I was looking at, etc. And then when a quarter of an hour arrived before class, I said to myself "no, I have to interrupt, that I have to stop.". My time, if you know math, my time had turned into what we call a Cantor set, that is to say that I no longer had, if you will, long enough continuous intervals to reflect. And that is the miracle of the CNRS, the miracle that allows young researchers to fully immerse themselves in a problem, in a math problem, for example, and by this immersion, if you want, by this kind of... I don't know, me, work, if you want, of penetration which occurs... One day... Indeed, for me, it had occurred in the 70s, I came back from having accompanied my wife to her high school, I was in a car, I was driving a small car, I was stopped at a red light and at one point, I did have an enlightenment. And this illumination was such that my brain was completely certain of the result, I didn't need to check it, I did not need paper, pencil, etc., I was fully lit knowing that was good, there was something extraordinary that was there. So when I returned to Paris after my military service, after my cooperation, in fact, I realized very quickly that I had made a terrible mistake and that in fact, it would have been better if I had become a postman or that, any job other than the job I had at that time, which made it impossible for me to have a long enough time to think ; at that time, I re-applied to the CNRS, it is true, I mean, a year after accepting the position in Paris, I re-applied at the CNRS because I said to myself "but I made a huge mistake". I re-posed my candidacy for four consecutive years, to the CNRS, without being accepted, but the CNRS gave me the silver medal, and later, in 2004, CNRS gave me the Golden medal. At that time, we could see that they were not too happy not to be able to see me get in but they couldn't, well, I mean. And then finally, at early 80s, I was taken over by the CNRS and it was again a period absolutely incredible, of incomparable creativity, compared with the time I had when I was at university, for exactly that reason, exactly that reason.

In the meantime, therefore, I had nevertheless perceived that there was something, as a researcher at CNRS, something that did not stick completely : what did not stick completely was that the job of a math researcher is a job where there is no lab, where there is not generally, well it must be said that if we really want to make a breakthrough, you have to be alone, and so it's a job which, at the level of human contact, is very frustrating. That is to say that in fact, most of the time, there is no illumination, I mean, I remember the story of de Valéry who asked Einstein if Einstein had a small notebook in which he could write down his great ideas. And Einstein replied "I had two big ideas in my life". So, I mean, It's obvious that most of the time, a math researcher spends his time to be frustrated, that is to say, we do not understand something, we try to understand, in fact, for real work, we don't do math because we want to have fun doing it, no, we don't do math because we want to make money, no, we do math because we are trying to understand. So most of the time, we are trying to understand, it's difficult, in the meantime, we're spending an extraordinary time, if you want, to take examples, to search, etc. The purpose of the manipulation, well, sure, is to understand, but it's also to create concepts, because by doing that, we create concepts. So I noticed something, which was a bit of a gap, at the time, of the CNRS, and that was that a math researcher was very isolated, and in fact did not have, if you will, the given opportunity to transmit his knowledge. And what makes that when the Collège de France asked me, in 1984, to become a professor at the Collège, I found, there, at the Collège de France, if you will, a combination that was really ideal, in the sense that we gave the teachers all the time it took to search, but there was an homeopathic dose, I would say, of teaching that is to say that each year, we have a teaching to do, and I think that since, if you will, the CNRS has amended in this direction, that is to say that the transitions between the CNRS and the university, which precisely fill this void, that is to say to transmit knowledge. It's still something essential, Nicole Le Douarin talked about it, it's still essential if you want, for a researcher, not to not remain completely isolated in his bubble, and to be able to transmit his knowledge.

What is obvious is that when we pass on our knowledge, the moment we make the effort to transmit it, we also make progress. That is to say, we realize that we had not understood something, simply because we give lectures on this. So my experience of course for the CNRS, I owe it everything, everything I have found in my scientific career, but I got there at some point blessed, which was at the beginning of the 70s, there had been this doubling of CNRS credit in the early 60s, well, I mean, unfortunately that's not the case anymore, because I saw, if you will, with the example of a lot of students that I had, the current difficulty of entering the CNRS is now much more difficult.

[...]

No, if you want, I think there is an error at least in my field, in mathematics, you can't generalize, but having copied the Anglo-Saxon system which is the system of having a Grant from the NSF and which is to make requests for projects, etc., it's disastrous for mathematics for the following reason : I think there is a comparison that is quite striking who says "mathematicians are fermions, physicists are bosons.". So for people who don't know what that means, you know, fermions is related to what reveals the periodic table of the elements, fermions have the property that they cannot occupy the same state. So what does that mean, it means that in general, mathematicians choose a small box, and they put themselves in there, and they work alone, unlike physicists who in general, well can... Of course, there are often physics modes, which make that there is a very large number of theoretical physicists of whom I know the capacities, which agglomerate on a subject. So what is the difficulty when we imitate the Anglo-Saxon system ? This is that in fact, when we make these requests for projects, etc., what will happen ? It is going to have a gregarious effect, that is to say that in fact, we are going to create feudalities. It ensues there will be a number of subjects that will develop at the expense of others, and for the reason that, ultimately, it is the people of these subjects who will be appointed in the appropriate commissions, and who will only recruit people from their own subjects. It happened in a completely obvious way in the United States, in mathematics. And in France, we escaped this defect, we really escaped this default, in Europe too, in general. And unfortunately, when we imitated, when we tried to imitate this Anglo-Saxon system with the ANR in particular, we fell in the panel, that is to say that this freedom that there was, this possibility if you want researchers to work on a topic that is truly original, and that does not correspond at all to one of these feudalities, has disappeared. And that, at the CNRS level, it's very, very disastrous for mathematics, in the sense that if you want I see what we need, for mathematics, I only talk about that subject. What we need is... I know a very large number of young and talented people, talented researchers who now spend their time writing research proposals, and so we know very well that in fact, what they write is because when we do research, what we will find, we cannot say it before, what we will find, we will search on a subject, then we will find something, something which did not correspond at all to what we said at the start. So they write, they spend their time writing this, they spend their time looking for a job, a year, on years, etc. instead of... In my day, I was given, I don't know, five years, five quiet years at the CNRS, I was not I was not permanent, not at all, I was a Trainee, after I was in charge, so at that time, it was before the 81s, when we established researchers, but we could take researchers, for a limited period, they were contractual, they did not hold for the rest of their lives. So we didn't have this infinite difficulty in choosing them, knowing that for the rest of their lives, they would continue to find, it was impossible. But on the other hand, we gave all these people the opportunity to realize themselves. Among them, there were some who couldn't, but hey, there were some who did. But if you will, it's a system that worked wonderfully better, than the current system in which we create these feudalities and these feudalities, what do they do, they only self-reproduce themselves and often in a sterile manner, after a moment.

Alexander Grothendieck, creator refugee in himself
Alain Connes

So, Alexander was a math giant, a French mathematician who died two years ago, in November 2014. And in fact, if you will, when I was asked to make a presentation, I gladly accepted, with the main motivation that of restoring a fragment of truth in front of a book which was written on Grothendieck, that I will not quote, by a non-mathematician, fascinated by the character, but whose the judgment on Grothendieck's writings, in particular on *Crops and sowing*, that he thinks he can summarize in one sentence, appeared to me as an insult to the memory of the great scientist.

I gave the title Alexander Grothendieck, creator refugee in himself. What I had in mind when giving this title, it was his journey, from his childhood of refugee, his prodigious creativity, both mathematical and literary, and then of this second half of his life, which led him in the past 25 years, to take refuge in himself in a small village in the Pyrenees, that of Lasserre, where he wrote thirty-five thousand pages.

The correspondence between Jean-Pierre Serre and Alexander Grothendieck, who was published in the form of a magnificent volume, shows how their ideas upset algebraic geometry. They bear witness of a deep friendship and of what was Bourbaki's spirit in those years. Unlimited dedication to beauty of math, completely free of all individualism. After an episode of discouragement due to the death of his mother in 1957, Grothendieck had a period of radiant creativity, which resulted in particular in the concept of topos. This notion was implicitly present in an article which, at the outset, was, in quotation marks, an "annoying draft" intended for Bourbaki and which, in fact, when it was published, made famous the newspaper in which it was published, at point that we designate the article simply by the name of Tohoku (the newspaper is called *Tohoku Maths Journal*). Were there already... so I'm going to talk to you a little bit of math, but it won't last very long. So there were already the categories of diagrams and that of sheaves of sets, but Grothendieck had not still released the new principle which makes it possible to include these two examples as special cases of the same concept, that of topos. So let's listen to him. actually, I will spend most of my talk quoting Grothendieck.

"The point of view and the language of the sheaves introduced by Leray led us to look at spaces and varieties of all kinds in a new light.

Lecture given at the Collège de France within the framework of the conference Migrations, refugees, exiles, October 13, 2016.
https://www.youtube.com/watch?v=FkBtSRyv6l4

However, he did not touch the very notion of space, contenting himself with making us understand these traditional spaces already familiar to all more clearly with new eyes. However, it turned out that this notion of space is inadequate for report on the most essential topological invariants that express the abstract algebraic varieties shape. For the expected marriages of the number and greatness, it was like a decidedly narrow bed where only one of the future spouses, namely the bride, could at the very least find nesting as well as wrong, never have both at the same time.

The new principle, which remained to be found, to consume marriage put by auspicious fairies, it was none other than this spacious bed that was lacking future spouses without anyone having noticed it before. This double places bed appeared as if by magic wand with the idea of the topos. This is the theme of the topos and not that of the diagrams which is this bed, or this deep river where come to marry geometry and algebra, topology and arithmetic, logic mathematics and category theory, the world of the continuous and the one of discontinuous or discrete structures.

If the theme of diagrams is like the heart of new geometry (it was an invention of Grothendieck too), the theme of the topos is the envelope or the abode. It is ... (therefore, it's always Grothendieck who speaks, in heard) *... what I designed larger to grasp with finesse, using the same language, rich in geometric resonances, a common essence to the most distant situations from each others, coming from such or such region, from the vast universe of mathematical things."*

So if you want, at that time, Grothendieck made an extraordinary discovery. He unveiled a concept of incomparable reach, both by wealth infinite spaces it allows to cover, but also and above all because in fact, if you want, it shows how ... how ... what is the true nature of a geometric space, which should be used simply as a parameter space for a variable set. And then, one of the great, one of the wonderful discoveries, precisely, of the notion of topos, is that when you work in a topos, it's exactly like if we worked in set theory, except that we can no longer apply the excluded middle rule.

We cannot say, we cannot reason by the absurd, but all intuitionist reasoning continues to work. *"So we have a wonderful example of a concept from pure mathematics, but whose scope, if only by its relations with logic, is no longer limited to this field of science."* For example, if you will, the notion of truth becomes a much more subtle notion in a topos.

And I mean, it's a notion that should be much better known than it is. So I come, I pass, now, so it's over for the mathematical part, I move on to Grothendieck's relationship with the world of mathematicians, which is in fact one of the main themes of *Crops and Sowing*. Let's hear it, at new. This is what Grothendieck says :

“It seems to me that the time has come to express myself about my relationship with the world of mathematicians. This is a very different thing from my relationship to mathematics. It existed and was strong from a young age, long before I even doubt that there existed a world and an environment of mathematicians, a whole complex world, with its learned societies, its periodicals, its meetings, colloquia, congresses, its prima donna and its hardworkers, its power structure, its gray eminences and the no less gray mass of cutters and laborers, in need of theses or articles. And of those also rarer, which are rich in means and in ideas and come up against closed doors, desperate to find the support of one of these powerful men, in a hurry and feared, and who have this magic power : have an article published. I discovered the existence of a mathematical world when landing in Paris in 1948 (Grothendieck was born in 1928) *at the age of 20, with in my meager suitcase, a license in science of the University of Montpellier and a manuscript with tight lines, written on both sides borderless, the paper was expensive, representing three years of solitary reflection on this which, I learned later, was then well known as the theory of measurement or the integral of Lebesgue.*

I had juggled with the sets that I called measurable, without having met besides of sets which are not, and with convergence almost everywhere, but I did not know what a topological space was. I hadn’t heard of it yet, in a mathematical context at least, strange or barbaric words, like group, body, ring, module, complex, homology and so on, which suddenly, without crying out, surging over me all at the same time. The shock was severe. If I survived to this shock and continued to do math and even make it my job, is that in these distant times, the mathematical world hardly looked like what it is since then.

It is also possible that I had the chance to land in a more welcoming corner than another in this unsuspected world. I had a vague recommendation from one of my teachers at the faculty of Montpellier, who had been a student of Cartan. Like Élie Cartan was then already offside, his son, Henri Cartan, was the first congener that I had the time to meet. I had no idea, then, how happy it was augurs well. I was greeted by him with this kindly courtesy which distinguished him, well known from the generations of normaliens who had this chance to do their very first weapons with him.

He must not have realized, moreover, the whole extent of my ignorance, judging by the advice he gave me then to guide my studies. What in any case, his kindness, obviously, was addressed to the person, not to the luggage or to possible donations, or to a reputation or notoriety. In the year that followed, I was the host of a Cartan seminar at the school to which I clung firmly. I also assisted to the Cartan seminar, as a dumbfounded witness to the discussions between him and Serre, with big hits from spectral sequences... Grothendieck writes in parentheses (Brrrrr !) *... and drawings called diagrams, full of arrows covering the whole painting. It was the heroic epoch of the sheaves theory and a whole arsenal whose meaning was totally eluding me, while I* *somehow forced me to swallow up the definitions given, and to check the demonstrations. On Bourbaki seminar days, bringing together a small twenty participants and listeners, we saw disembark, like a group of friends a little noisy, the members of this famous Bourbaki gang.*

They were all familiar, spoke the same language which almost escaped me, also smoked a lot and laughed happily. Only were missing the boxes of beer to complete the ambiance. It was replaced by chalk and sponge. At the time, I went to see Mr. Leray at the Collège de France to ask him, if I remember, what his course would cover. I don't remember either the explanations could give me, nor if I understood anything there, only that there too, I felt a warm welcome, addressing the first stranger to come. That's it, and nothing else, surely, that made me go to this course and hold on to it bravely, as at the Cartan seminar, when the meaning of what Leray exposed there almost completely escaped me. The strange thing is that in this world I was in newcomer and whose language I hardly understood and spoken even less, I did not feel like a stranger when I had little opportunity to speak and to cause, with one of these happy guys, I still felt accepted, I would even say almost one of their own. I don't remember a single occasion when I was treated with condescension by one of these men, nor occasion where my thirst for knowledge, and later, again, my joy of discovering, was rejected by a self-importance or by a disdain. If it had not been so, I would not have become a mathematician, as they say, I would have chosen another profession where I could give my measure without having to face contempt.

Whereas objectively, I was a stranger to this world, just as I was a stranger in France, a bond, however, united me to these men from another background, another culture, another destiny, a common passion. I doubt that in this year crucial where I discovered the world of mathematicians, one of them perceived in me the same passion that inhabited them. For them, I had to be one among a mass course and seminar listeners, taking notes and obviously not well in the shot.

If perhaps, I distinguished myself in some way from the other listeners, it is that I was not afraid to ask questions, which most often had to denote above all my phenomenal ignorance, both of language and of mathematical things. The responses could be brief, even astonished. Never the amazed howl I was so did not meet with a rebuff, a rebate in my place, nor in the middle without the Bourbaki group's manners, nor in the more austere context of Leray's lessons at the Collège de France.

During these years, since I had landed in Paris with a letter for Élie Cartan in my pocket, I never felt like I was in front of a clan, of a closed, even hostile world. If I have known, well known this contraction inner face of contempt, it's not in this world, not at that time, at least. Respect for people was part of the air I breathed. There was not to deserve respect, to prove yourself before being accepted and treated *with some amenity. Strange thing, maybe, it was enough to be a person, to have a human face."*

So Grothendieck continues. You should know that Grothendieck left deliberately-the mathematical world around 1970. This is what he called the great turning point. *"It was only after the great turning point of 1970, the first awakening should I say, that I realized that this cozy and friendly microcosm represented only a very small portion of the mathematical world and that the traits I liked to lend to this world, which I continued to ignore, which I had never thought as presenting an interest for me, were fictional traits.*

During these 22 years, so between 48 when he arrived in Paris and 70, this microcosm itself had changed its face in a surrounding world which also changed. Me too, surely, over the years and without suspecting it, I had changed like the world around me. I don't know if my friends and colleagues were more aware than I of this change in the surrounding world, in their own microcosm, and in themselves. I cannot say either how it did that this strange change probably came insidiously, discreetly.

The notorious man was feared, myself I was feared, if not by my pupils or my friends, or by those who knew me personally, at least by those who only knew me through notoriety and who did not feel themselves protected by comparable notoriety. I learned about the fear in the mathematical world that the day after I wake up, there are almost fifteen years." (When he wrote *Crops and sowing* and the meaning of *Crops and sowing* is exactly that : he reaped what he sowed. It was in 85, fifteen years later.) *"During the fifteen years that had preceded, gradually and without suspecting it,* (That was before 70.) *I had entered the role of the big boss in the Who is who mathematical world. Without suspecting it too, I was a prisoner of this role which isolated me from all except a few peers and a few students. It is only once that I came out of this role that at least part of the fear that surrounded it felt, tongues untied, which had been mute before me for years. The testimony they brought to me was not only that of fear, it was also that of contempt, especially the contempt of people in place towards others, a contempt that arouses and fuels fear. I had little experience of fear, but that of contempt, in times when the person and the life of a person do not weighed not heavy. He no longer had to forget the time of contempt and lo and behold, it remembered itself to my memory. Maybe he never stopped, when I was satisfied just to change the world, as it seemed to me, to look elsewhere, or just pretending to see nothing, hear nothing, apart from passion-endless mathematical discussions. In those days, I finally accepted to learn that contempt was rampant all around me, in the world I had chosen as mine, with whom I identified myself, who had my deposit and who had pampered me."*

So if you want, that's a summary of what's said in the main topic of *Crops and* *sowing*, of course, which is Grothendieck's relationship to the mathematic. I pass to an absolutely essential text, another text by Grothendieck called *The key of dreams*. And when I prepared this talk, I was rereading *The Key of dreams* and I noticed one thing. I understood in fact that without knowing it and without wanting it, I had, by giving my title, left ajar the possibility of a completely different interpretation which actually touches the heart of the book which is *The Key of Dreams* and where the word creator seems in a sense that I let you guess, as I read his testimony. And I'm going to read you Grothendieck's testimony, we've heard a lot about Grothendieck's childhood, etc. But of course, it's much better to hear what he has to say about it himself.

I'm going to read you Grothendieck's testimony about his childhood, which is in *The key of dreams*. *"I lived the first five years of my life with my parents and with my sister in Berlin.* It's Grothendieck who is speaking, of course. *"My parents were atheists. For them, religions were archaic survivals and churches and other religious institutions, instruments of exploitation and of domination of men. Religion and Church were destined to be swept away without return by the world revolution that would end social inequality and all forms of cruelty and injustice, and would ensure the free development of all men.*

However, since my parents were both from religious families, this gave them a certain tolerance for religious beliefs and practices with others or towards people of religion. They were for them people like the others, but who happened to have that kind of anachronism, as others also had theirs. My father was from a Pious Jewish family in a small town in Ukraine, Novo Zubkov. He even had a grandfather who was a rabbi.

Religion, however, should not have had much control over him, even in his childhood. Very early on, he felt solidarity with peasants and ordinary people, more than with his middle class family. At the age of 14, he took off to join an anarchist group who crisscrossed the country preaching revolution, sharing land and property, and the freedom of men. Enough to make a general heart beat red and bold. It was in Czarist Russia in 1904. And until the end of his life again, and upside down, he saw himself as … his name was Sacha Piotr, that was his name in the movement … anarchist and revolutionary whose mission was to prepare the world revolution for the emancipation of all peoples. For two years he shares the hectic life of the group he had joined then surrounded by … (so, it was in 1906) *… surrounded by the police, and after a hard fight, he was took prisoner with all his comrades. All are condemned to death and all, except him, are executed. For three weeks he waits day after day to be taken to the peloton.*

He was finally pardoned because of his young age and his sentence commuted to that of life sentence. He stays in prison for eleven years, from the age of 16 to the *age of 27 years old, with hectic episodes of escapes, revolts, hunger strikes. He was liberated by the revolution in 1917, then participated very actively in the revolution, especially in Ukraine, where he fought at the head of an autonomous group of well-armed anarchist fighters in contact with Makhno, the head of the Ukrainian army of peasants. Sentenced to death by the Bolsheviks and after their stranglehold on the country, he left the country clandestinely in 1921 to land first in Paris, just like Makhno. During the past four years of intense militant and fighting activity, he also has a fairly tumultuous love life from which a child came, my half brother Dodek."* (It's of course Grothendieck who is speaking.) *"In emigration, first in Paris, then in Berlin, then again in France, he makes a living as best as he can, as a traveling photographer who assures him his material independence. In 1924, on the occasion of a trip to Berlin, he met there the one who was to become my mother. Love at first sight on both sides. They remained indissolubly attached to each other, for better and especially for worse, living in common-law until the death of my father in 1942, in deportation to Auschwitz.*

I am the only child resulting from this union in 1928. My sister of 4 years my elder, was from a previous marriage. My mother was born in 1900 in Hamburg, thousand well-off Protestant who had experienced an inexorable social decline throughout of her childhood and adolescence. Like my father, she had an exceptionally strong personality. She begins to emerge from the moral authority of her parents at the age of 14. At 17, she goes through a religious crisis and emerges from the naive and problem-free faith of his childhood, which gave her no answer to questions from her own life and the spectacle of the world.

She spoke to me of it as a painful and necessary wrenching. As well my mother and my father had remarkable literary gifts. My father even had an imperative vocation there, which he felt inseparable from his revolutionary commitment. From the few fragments he left, I have no doubt that he had the makings of the great writer." In fact, if you think about it, you will see that Grothendieck realized what his father had not had time to do, that is to say this writing. *"And for many years after the abrupt end of a huge epic, he carried within him the work to be accomplished, a fresco rich in faith and hope and pain and laughter and tears and shed blood, thick and vast like his own untamed life ; and alive like a song of freedom.*

It was up to him to embody this work which was becoming dense and heavy, and which grew and demanded to be born. It would be his voice, his message, what he had to say to men, what no one else knew and could not say. If he had been faithful to himself, that child who wanted to be born would not have asked for it in vain. While he scattered to the four winds, he knew it well deep down and that if he left his life and his force to be nibbled by the petty things in the lives of migrants is that it was connivance. And my mother also had blessed gifts that predestined her for great things. But they chose to neutralize each other in a past confrontation endless, both selling *their birthright for the satisfaction of a flamboyant married life with great love, superhuman dimensions and neither one nor the other, until their death, will take care to update the nature and the true ones springs. After the advent of Hitler in 1933, my parents emigrated to France, land of asylum and freedom for a few more years, leaving my sister on one side in Berlin, me on the other in Blankenese near Hamburg"* (So Grothendieck spent six years of his early childhood alone, without his parents.) *"and without much regard to their cumbersome offspring until 1939. I joined them in Paris in 1939, the situation for me, in Nazi Germany becoming more and more dangerous, a few months before the outbreak of the world war, it was time. We are interned as unwanted strangers, father in winter 1939, my mother with me in early 1940.*

I stayed two years in the concentration camp, then I was welcomed in 1942 in a Swiss children's home in Chambon-sur-Lignon, in the protestant Cévennes region, where many Jews are hiding, hunted like us by the deportation. The same year, my father was deported from the Vernet camp to an unknown destination. It is years later that my mother and I will have official notification of his death at Auschwitz. My mother stayed at the camp until January 1944.

She died in December 1957 from contracted pulmonary tuberculosis at the camp." So, I think it's better that I skip a little passage. I will come back afterwards, possibly because I want to read you, probably the most important texts that I collected in *The key of dreams*, and which, I hope, will give you the meaning, the second meaning of the title. Again, Grothendieck speaks, and he will tell an episode that happened to his father.

So that's what Grothendieck is telling us. *"In recent months, a such density by the action of God in me, I sometimes thought of an event in my father's life, which took place long before I was born and which I had rarely had the opportunity to think to. They never spoke to me about it elsewhere, nor to a soul who long live, moreover, except to my mother, in the weeks of tumultuous passion which have followed their meeting in 1924. She was the one who told me about it and years after his death. This is an experience he had in prison, in his eighth year in captivity, therefore around the year 1914. It was after a year of solitary confinement that he had worth an escape attempt, during a transfer from one prison to another. It was surely the hardest year of his life and which would have destroyed or broken or extinguished more of one, total loneliness, with nothing to read, write or occupy, in an isolated cell in the middle of a deserted floor, cut off even from the sounds of the living, except the immutable and haunting daily scenario, three times a day, the brief appearance of the guardian wearing the pittance and in the evening, a flash appearance from the director coming in person inspect the hard head of the prison. Every day stretched, like a purgatory without end. There were 365 to pass, before he was again attached to the world of alive, with books, a pencil. He counted them on those days, those eternities he had to cross, but at the end of the 365th, he barely could grasp that it was indeed the end of* *his endless ordeal.*

And for the next three days, nothing. At the end of the third, to his deasks "The year has passed, now ... When will I have books ?", a terse "Wait" from the director. Three days later, the same ! We were playing with him, who was delivered to their mercy. But the revolt brooded, ulcerated, in the man pushed to the limit. The next day, hardly pronounced the same impassive answer "Wait", the heavy spittoon in copper with sharp edges almost smashed the head of the imprudent tormentor.

Throwing himself to one side just in time, he felt the breath in his temples, before the projectile crashes on the opposite wall of the corridor and it hurriedly rejects behind him the heavy clad door. It is a miracle for me that my father was not hanged on the spot. Perhaps a scruples of conscience from the director who "feared God" and who felt confusedly by the very death that had grazed him so closely, that he had gone too far. Still, the young rebel is beaten like plaster, it was the slightest of things, then thrown into irons, into a stinking dungeon, into the dark total, for an indefinite period. Every third day, we open the shutters, and during the day relay the moist night. However, the revolt is not broken. Total hunger strike, without eating or drinking, despite the young body that stubbornly wants to live, the ulcerated soul, eaten away by the impossible revolt and humiliation of helplessness, and swollen flesh protruding into glass beads around the iron rings on the wrists and ankles.

It was the days when he reached the bottom of human misery, aware of it, even, that of the body, that of the soul. It's at the end of the sixth dungeon day, day with open shutters that took place the incredible thing, which was the most precious secret and the best kept in his life, in the ten years that followed. It was a sudden wave of light of unspeakable intensity, in two successive movements, which fills its cell and penetrates and fills it, like deep water, which soothes and erases all pain and like a burning fire burning with love, boundless love for all the living, all distinctions of friends and enemies swept away, erased.

I don't remember that my mother had a ready-made name to name this someone else's experience. I will now call it an illumination, exceptional and fleeting condition, close to what the testimonies of certain sacred texts and many mystics. But this experience takes place here outside of any context that is commonly called religious. It was over ten years surely that my father had detached himself from the grip of a religion in order not to never come back to it.

It is safe for me, even without having precise details on this subject, that this event has profoundly transformed his perception of things and his whole inner attitude in the days and weeks at least that followed. Days of very hard trials, surely. But I have good reason to believe that neither then nor later did he attempts to locate what had *happened to him, in his vision of the world and of himself. It was not for him the beginning of a deep and long inner work breath, which would have made fruitful and multiply the extraordinary gift that had been given to him done and entrusted. He must have reserved a very separate hut for her, like a jewel that we hold in a closed box, taking care not to put it in contact with the rest of his life. However, I have no doubt that this incredible grace, which had changed in an instant the excess of misery in an unspeakable splendor, was intended not to be kept thus locked up, but to irrigate and fertilize his whole subsequent life.*

It was an extraordinary chance which was offered to him, and which he did not seize. A bread which he ate only once in the mouth and to which he no longer touched. Ten years later, the way he opened up to my mother, in the intoxication of his first love with a woman who was going to tie him, feet and fists, it was good like a jewel unusual and very precious, which he would have given her the first. And when she has me spoken, more than twenty years later, I knew that she had really appreciated, and appreciated again, this tribute thrown at her feet, and which she had greeted with emurgently and as a brilliant testimony of total communion with adored man, and of an intimacy which has nothing more to seal. And myself, hearing it, young man of 17 or 18, read it with eagerness moved quite similar. I also saw the jewel that also makes the shine for me again and again from this prestigious and incomparable hero father, at the same time as that of my mother who, alone among all the mortals, had been judged worthy to take part in it. So the bread given by God as inexhaustible nourishment of a soul, which, perhaps, would grow and would nourish other souls still, ended up becoming a family adornment, coming enhance the splendor of an expensive myth and fuel a common vanity."

Here. I'm done, thank you.

Imagination and infinity
Alain Connes and Alain Prochiantz

Imaginations, a series of interviews proposed by Alain Prochiantz, neurobiologist, professor at the Collège de France.

Alain Prochiantz : We are in the summer grid, on the program Imaginations , with philosophers, sociologists, and scholars, and artists, on the probably most unifying theme, between scientists and artists. It must be said that this is how we thought in any case thing. And today I have the pleasure of welcoming Alain Connes. So, Alain Connes is a very great mathematician. He is a professor at the College of France, where he held the Analysis and Geometry chair. He is a recipient of the greatest reward you can have in math, which is the Fields medal. And he has this interest not only in mathematics, of course, but also for music and for art, which makes it really one of the people who can make the very strong link today between art and science on a mode which is not a flat mode but which is a mode which intellectually engages those who make art or those who make science, that is to say, a real reflection on the subject of art and on the subject of science. He is a specialist in what is called non-commutative geometry and if I say that, it's probably because it's not unrelated to his interest in time, and through interest in time, interest in musical creation.

Alain, I have the duty to try to extract from you, not everything because it's inexhaustible, but in any case elements for reflection on this question of science, math, beauty in math, and its connection with artistic beauty.

Alain Connes : Yes. In fact, therefore, I thought a little bit about what is simply the imagination. And the first thing that struck me was that finally, radio, as a means of communication, is a means which is much more interesting, in terms of the listener's concentration, level of listening precisely, that another means of communication like television. Why ? Because in the term

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imagination, there are the pictures, and it is extremely important that the listener does not have a purely passive role, does not receive the image as we want to impose it, but be able to create it himself, and create it from speech, from language. So this is the first thing that struck me, it was how much more appropriate a radio shows, to speak of imagination, only if we tried to illustrate it directly. The second thing that also struck me a lot is how the mathematicians have a use of the imagination which a priori is very different, very special, very different from what happens in other areas, that's just that I want to try to explain. So what I mean is that a mathematician uses imagination a lot, but he uses it in a very special way. That is to say, in fact, the role, the first role of the imagination for a mathematician, which is an essential role, is that of creating mental images.

And this role, in fact, of course, is absolutely not something passive. It's a... how to say... we can't do it only when you dry on a problem. So there is an essential virtue... for a mathematician, for example, if he is to read a book to take for example a theorem which is in a book etc. and above all, he must not to watch the demonstration, but to try to demonstrate himself. Why ? Because when he does that, in fact, he will create in his brain, I say a mental image but in fact, it's... well, it is not always something geometric, it is not always something that can be described as an image but, it's a certain assembly in his brain which then will do that when he is confronted to a page of formulas, well, that page of formulas will speak to him. And in this page of formulas, he will see actors. He will see things that will "resonate" with each other, etc. The comparison that I always want to take, is that suppose for example that you are in the metro, and that you see a subway passenger who is in the process of read a music score. When you're not a musician, this score of music tells us nothing, nothing at all. When you're not a mathematician and you see a passenger reading math formulas, you get the impression I mean that we are completely excluded and that we have no chance to understand. And in fact, the reason is precisely this fabrication of mental images and this fabrication of mental images, it involves this ability absolutely essential to imagine. This ability to imagine, there, plays an absolutely fundamental role. And I always give the following advice, I mean, for example, to a young mathematician : the advice, it is if one is confronted for example with a calculation even very difficult, of course, these are abstract calculations etc., the right method is not to rush to the computer to try to do the math, or take a spreadsheet, no ! The right method is to go for a ride on foot and try to get by with what you have, for precisely, by there reflecting, creating in the brain these mental images, and whatever complexity of the problem.

Whatever the impossible character at the start of this creation, it is in drying, and rightly, by gradually appropriating mental objects that will correspond to the problem that we will progress. So there is an active part which is absolutely essential and from my point of view, that is really the first, the fundamental role of imagination in mathematics. In this sense, it is very different from other areas, because, of course when we do physics, and when we talk about the universe, well, everyone has a mental image at the outset, of what the universe is, therefore, we are not going to need to create something out of nothing, when in math, really, we are faced with this thing. So from my point of view, there is this essential point, which is that one must be active, and one is all the more active that one is not given an already pre-established model. And if for example we were on television, we would try to illustrate concepts of mathematics by pictures but, every mathematician, every person, is a particular case and will create in his brain, a very particular image, a very very particular assembly, and it is impossible to give a generic shape of that.

Alain Prochiantz : But when you talk about mental images, I think that at some level, even outside of math, there is a moment where it is necessary to have recourse to this kind of fabrication of mental images...

Alain Connes : Absolutely.

Alain Prochiantz : of bricks that we handle...

Alain Connes : ... which fit together...

Alain Prochiantz : which interlock or which do not interlock and that, it is perhaps because precisely, mathematical thought is the only way to think, outside the formulas. It's not just formulas, it is a way of thinking which is a natural language for the scientist of a some type. So the question I wanted to ask you, to shed some little light, for me moreover, but also perhaps for those who listen to us, that is, I would say "What is the image of these mental images ? What does it look like a mental picture ?"

Alain Connes : Okay, so first, there are the most simple, of course. That is to say that if we talk about ordinary geometry, for example, plane geometry, there is a theorem that I really like, it's Morley's theorem. So there, the listener must first try to imagine a triangle. So each listener will imagine a different triangle, but whatever, okay. So we start from a triangle. And what does the theorem of Morley, what does Morley's theorem say ? Is that we cut each angle of the triangle in three equal parts, therefore in three equal angles. And we intersect the corresponding lines. We get a triangle inside, in general, triangle from which we started, and the wonder is that the triangle we get at the interior is always, always, an equilateral triangle. So this is mernight light ! This is called Morley's theorem. And when we stated this theorem, we have a mental image, of course. And in fact, the mental image that we have is much better than if we were drawing a triangle on paper, and that we have drawn the equilateral triangle in the middle. Why ? Because we would be automatically disturbed by the grain of the paper, by the pencil with which we wrote, etc. So there, there is already an abstraction that has occurred. But... there is an essential point precisely in mathematics. The essential point is that I was able to explain to you that a mental image was very simple, in the case of plane geometry. But in mathematics, we manipulates geometries which are much more complicated than that, and which in general, are geometries that have dimension much larger than 3, and even in general, of infinite dimension.

That is to say, we understood, thanks to physics for example, that the quantum mechanics, it's mechanics that occurs in a space that is called the Hilbert space, which is a space of infinite dimension. So, how is it that the mathematician can have access to this space, from infinite dimension ? This is the question, this is the essential question. And this essential question, in fact, it has a very, very, very fundamental answer. This answer is the duality between geometry and algebra. So to explain it, I'll take an example. I will take a simpler example than Morley's theorem. Suppose for example that we are struck by the fact that the medians of a triangle meet at one point. So it's okay when we do it in plane geometry. We see well what it is. We have an analogous statement when we go into geometry in the space. But what about when we look at geometry in arbitrary dimension ? It seems impossible because how are we going to represent a space of dimension 4, an object corresponding to a triangle in 4 dimensional space etc. or in a larger dimension :

the answer is wonderfully simple : the way of understanding in algebraic language, by a formula, why the medians of a triangle meet, is just to write the coordinates of what's called the barycenter of the 3 points. That is to say that we take the coordinates of the points. And then we add them up and divide it by 3, because we are in dimension two ; if we were in dimension 3, we would divide by 4 and so on. So what, the wonderful thing is that when you have algebra, it's a bit like the two hemispheres of the brain, i.e. there is the right hemisphere and the left hemisphere, they communicate with each other. It is the right hemisphere which has themental image of the triangle, as I explained to you at the beginning. That is to say, it sees it, sees Morley's triangle, etc. And then after, we communicate.

We communicate with the left hemisphere. In the left hemisphere, there is a formula. It's a formula that allows... And this formula is completely insensitive to dimension. That is, once we have it written in dimension 2, it will exist in dimension 3, in dimension 4, and even in infinite dimensions. So there is a wonder that is happening, which is that in mathematics, we are able to climb, precisely, because it is something that strengthens the mental image, that helps give it security, and that's the formula. And once we have this formula, afterwards, we will operate in another mode. And this mode is no longer the visual mode, and that's why the notion of mental image is reductive, because it reduces everything to a geometric vision. Now, in mathematics, there is a duality, between geometry and algebra. Which means on the one hand, we have this geometrical vision and there, in general, the geometrical vision, it is something that will impose itself immediately... That is to say, when you have a figure, that figure will speak to you, but it will speak to you right away.

While algebra is something entirely else. And it's something which will evolve over time, and this is something in which the calculations will be algebraically, and I admit that I, for example, am persecuted. The last night I woke up, I said to myself "in this formula, I was not mistaken ?". Why ? Because my brain keeps on function...

And it continues to do the calculations etc. And this is something that takes place over time, and which is not at all of the same nature, as a static mental image, a geometric image, which exists and is frozen once and for all and that we understand immediately.

Alain Prochiantz : Mental images are not necessarily static ticks. I imagine that we move them, we turn them over, we combine them.

Alain Connes : We can move them, we can return them. There is the reversal of the sphere for example.

Alain Prochiantz : But probably not just anyhow. Is there a grammar of the combination of mental images ? What is permitted in the manipulation of mental objects, of these images, and that makes it not just anything, there is a kind of grammar behind.

Alain Connes : There is of course a grammar behind it. I think that, arguably one of the most important facets of grammar is the power of analogy. And then metaphor. But I think that analogy is something extraordinarily powerful, and which for the moment, is completely inaccessible to processes like the Machine Learning, Artificial Intelligence, etc. So it's an extraordinary tool...

Alain Prochiantz : For you, it's intuition, analogy ?

Alain Connes : No, it's more than that. In other words, what happens, precisely, through mental images, is that at some given point, the brain realizes that two mental images that would appear extremely far from each other (this was what had happened to Poincaré when he got on the bus), mental images far apart of the others, I mean, he was talking about two completely different things, in fact he realized at one point that there were extraordinary similarities between the two. And the fact that there were these similarities between the two, he was able to develop an analogy afterwards between two domains which a priori have nothing to do with each other.

And so, an analogy is something that is extremely delicate to manipulate, because it's not a simple dictionary. If it was a simple dictionary, that would be razor-sharp, that is to say if one could say "such thing corresponds lay on such other thing etc., etc.". It's a kind of... There is a Japanese mathematician, whose name is Oka, who had described this wonderfully, it is a kind of transplant, a kind of... We have a little flower and very lawfully, we are trying to transplant it to another place, and why this is something incredibly fruitful and effective, it's because in general, precisely, the things that we understand on the one hand, we don't understand on the other and vice versa. So that means that we will be able to transplant the understanding that we have on one side, and see how, when we try, we make tests, we see etc. but above all it is not necessary, at that time of development, we must not try to be too rigorous, because if we are too rigorous, everything will collapse. And it is a moment, precisely, which has a poetic and artistic aspect. Why ? Because when we transplant little flowers, when we do this, if we try to be too smart, too fast, etc., we will say "well, it will not work, but it will not work for such reason etc." And then we give up, and we ruined everything.

Alain Prochiantz : It is a kind of correspondence, in fact.

Alain Connes : It's a kind of correspondence, of analogy. And we cannot try to codify it too precisely at the moment when it is find out. It is something extremely fragile and this fragility makes that, for example, when we discover it, we try to say it, we try, you have to, you have to know that mathematicians are very hard people, that is to say, in mathematics, the dream is excluded. I will come back to that. But if we have perceived an analogy between two subjects, and if we try in a way too fast, too premature, to say it, it will be destroyed. So there is a part of developing a new theory like that, in which we must protect themselves, we must protect themselves, it's like a little child who must be protected, etc. and only after a while, when it has done proofs, when it has grown enough, there we can reveal it.

Alain Prochiantz : You have to let the analogy mature, the correspondance mature, to make it strong enough, to face the test of truth.

Alain Connes : So the test of truth is something absolutely terrible. So what you need to know, when I was talking about the imagination in mathematics, naively, you would think that, imagination in mathematic is to imagine things and then try to demonstrate them, etc. But in fact, there is a yoke in mathematics, which is absolutely terrible, and which, I think, is largely similar to that of physics, but of a very different way. That is to say, in mathematics, what happens is that we can have imagination, we can imagine a new theory etc. But, the problem is that, very quickly, we will come up against a reality, which is mathematical reality, and this mathematical reality, it is terrible, in the sense that, I mean, if we don't have all the elements of a demonstration, if we don't have for example, I mean, the possibility to check things out on a computer etc. we actually realize that the freedom we enjoy is absolutely minimal. So that's why I have insisted that the role of imagination in mathematics is not to imagine new things, etc. Not at all. Is to create an image mental inside the brain. There, it is really useful. It's something essential. Against that, after, there is such a straitjacket in terms of imagination, compared to other subjects, I think of artists, I think of novelists etc., this yoke is so hard, so constraining, that in fact it prevents, it just annihilates any possibility of freedom.

Alain Prochiantz : Very good. We may be going over a first variation on a German national tune by Chopin, interpreted by Nikita Magaloff, on which you will give us a little comment.

Alain Connes : Exactly, of course, yes yes.

(Musical interlude) : On a German national tune

Alain Prochiantz : Can you tell us why you chose this track ?

Alain Connes : So, then, why did I choose this tune, these variations. Of course, for the interpretation of Nikita Magaloff, whom I like a lot. But, in fact, my reason is a very personal reason, and which has to see, how to say, with the structuring of the child's imagination. What I'm going to say, it's something very personal, so, but whatever, I think it's something generic. In fact, my two mother's grandparents come from Constantine in Algeria. They were from this city. And my childhood was lulled by the fact that my maternal grandmother was a pianist. And she was an orphan, she had become an orphan at the age of 6. Both of his parents were dead. She was in Algeria, she had been taken in a convent, and she brought me back often told stories when I was a kid, when I was very little, stories from his father. And her father, when she was very young, had offered her a piano, and he had played, on the piano, the part of Chopin's variations which is so beautiful, not the very beginning, but the major theme and that is introduced by Nikita Magaloff in an incredible way, because it's a song that might interpret it as a piece of technique but not at all, he understood exactly how much the theme exhibit should be preceded by a slowdown, etc., and how beautiful the theme is.

And in fact, my childhood was rocked by this air, which I had a lot hard to find afterwards, when I wrote the family genealogy, and so actually what I meant was I think the imaginary of a child is structured very early especially by music, and by this time, the imaginary, in the naive sense, of what the child can imagine when he hears stories like that. So I never went to Constantine. Indeed, I never went to Algeria, but I always had in my head, a extremely interesting picture, precisely, of this moment when the father of my grandmother who was a doctor, in fact, had given him this little piano. And the role it played...

Alain Prochiantz : And does that have to do with the way of thinking as a mathematician ?

Alain Connes : Well, let's say it's very well known, each mathematician is different, so I don't want to generalize. But in fact, it is very, very accepted, that in general, mathematicians are very interested in the music, failing that, necessarily, being musicians, since we don't have much of time when you're a mathematician, so if you want to practice an instrument is something that would take too long. But in general, they are very sensitive to music and it's true, it's true, and I continue what I said earlier, it's true that there is a very strong analogy between algebra and music, by the course in time...

I always explain of course the fact that language itself, language that we use all the time, is non-commutative since we cannot commutate the letters between them, unless you make anagrams but... so there is a very strong relationship actually between music and algebra, I think.

Alain Prochiantz : And therefore temporality...

Alain Connes : And temporality, of course, of course.

Alain Prochiantz : You can explain this question to us a little bit temporality and a little bit non-commutative geometry, how does that get in there ?

Alain Connes : Yes, of course. So let's say we wrote two books, with Dany Chéreau, who is my wife, and then Jacques Dixmier, who was my thesis teacher. And precisely, in these two books, we continued this theme which is an essential theme in what I have done, in my life of scientist, and that started with the discovery that when you do algebra, but in a non-commutative way, that is to say that we do not allow ourselves to swap letters between them. So of course, this is something that is essential because that's what Heisenberg found when he discovered quantum mechanics. When he discovered quantum mechanics, he understood that when deals with very small systems, microscopic systems, in fact, contra what you do when you do classical physics, where you write $e = mc^2$ or $e = c^2m$, it's the same thing. When working with a system microscopic theme, we no longer have the right to swap letters, it's extraordinary, he made a fantastic discovery. And besides, he made this discovered at 4 a.m., when he was isolated on the island of Heligoland and, what's wonderful actually, is how far discoverers get to convey, in their writings, Heisenberg did so in his memoirs, the former the traumatic vision he had when he made this discovery. And he said it was a vision that was scary, because of the fact that it was the first to see this, in fact, he had before him a landscape that was almost the whole landscape, and that was scary. And he describes it wonderfully. And he is not the only one to have been able precisely, by being the first discoverer, to transmit that.

So on my side, therefore, what I had perceived if you want, is that, in fact, when we do non-commutative geometry, automatically, it's a certain type of algebra that was discovered by Von Neumann, automatically, algebra itself secretes its own time so does that non-commutative algebra secretes time, the passage of time and this is something absolutely overwhelming in a way, and for years and years, I had been fascinated by this fact. For sure, mathematically, it has a lot of consequences because for example, algebra has periods, etc., but I had always been unable to have an idea of how this, if you will, could find its place in physics and so this is the subject of our first book, which is called The quantum theater, which is published by Odile Jacob and in which, precisely, we tried, me and my two co-authors, to transmit this find, but so that it can be seen by the public.

It's difficult, it's difficult. And in the second book then, *Le Spectre d'Atacama*, we went much further in the sense that there, we tried to precisely transmit this link between forms and music, which is a link also extremely important, and that says that when we try for example to explain where we are in space, well actually, we are faced with a mathematical problem which is not at all trivial, which is not at all obvious which is the problem of being able to give a space or a form, more generally, invariantly. And what math teaches us is that if we want to give a form invariant, the first thing a form gives us is a musical range, it seems something quite surprising therefore : a form gives us a musical range called a spectrum. And after, of course, there is a whole development from there. The second book is called *Le Spectre d'Atacama* because precisely what is happening, and which is quite astonishing, is that we can calculate the spectrum of ordinary forms. So if we look at a space as a drum, it's Marc Kac who had long posed the problem if you want : can we hear the shape of a drum ?

That is to say that a drum has vibrations, when you hit it, and one does not believe that you will always get the same sound ; that's what do know people who play drums for example ; so we have very different sounds but these sounds form a range, and an obvious question is "Can we recognize the shape of the drum from the scale ?".

So this is a question that has a mathematical answer but the thing is really amazing, when we can calculate the range of a geometrical object, of a geometric shape that we know, there are spectra, so I identify the range if you want with the frequency range. And those frequences, we will represent them by spectral lines. There is a problem which arises in an absolutely insolent way. In fact, there are spectra, which appear completely naturally, and here we have really a considerable difficulty, but in some cases, we get there, to find the form, the physical form, whose spectrum is the spectrum. And then there is an example that we explain and that will justify the second piece of music which I will talk about later... This is an example that we explain in large detail in the book, this is called the spectrum of the guitar.

So I'm going to try to explain it, but again, the auditor is preparing to construct mental images himself in what I will explain. So the first mental image is to imagine a guitar. Well. You have a guitar. You may have seen guitars, so you can imagine in your head what a guitar is, I don't need to show it to you. So if you look at a guitar, you'll see on the neck of the guitar, lines that are perpendicular to the neck, and which are called frets. So if you look closely at these frets, you'll see... Look at them in your head. You will see that at the start, there are no frets, there is a kind of hole, good, which will allow resonances. And then there, the frets begin. And they are not at all evenly spaced. You would have thought that simply, that when we look at the neck of the guitar, if you want, the frets will be equally spaced. In fact, they are not at all equally spaced. And the mathematician, when he sees the spacing of the frets, asks himself the question “but why did we not space the frets of equal intervals ?”. So the answer is a mathematical answer, but it’s an answer which is wonderful, because it will give us a spectrum. And this spectrum, afterwards, we will have to look for the form of which the spectrum is. So first, what is this spectrum ? Well, when we make music, we notice a very important thing, which is that the ear is not all sensitive to 1 2 3 4 5, etc. she is not sensitive to add up, she is sensitive in fact to multiply a frequency by something, that is to say if we take a frequency and multiply it by 2, that corresponds to the stepwise to the octave. The ear is sensitive to the passage to the octave, it feels a correspondence between the two frequencies, it feels a harmony between the two frequencies. It is the multiplication by 2. The ear is also sensitive to multiplication by 3 : when we take a frequency and when we multiply it by 3, the ear hears a resonance, it hears something that matches. So now, how does that explain the spectrum of the guitar ? It explains the spectrum of the guitar because when you raise the number 2 to the power of 19, we get practically the number 3 raised to the power 12. It can’t be a tie because when we raise 2 to the power 19, we get an even number. When we raise 3 to the power 12, we get an odd number. So it can’t be an equality. In fact, what happens is that if we look at the twelfth root of 2, it is a number equal to 1.05 etc., and it’s practically the same thing that the 19^{th} root of 3, what does that mean ? It means that in musique, what we did, with the frets of the guitar, is that we managed to make it seem as if these two numbers were equal and the 12 in question, these are the 12 tones of the well-tempered scale. And all the music is based on it. And what is the spectrum of the guitar ? Those are the powers of the number, which is the twelfth root of 2, and which is practically the 19^{th} root of 3. So it’s something extraordinary. And then, here we are faced with a problem because we have this so evident spectrum. This spectrum is before us and we wonder what is the object from which it is the spectrum. So when you’re a mathematician, you have a bunch tools to watch this ? Why ? Because when we look at the spectrum which corresponds to the drum, or the spectrum which corresponds to a form which is two-dimensional, which is of dimension 2, we notice that its range, it grows like a parabola. If we looked at a 3-dimensional object, that would grow with a power of 3, etc. And then, we now look at the spectrum guitar ? *(Clicking of interrogative language)*. Ah ! He is extremely weird ! Because if we calculate its size using what I told you before, we obtain that it is an object of dimension 0, an object of dimension

0 in the sense that its dimension is smaller than any number, not zero but positive. Ah ? ! So the wonder is that in fact there is an object whose spectrum is the spectrum of the guitar, but it's an object of non-commutative geometry. So we fall back on our feet. And so in fact, the book, the book we wrote on *Le Spectre d'Atacama*, this is a book which is entirely based on the fact to try to understand a spectrum. This spectrum was observed by the Observatoire of Alma in Chile and throughout the book, there is a hero, finally, there are several, there are three essential characters, there is a mathematician, there is a physicist who was there in the first book, who escaped a quantum stay and the whole book is based on trying to understand this mysterious spectrum in the Atacama Desert.

Alain Prochiantz : In the Atacama Desert. So let's listen now *Hi of Love*, from Elgar, played by Itzhak Perlman and after, we will resume our discussion.

Alain Connes : Of course.

(Musical interlude : Hi of Love)

Alain Prochiantz : After this *Hi of Love*, I would like to remind you that we receive today, as part of the series of interviews on the theme Imaginations, Alain Connes, mathematician and professor at the Collège de France, holder of the Analysis and Geometry chair. Alain, I think you wanted to betake precedence over this piece.

Alain Connes : Absolutely. Why did I choose this tune ? It's to illustrate exactly what I said earlier but the difference between the violin and the guitar. So the guitar, I talked about the spectrum of the guitar, and frets on the neck of the guitar. Well. It is obvious that when we have a violin, we don't have frets, and the extraordinary difficulty of the violin comes from the fact that precisely, we do not have a discrete spectrum in the mathematical sense, that is to say we do not have... If you want, an infinitesimal displacement finger on the violin neck will make all the difference. And this, in the interpretation of Itzhak Perlman, is wonderful, and that's why I wanted it to be listened to, it is wonderful because of the infinite precision it happens to have, in the sounds it produces ; mathematically, what we say, if you will, it's that the difference between the guitar and the violin is that the guitar has a discrete spectrum, which I explained earlier, the violin has a continuous spectrum. But there is in the spectrum of the violin the same difficulty as in the guitar. That is to say that there is an exponential scale, that is to say that when you go from one note to another, naively you would think that it should be done by spacing the fingers of equal length, no, it must be done by spacing of an exponential length, since that corresponds to the powers from the number before.

So in fact, there are always, always, between two violinists, infinitesimal differences and this interpretation of Itzhak Perlman is, in my point of view, a marvel because there are very small nuances, tiny, that the ear perceives of course, and that makes this adaptation wonderful.

Alain Prochiantz : Thank you very much, Alain Connes, I would have liked to come a little bit on the question of the mathematician, on the question of the demonstration in fact, because there are conjectures. How it comes a guess ? And how can we pass, afterwards, from the work on the conjecture at the demonstration of the thing and these are two different ways of doing mathematics. Are these two different types of mathematical minds ?

Alain Connes : In fact, it's always amazing what happens with conjectures. That is to say, in fact, a mathematician discovers something totally new. The example I have in mind, of course, we talk about it a lot in the book, it is Riemann, in the last century, in the XIX$^{\text{th}}$ century, more specifically, it was not the last century, that he made a discovery absolutely phenomenal. In fact, he found that we could understand the prime numbers, so understand the randomness of prime numbers, the randomness that was not controlled at all, from a function called the zeta (ζ) function and after demonstrating a formula, so it gives a formula exact if you want, for the number of prime numbers smaller than n. It's not so much the fact that there is an exact formula, because there are others, but it's the fact that this formula actually describes exactly the behavior of prime numbers. And he actually saw in this formula that he demonstrated, that there was a music of prime numbers, that is, he showed that there was a dominant term, which is easy to understand because basically the prime numbers get more and more rare, by and large, like the inverse of the number of digits of the number keep.

So when we look at prime numbers for example, between 10 000 and

100 000, or between 100 000 and 1 000 000, the proportion will be divided by 2. In fact, this thing, if you will, this phenomenon, leads to a function called the integral logarithm, which is actually the first term in Riemann's formula. But afterwards, the hazard of prime numbers manifests itself precisely by a spectrum. And it manifests itself precisely by what we could call the music of prime numbers.

So in fact, when Riemann found that, he saw himself making calculations, he did calculations, he realized that the zeros of his function, which rightly governs the spectrum, seemed to be all on a certain line. And the fact that they are on this line plays an essential role, because the fact that let them be on this line says that the formula he gave is a formula extremely precise. If there were any who were outside this line, there would be a kind of chaos which would be introduced, it would not be at all a nice thing. And he surmised that all his zeros were there. This conjecture was made roughly in the years 1850-1860, so it's been a considerable time that it was made, but, I think it was basically sure it was true, and basically he wanted to go on and make a guess is to be pretty sure that a result is true, and to go beyond.

So now, this Riemann conjecture, it has been verified with the computer, because with the computer, we can go very very far ; in fact, we have a way of calculating, which is very very efficient for this function, and we checked it for billions of zeros ; so at the verification level, we have a very strong indication. We do not know if it is true because there have been other conjectures that seemed to be true like that, but that are not true for very very large numbers, so we don't know if it is true but the way he found it was that he didn't want to stop there, if you will, and he wanted to go further.

And after him, there were a very large number of mathematicians who were interested in it, there were for example mathematicians who, when took the plane or etc., sent a letter saying "I demonstrated... etc." thinking that if the plane broke its face... , then... *(laughs)*. Here. So I have all kinds of stories, around this conjecture. But let's say that what is extraordinary is that with a conjecture like this, there is something extraordinary, it is that in fact secretly, it motivated most of the most interesting developments in mathematics in the XX$^{\text{th}}$ century. That is to say that if we know enough things in mathematics, we realizes that, a quantity of developments which a priori have nothing to do with guesswork, in fact were motivated by that one ; a typical example is the whole theory of almost periodic functions of Bohr, the footballer, the brother of the physicist, so I mean, it's amazing, it's amazing. And about that, we explain it in detail in the book, what is important to know is that a mathematician faced with a problem, always has a technique which doesn't make him unarmed and what is this technique ? It is a very interesting technique which I think does not only apply to mathematics, it applies, I think, in fact to all kinds of fields ; and that's why I want to explain it.

It is a technique which consists in saying, faced with a fixed problem, for example Riemann's conjecture, instead of being there looking at the problem and then being unable to do anything, no. What we are going to do, the first thing we're going to do is criminal in some way. That is to say, we will take the problem and we will generalize it. So it sounds completely silly. It seems completely silly to replace a particular problem by a much more general problem. And the example that we take from the book, this is the example of chocolate bars. That is to say imagine we are there, we look at you, and then we ask you "what is the optimal way to break a 6×8 chocolate bar for example in small tiles ?". And then, the interest of generalizing is that we are going to hold power to specialize the problem generalized to much more simple cases. This is great because if you are faced with the problem of a 6×8 tablet, you're completely stuck because you say "but it's too complicated, I'll never get there." However, if you replace 6 and 8 with l and m, it seems weird. But now you take $l = 1, m = 3$, you have a shelf of three tiles, three tiles. Well, to break it, it's not very difficult. So in fact, in mathematics, we do that and we did that for Riemann's hypothesis, and it was something extremely fruitful because that is what allowed André Weil precisely, to demonstrate a generalization that had been made and to demonstrate that it was true in this case. So it gives confidence and in general, precisely, that makes it possible to give an anchor point, for what I was talking about earlier, that is to say the analogy. That is, once we have demonstrated a special case of the generalized problem, we have an extraordinary tool which is the analogy. That is to say that we imagine that the demonstration that we made in the particular case will be able to transplant, I do not say to transpose, I say to transplant, as I said earlier, with the small flowers which are very fragile. So she will be able to transplant herself in the case that really interests us. So the creative power of conjectures is not at all negligible, it is a kind of way of having seen further than the others, and after all, well, after, you have to get into the tough stuff, you have to try to demonstrate the conjecture.

Alain Prochiantz : But the conjectures are still proven ? Or are there some that are false ?

Alain Connes : Yes, of course, there are some that are false.

Alain Prochiantz : And if we work on a conjecture that is false, and if we draw interesting results, and we prove that it is false...

Alain Connes : In fact, what happens in mathematics is that there are two aspects. There is the aspect where there is a problem, is this problem resolved or not, well... This is an aspect of mathematics. But there is another aspect that is largely as important and it is the aspect of building theories. And for example Grothendieck was very well known for when he was asked a question, etc. he always tried to formulate the question in the good framework, and then he built a theory that made the question resolves by itself. Serre used the best metaphor in relation to that : he said that when we asked him a problem, he was trying to let it dissolve in a rising tide of general theories. So it says well what it means. And so in fact, there are the impetus which is given by a question such as a conjecture etc. Very often precisely, the most creative, the most positive aspect of a conjecture, it is the construction of the theories which will either resolve it or say it is false. It could very well happen. And besides, what we wants is to know the truth. We do not necessarily want to demonstrate. Actually, in the book, we tell a story that I don't want not to tell because the book, *Le Spectre d'Atacama*, ends with this story ; this story is the story of an aging mathematician, well, think about whoever you want, who has been tackling a good guess for years, and who finally decides, because he sees that he is running out of time before him, to sell his soul to the devil, to know the answer. One says initially, to know the answer.

Alain Prochiantz : It's a known story, that.

Alain Connes : Uh, not so much the one we are telling...

Alain Prochiantz : The story of selling your soul to the Devil, in all case.

Alain Connes : Of course, of course. Selling your soul to the Devil is a known phenomenon, but in this case what happens is quite surprising because he ends up having a date with the Devil and moreover the Devil is embodied by Machine Learning. Huh ! *(laughs)*. So he ends up have a date with the devil and then when he meets the devil, they encounter in an ill-famed suburb of Naples and the devil begins by making him sign the papers that he sold his soul to the devil and the mathematician does not realize that because he signed the papers and that, he gave his soul to the Devil, he will change his behavior. So the Devil says to him "but hey what is your wish now ? You must give your wish..." And the mathematician says "I wish the hypothesis of Riemann's false" *(laughs)*. And it's only when he gets home he realizes that in fact what he just said is because he had sold his soul and that therefore, instead of wishing it to be true, etc., he wished it be false.

Alain Prochiantz : Dear Alain, I think we will close soon this interview which was really fascinating ; I would like to ask you a question : "Is mathematics a natural language for you ?"

Alain Connes : Well I think that not only it is a natural language, but I think it's the only language that will allow us to understand and to communicate with extraterrestrial intelligence. And it joins the book but I'm sorry to mention it too much, but well, what happens in the book precisely, it is that this message which is received and which is the spectrum of Atacama, it is received alternately with prime numbers, and an intelligence earthly, a mathematician, cannot fail to recognize an intelligence external to us, and which is manifested by this understanding which is extraordinary, which was made by Riemann in the XIX$^{\text{th}}$ century, so this is what I claim,... and there is a language that was invented called Lincos... , but what I claim is that we can communicate precisely with extraterrestrials thanks to mathematical language, why ? Because it is the only language that is not self-referential. It's the only language that is not self-referential, that is to say that, unlike a dictionary which when looking for the definition of a word, it refers to another word, which itself refers to another word etc. etc., the mathematical language is not self-referential.

Alain Prochiantz : But this language is composed, therefore, to return at the starting point, mental images.

Alain Connes : Uh no, this language is composed, at the start, by example of signals, which we send spectrally, which we send repetitively

Alain Prochiantz : But for example you, when you think ?...

Alain Connes : Ah when I think, of course, I think through mental images, of course.

Alain Prochiantz : You never think in natural language ?...

Alain Connes : No, no, no. Natural language, I mean, is a language which afterwards painfully tries to transcribe our mental images, our ways of thinking, etc., but I say "painfully" because in general, I can't get it to pass in a really satisfying way, I try to orally etc. there are people who are really strong at it and I think in particular of Grothendieck. Grothendieck was capable when, what we were talking about earlier, that is to say about an idea that was not still ripe, he was able to start writing on it and that,...

Alain Prochiantz : Did it ripen it ?...

Alain Connes : It made it mature, but I think it's not given to everyone to be able to write about an idea that is not ripe core and make it ripen, I mean.

Alain Prochiantz : To get it out of its cocoon...

Alain Connes : To take it out of its case, out of its cocoon. And there is anything else I wanted to say, before we finish, it is, I don't know if it was mentioned in another dialogue on the imagination, but there is an extraordinary example, it is the example of Eureka by Edgar Poe. So this example is still wonderful, to know that a poet has been able, in the XIX$^{\text{th}}$ century not only to have the intuition of the Big Bang, but to imagine the fact that the universe could then have a Big Crunch, etc., and that it could be mocked, he was mocked for over a century, until finally, we realize that in fact, he was right, but he was right by an intuition purely brilliant, and purely poetic.

Alain Prochiantz : Well, well, listen, I think it's the best way to end this interview with, I remind, you, Alain Connes, holder of the Analysis and Geometry Chair at the Collège de France. Thank you Alain, for coming today and see you soon.

Transcript of a conference given by Alain Connes "Duality between shapes and spectra", given at the Collège de France, on October 13, 2011
The conference can be viewed here :
http://www.college-de-france.fr/site/colloque-2011/symposium-2011-10-13-10h15.htm

So then, my talk will focus on the duality that exists between the shapes and their spectrum, that is, if you like, the range that is associated with a shape. That, of course, partially answers the question of the relationship between shapes and time, since the vibrations of a shape take place over time.

And I will start my presentation by explaining why it's absolutely fundamental to ask the following question "how can we define invariants of a shape ?". Imagine you have a shape in the very naive sense of the term ? You can talk about its diameter, you can talk about its size and volume, things like that, but of course, to get the fully shape, you need invariants much more subtle than that.

And among these invariants, there are precisely the frequencies, the possible range produced by a shape, this is what we're going to talk about. And then, we not only want to know how to characterize a shape, but we also want to know how to characterize the position of a point in relation to this shape. And what we will see is that basically, if you want, a point, it is characterized by a notes chord, in this range. So to present things to you in a somewhat naive way. So we will talk about vibrations, shapes, I will make you hear simple shapes and then, by reference to a famous article by Mark Kac in the 60s, who asked "Can we hear the shape of a drum ?", we will speak of an additional invariant which makes it possible to complete the table, that is to say which allows, if known, to know the shape. And finally, I will finish, I really want this little addition, because in preparing my talk, I realized that I had tried to play *Standing in the moonlight* over the range that is produced by the simplest shapes like a sphere or things like that. And I realized that it gave a result which was not good at all. And I realized that in fact, the range, the real musical scale, the one that is sensitive to the ear, well, there is no simple shape to which it corresponds, i.e. a shape whose frequencies correspond to the musical range as we know it. And I had fun looking for an object and there is a really interesting object that seems to answer the question and which I will talk at the end, which is the quantum sphere.

So that's the program.

So what ? So, to start, we will try to think in an intrinsic way to the concept of shape or to the concept of position in relation to a shape, asking a question which is a very simple question and which is "where are we ?".

See, to this question, you can answer "we are at the Collège de France, in the Marguerite de Navarre amphi". But if you want to pass this information to another civilization will be inaudible.

How can we transmit where we are in an intrinsic manner ? So, of course, the men tried and sent the Pioneer probe in the space. And on this probe, they gave a certain amount of information. What are those informations ?

Well, of course they showed what they looked like. This is the drawing that is there.

They also gave a small overview of the solar system. We see, below, the Sun, we see a first planet Mercury. We see a second planet, Venus. We see the third from which the probe left. That's why they put the little one drawing, and so on.

But it is obvious that for the moment, you have information which is almost zero because there will be an infinity of planetary systems which will have little almost the same pace. And so, in fact, you will have absolutely no idea where you are.

So, in fact, there is, in the drawing that they sent, something which is much more interesting, much more cryptic and much more informative, and that is what which is in the middle, on the left, you see.

And what is it ?

These are the directions from the Sun relative to 14 pulsars and to the center of the galaxy. And in addition, they indicated for each of these directions the corresponding frequency laying. So we will see that this is very, very, very close to the answer we are going to get from an abstract mathematical reflection on the abstract problem. And the abstract problem, it can be formulated as follows, it can be formulated in the form of two questions, excuse me, from time to time, I will put slides in English because I know there is a simultaneous translation and I take advantage of it, So, to put some slides in English, the deal is... so, the first question is "can we find complete invariants for geometric spaces or, if you like, shapes, more generally ?"

And second, "can we invariantly specify where is a point with respect to a shape ?". So the essential thing, we can see it in the example that I gave you, when we wanted to give our position relative to the universe, is that if you see someone very learned, you would say "But to give your position in the universe, you just have to give your coordinates in relation to a reference system.". Yes, but where is the origin of the reference system ?

You have to say where it is. And to do that, you have exactly the same problem as in the original problem. And so on. So you see, it is not at all simple. It is not at all obvious. We could you say "Well, I know general relativity. I know a point is specified by his contact details, all that...". But these answers are null and void compared on the invariant and intrinsic side of the problem.

So the important thing, therefore, the important thing is that in fact, in a shape, therefore, there is a whole series of invariants already. And these invariant is, if you want, the range of the shape, then this is where we will see if the sound works, I hope that it will work. So I'm going to make a try. This morning, when I woke up, my computer had rebooted and therefore nothing was working. And like the program takes a very long time to get started, I was really scared. We'll see if it works. It works, so we hear the sound, so I'll start by the most elementary shape, the most elementary shape that exists, it is the interval.

If you want, it's a string that will vibrate, like a violin string, and it will vibrate. It will have a fundamental sound. And then it's going to have the multiples of this sound, like vibrations. The corresponding range will be extremely simple.

And we're going to have fun playing a little with this range. Okay, so if I do that. *(He clicks the numbered buttons 1 to 20 and means associated sounds.)* It sounds weird on the 7th. Well, I pretend that if you try on that range of play *Standing in the moonlight*, the first note must be 131.

So you see that it looks... Naively, we say to ourselves "But that's the range ! Of course ! Since these are the multiples of an integer...". No that's not true. It is not true at all. Big mistake. First naive mistake we would make. So that, it is for the simplest object. A slightly more complicated object, but even with this object, if you will. It has an extremely simple spectrum. When we want visualize frequencies, they can be represented in their visual shape, that is to say in their shape from the spectrum. And then, we can also represent them in the shape of a graph. The graph is interesting because we will see the multiplicity of a proper value in the graph. So now let's move on to a shape that is already more evoluated, which is the disc.

So what does disc mean ? The sounds produced by the disc, that means you take a round drum, you tap on that drum. There is going to be its fundamental sound. There is going to be exactly as in the case of the vibrating string. There are going to be harmonics, there are going to be other sounds. So the drum will produce a whole series of sounds that will no longer be as simple as the integers whose I spoke earlier, and that will give you a range.

And this range will still be extremely informative on the drum, that is, the lowest note will give you the diameter, will give you immediately the diameter measurement. And then the behavior, for example, of much larger notes, will give you the size of the drum, etc., etc.

So I will give you a bibliography at the end. I mean, for all mathematicians who have been involved in this kind of thing, but I'm not going at all to tell you "this is due to x or this is due to y". I will give you the bibliography at the end, but let's listen to the drum a little.

(Clicks do not produce any sound.) So there, I did not put sounds precisely, so I didn't take long because there are sounds that are very sharp, let's look how it's vibrating right now.

Okay, you see, I hope. You see how it vibrates : whenever you have a picture like that. The drawing is not at all as simple as it would appear, because the functions that are involved are what are called Bessel functions. And if you want, precisely, when you type more or less, to a sufficient place on the drum, etc. We're going to make it vibrate. According to one of his harmonic frequencies. We will listen to them, let's listen to them. So we can calculate them. These are numbers that are not at all trivial. They are not at all numbers like integers. These are zeros of a function which is quite complicated that we call the Bessel function, which are parameterized by two integers and that can be calculated. We can calculate them with as many decimal places as we want. But these are not simple numbers and that's the range of the disc. So the disc has a range like that.

I show you the first notes. It has a spectrum that is like that and now, we will hear it. *(AC varies the values of the two sliders, and we hear more or less acute notes, and we see at the same time, the circle colored in blue gradations, having more or less numerous radial and angular colored divisions.).*

So, I hope it's a note not too high, because I didn't want to make you hear too sharp notes..., I was nice, I did not put notes too acute. It goes on of course, as much as we want, etc.

And so we get, you want, so, a spectrum which is the spectrum of the disk that looks like this, so it continues indefinitely, it continues indefinitely.

And you can see, of course, that it has nothing to do with the spectrum we had just now for the interval. So now let's go a little further.

Take an object always of dimension 2, let us take an object which is a square now. It's like taking a piece of skin, stretching it out in between if you want a frame, like that, a square drum, and you tap on it.

And now the vibrations you get have the following look. It will do noise twice before giving it the right amount. We're going to climb a little higher. *(Square sounds).* Okay, so we see a spectrum again, the spectrum looks like this.

It is very, very different from what was happening in the case of the disc, because if you want for the case of the square, it is not very difficult to do the calculation. We realize that the corresponding frequencies are the square roots of the sums of two squares, so the numbers are of squareroot form of n^2 plus m^2. So this is something which is very simple to understand, which is much simpler to understand than the numbers that came up for the record. They are very different, but they have, if you want the same sort of infinite distribution. We can change the color if you want. But now, let's come to the sphere, so, all of this, these are 2-dimensional shapes and a priori, they are very banal shapes.

When I speak of the disc, when I speak of the square or when I speak of the sphere, I mean the title of the conference, it's *La vie des formes*. So you have to make them live.

And to make them live, you have to make them vibrate. And from the moment we make them vibrate, we realize that, although they are shapes that look extremely simple, extremely common, extremely elementary, when vibrated, the vibrations themselves decorate these shapes in an extremely harmonious and extremely non-trivial manner. So if we take the 2-sphere, if we take the round sphere, its spectrum, this time, is very, very simple. It is also formed by integers, exactly as in the case of a string.

But these integers appear this time with a certain multiplicity, that is to say that it's not exactly integers. It is more exactly the root of $J(J+1)$. It's practically an integer, so it looks a lot like what was going on in the case of the circle or the interval. But they appear with a multiplicity.

So now, if we take the sphere, then there, I am afraid that it is too acute.

See what happens, what happens is that if I take for example the Spin $= 6$, there is a certain number of frequencies, how to say, of what are called eigenfunctions that exist, but that have exactly the same, the same frequency.

How to say ? The shapes on the sphere are different, the sound we hear is the same. And that is what is called spectral multiplicity, that is to say that in the spectrum, what will happen is that we will have the same value, but it will happen multiple times. So this is what happens to the sphere... I will come back to this for the musical shape, that, we will see that later.

So now I'm going to go to the normal course starting from the pdf. So I go and do that.

So we have these two questions, we have these two questions, to define a full invariant of a shape.

So in fact, we have known since a famous article by John Milnor in the 60s, that the spectrum of a shape is not sufficient to characterize this shape.

It's a wonderful math article. It is one of the rare articles of math that has only one page. And what Milnor did was something remarkable. He used a Witt result to see that there are toroids, these toroids are of fairly large dimensions, they are not toroids of low dimensions. But there are toroids, which are geometrically different, but which have exactly the same range. And that comes from a result of number theory. Because, of course, the range associated with a shape with all its subtlety, as we have just seen in the examples that I have shown you, this range has of course a very deep relationship with arithmetic, arithmetic in the most naive sense, the arithmetic of the first range of all these integers. But basically, each shape has an arithmetic associated with it, and that is the arithmetic of the range it gives us naturally through the sounds it produces.

So, what is very, very interesting is that, as I will explain, the invariant which was missing compared to the spectral invariant, it is an invariant which one will see that is connected, in fact, to what physicists call the Cabibbo-Kobayashi-Maskawa matrix. And it's an invariant that actually measures an angle between two algebras and which generalizes a little what physicists did when they looked at what's going on with quarks. And that I will talk about.

But don't worry about the technical side of this page at all. So, we have considering the vibrations of the disc.

We have seen examples of disc vibrations, we have seen the natural frequencies of the disc which, as I told you, is not as simple as it seems. We saw its spectrum with the start of the spectrum. We saw, now, I put more natural frequencies for the disc and you see that it starts to look like... This curve there, it starts to look like what...

it starts to look like a parabola. And the more frequencies we add, the more we go at high frequencies, the more it will resemble a parabola. See if you take the disc where the eigenvalues are difficult to calculate, the range is difficult to calculate, but if you look at the range, but now from a great distance, i.e. you look at the highs frequencies and you put all these frequencies together, you see that that it seems more and more to a parabola and there is a famous theorem of Hermann Weyl which dates from the 30s and which says that this parabola has an invariant, if you will, which is how it is angulated or not, this invariant gives exactly in the case of surfaces, in the case of 2-dimensional shapes, gives exactly the surface of the shape, okay. So you see, we can measure the area of the shape of dimension 2 simply by looking at this parabola. This is what Hermann Weyl demonstrated.

So we also saw what is happening for the square. I had shown you some vibrations of the square earlier. The spectrum of the square, as I said, is extremely simple. These are the numbers which are the square roots of n^2 plus m^2. That's the vibrations of the square.

Of course, I don't want to bother you with mathematical formulas, but that, it comes from the Helmholtz equation, we look at the Laplacian, and we look at the equation of waves.

So the spectrum of the square, we look at the natural frequencies of the square, see, they look a little bit, from afar, like what was happening just now for the disc. We look at high frequencies, we look at high frequencies, it oscillates a little. And then, now we're going to look at very high frequencies, very high frequencies, you see, it's incredible, we really see a parabola, it stands out with the naked eye, absolutely a parabola. This parabola, precisely, its invariant, is the area of the square.

So, we look at the sphere too. The spectrum of the sphere, as I said, is practically the integers, these are the numbers of the square root form of $J(J+1)$. Here if you look at the natural frequencies of the sphere, that gives you that, the frequencies proper quences of the sphere ? *(We see a parabola, but it has like stairs.)* Why it looks like that, by stages ? Well, it's because you have the same frequency that will repeat itself a bunch of times. Okay, so, it's something that's tiered like that.

So you say to yourself "But that doesn't sound like a parabola at all."

It doesn't sound like a parabola, but it's because you don't watch enough high frequencies. And if you now look at much higher frequencies, you see that there are still small floors, of course.

Okay, but it looks more and more like a parabola and it will give you the area of the sphere. Okay.

Well. So, what does this have to do with the problem we had at the start, so that was the problem to say where we are precisely ?

If we mean where we are. Two things must be said :

We must say in which universe we are and how far from this universe we are. To say in which universe we are in fact, what I pretend, is that what you have to give are precisely the vibration frequencies of this universe, the first thing to give. So how do we do it ? We did it if you will, there is a very interesting thing that happens, and goes back to Mark Kac, and to "Can we hear the shape of a drum ?, etc.", we are concerned with the equation of the waves and we are concerned with something called an operator that mathematicians call the Laplacian, which is called the Laplacian, which is called Δ but when you write the wave equation, if you want, in fact, we write this equation in the Δ form of a function plus k square times f equals 0. *(AC written on the blackboard* $\Delta f + k^2 f = 0$. *)*

This is the Helmholtz equation. And when we write this Helmholtz equation, we see that the number k that appears, it's going to be, if you want, what we call eigenvalues of the opposite of Δ, but that is not k, because it is k^2 which is an eigenvalue of $-\Delta$, and therefore in fact, the number k which appeared, in all the examples that I have given, it is a number which is eigenvalue of the square root of $-\Delta$.

So this is called an elliptical differential operator and its square root, it's not something very pretty.

And then fortunately, there is a physicist, who is Paul Dirac, who found a way to extract a square root from the opposite of the Laplacian in such a way that it is a differential operator in a aesthetical way. This is called the Dirac operator.

So what makes that in all the examples I gave you, in done, it's much more natural and important to give for a geometrical shape is not to give the spectrum of the Laplacian but to give the spectrum of the Dirac operator, that wouldn't make difference practically for all the examples I gave you. So this is very important. Well, this is the first thing, that is to say that what we are going to keep back, basically, it's a square root of $-\Delta$, so it's not going to change a lot. So we will, we will give all of its eigenvalues. We will give its range, if you want. And now, which is enough extraordinary is that there is a way to find a complementary invariant from this range.

And basically, this a complementary invariant, it's going to be a prescription, we're going to give the possible agreements on this range.

We are going to give a set of possible agreements, but the origin, the origin of this invariant : it comes from physics, and what is called in physics... flavor. If you want, in physics, there are quite complicated phenomena called weak interactions and in weak interactions, people realized that there was what are called currents which allowed to change "flavor", i.e. to change family. That is to say that, for example, for quarks, you have the quarks that we know that are the up and down, that are the main quarks, that form the neutrons, protons, etc.

But you have other quarks, there are two other families of quarks. Well, there are interactions in physics that allow to change..., that allow to pass from one family of quarks to another family of quarks, this is called "flavor changing neutral current", and physicists understood that what measured if you want, these currents which allow to change family, it was in fact an angle between two commutative algebras, but very simple in their case. It was first found by Cabibbo, then by Kobayashi and Maskawa. And this is called the Cabibbo-Kobayashi-Maskawa matrix. And what it does is that it measures, if you actually want, the angle between two algebras. And what is quite extraordinary, is that good, in fact, they are algebras in a space of dimension 3. So it is something very simple. And yet a complex number appears and that's it that made the violation of what is called CP in physics.

So now, what is quite fun is that if we generalize this idea, we get a solution to the problem just now, that is, we get another invariant which is not only the spectrum of the Dirac operator. So, the first invariant, if you will, is the spectrum of the Dirac operator.

This is very important *(AC writes Spec D in chalk on the board.)*. This is the range, if you want. Okay. Okay, but there is a second invariant. And what is this second invariant ? Well, this second invariant is also an angle.

It is not a number, it is an angle, it is an angle. It is a notion a lot more complicated. It is an angle between two algebras. Then there is the algebra of functions of the Dirac operator.

It means that you look at all the operators that are diagonal in the same basis as the Dirac operator, which is the basis of the eigenfunctions and another algebra, which is the algebra of functions on the space in which you are, on the shape in which you work.

And then there is a wonderful von Neumann's theorem that dates back to years, from very, very long and who says that the representation of this algebra in Hilbert space is independent of the shape you choose. That is, if you take any shape, be it a sphere, a disc, a shape of dimension higher, etc., well, the function algebra will always act the same way in Hilbert space.

So the only thing missing to complete the picture is going to be the relative position of these two algebras and the relative position of these two algebras, in fact, it is specified by a series of chords.

Okay, this is a continuous series of chords, which are formulated over the range and what happens is that... So, we have these two invariants and how should we interpret a point, so a point, in fact, if we look from this point-of-view, a point is given by correlations between different frequencies. So, you can think these correlations, these are complex numbers. But you can exactly think of these correlations between different frequencies like a chord, we take a chord between these notes and that's it, a point, it's that, okay. So think in your head, and keep a geometric object, a geometric shape is given by its music, by its scale, it is given by its range, and the set of points is given by the set of possible chords and a point is given by a chord.

Okay, so then if we continue, from this point of view, we realize *(we can dim the light, there, it's good).*

So we realize that it's quite astonishing to see how much this point of view which I have just spoken of is close to physical reality. Why ?

Because now, man has evolved, maybe by natural selection, sufficiently to be able to look at the universe. He has an eye. This eye is Hubble. His current eye is the Hubble telescope. With this telescope, man looks at the universe. I advise everyone to log into the NASA website every morning, NASA gives a new image every morning. It is done on a daily basis and each day, you can look at the universe and you will have a different picture of the universe.

And what's amazing, what's really amazing is that the information that comes to us from the universe, it is spectral. And not only did this information provides us information, by a spectrum, on the composition of very, very distant stars or intergalactic clouds, etc., simply by their spectrum, but in addition, it gives us signs on their origin. And how does it tell us about their origin ? Because by Doppler effect, the more things are distant, the more there is what is called redshift, then the redshift, naively, if you are very naive, you think the redshift, you will take the spectrum and you will shift it like that by a translation. But it is not true. Redshift is a multiplication. It's not an offset, it's a multiplication. That is to say that we take all the frequencies and we multiply them by the same number. And what's really amazing about redshift is that now we observe, we observe... So why do we know that this is the same thing we see ? Well, by the fact that the range is the same.

They look alike, of course. So you can see that on the left, you're going to have some provision in the range. It will end up on the right, but it will not be found in the same place and it will be offset, but not offset by a translation, offset by an homothety, that is to say that we multiply all numbers by something. And what is extraordinary is that it is thanks to this redshift that we can go back in time. We are now measuring redshifts which are on the order of 10, but in fact we expect redshifts on the order of 1000, etc., and they correspond, of course, to increasingly distant times.

So it's really amazing, it's really amazing that this view on shapes be as close in point of view as abstract mathematics suggests on the shapes.

And on the other hand, there is another thing that is extremely important : it is that, of course we can't get around at the moment, but probably for always, towards other galaxies. And so, it's an act of faith that we do, to know that these things exist somewhere. And this act of faith, it comes precisely because of the correlations that there are between the different frequencies and the image that I'm going to show you now, this is a picture of the Milky Way.

But it is an image which is not at all taken in the visible. It's an image which is caught in wavelengths that are completely invisible. Okay, so, it's absolutely mind-blowing and incredible that there are so many correlations between these different frequencies that in fact, these images are compatible. And here is an image, therefore, of the Milky Way, taken in frequencies which are absolutely not visible, but which, precisely, are correlated with the images in the visible and therefore ensure us that there is indeed a coherence, okay.

So now, when I was preparing this talk, it took me a long time to prepare this presentation. Why ? Because good, of course, I was given a rule which was that I should not show images like this that were not approved by the author of the image, etc. So, I said to myself "that, maybe I can do it.".

But on the other hand, I didn't have it for all the other images I wanted to show on the spheres. So I stuck doing it on the computer and it took me a lot of time. And then, at one point, I wanted to relax a little and I said to myself "Oh well, I'm going to play *Standing in the moonlight*" on the range of the simplest object, ie the vibrating string.

I tried and that's at that time that I realized that the first note that I had to use was 131. I said to myself "There is something strange".
So I asked myself the problem. I asked myself the problem of finding a musical shape.

So what do I mean by that ? Well I mean that when you make music, we'll see it right away, when you make music, in fact, it is not at all integers 1, 2, 3, 4, 5, etc., as frequencies which are used ? Absolutely not, these are the powers of the same number, the powers of the same number, that is to say we have a number q. And we look at the numbers q^n, that is what counts, because it is the relationships between frequencies that count. And the wonder that makes piano music exist, called *The harpsichord well temperated*, etc., it is the arithmetic fact that exists, which means that if we take the number 2 to the power of a twelfth, if you take the twelfth root of 2, that's very, very close to the nineteenth root of 3.

See, I gave those numbers. You see that the twelfth root of 2 is 1.059..., etc. The nineteenth root of 3 is 1.059... Where does 12 come from ?

The 12 comes from the fact that there are 12 notes when you make the chromatic range. And the 19 comes from the fact that 19 is 12 + 7 and that the seventh note in the chromatic scale, this is the scale that allows you to transpose. So what does it mean ? It means that going to the range above is multiplying by 2 and the ear is very sensitive to that. And transpose is multiplication by 3, except that it returns to the range before, i.e. so it is to multiply by 3 / 2, that agrees.

Well, that's the music, well known now, to which the ear is sensitive, etc. Okay. But... there is an obvious question ! It is "is there a geometrical object which range gives us the range we use in music ?". This is an absolutely obvious question.

If you look at what is going on, like these are the powers of q, you notice that the dimension of the space in question is necessarily equal to 0. Why ? Because earlier, I had shown you its limits. *(pauses to tell someone "I'm coming.")*. So I had shown you *(pauses to tell the person "I still have 5 minutes.")*. I had shown you earlier that the objects had a range that looked like a parabola when they were of dimension 2. When an object is larger, it will be a little more complicated than a parabola.

For example, if it is in dimension 3, it will be $y = x^{\frac{1}{3}}$, okay, but here, it's not at all a thing that is round like a parabola like that *(AC draws a parabola in the air)*. This is something that pffuiittt ! *(AC makes the gesture of an exponential in air.)* that gets up in the air like that. And what it tells you is that the object in question must be of dimension 0. So you say to yourself, "an object of dimension 0, What does it mean ? etc. Well.".

Well, when you develop geometry, which I have done for years and years, from a spectral point of view and for the development of what is called non-commutative geometry, etc., well, you actually see that these objects exist with a small nuance, it is that the algebra which is going to intervene is not going to be necessarily commutative.

And the wonder of wonders is that I realized, while preparing my talk, that there was an object, well known to mathematicians, that study non-commutative geometry or quantum things, which works for that thing and which gives you the right range.

And what is this object ? It is none other than what is called the quantum sphere S^2 index q. So this object is therefore a more delicate object. It was considered in particular by these three names *(on the transparency are noted the names Poddles, Brain and Landi)*. It has a spectrum, it has a spectrum. And this spectrum ? If you choose the number q carefully, it will correspond exactly to the musical spectrum. So, I return now to my experiments and I finish on it. So I will try. I hope that it will work. So we're going to go back to the experimentation. So we made the sphere and now we're looking for this musical shape which is going to be dimension 0. Okay, so we're going to try to play *Standing in the moonlight*. As I am tired, I will surely be wrong. But it's not a big deal. So see. *(AC returns to a rainbow spectrum and plays* Standing in the moonlight *on a keyboard with buttons indicated by integers, 25-25-25-27-29-27-25-29-27-27-25. 27-27-27-27-24, he makes a mistake, plays a B instead of an A, laughs, takes it up, etc... applause and ecstasy !).*

So now what is absolutely extraordinary with this range is... Can someone give me a number, above 10 anyway ? *(We see Jean-Pierre Changeux who waits positioned at the office to give his own Classes.)* 13, very good. Well I will see, I will try again but I promise nothing *Standing in the moonlight* from 13 *(AC plays 13-13-13-15-17-15-13-17- 15-15-13...)*

(When looking for the lowest score discarded from the others, AC says "So it's here I must not be mistaken, otherwise you will yell at me...". Re-applause.)

I will finish by saying the following thing, if you will, is that "nothing is too much beautiful to be realized in Nature". Recently, the Nobel Prize in Chemistry has been awarded to a chemist who discovered quasicrystals, which have a wonderful mathematical history, in Nature. What I hope is that one day, we will find the non-commutative sphere S_2^q in nature and one will be able to use it as a musical instrument, and it will be a wonderful instrument because it will never detune. Here. Thank you.

(Applause).

An interview with Alain Connes

1 Noncommutative geometry

The subject that has occupied me for all these years is very, very far from being exhausted. It is a subject that begins with the discovery of Heisenberg. It is a discovery of physics in the years 25, 1925 of course, and what Heisenberg discovered is something quite extraordinary. He discovered that when we do physics with microscopic systems, well, we can no longer do calculations as we are used to, that is to say what is called commutative algebra.

We can no longer use commutativity. So commutativity means that if you write mc^2, or c^2 times m, it's the same thing, but when you do quantum physics, you can't. So what does that mean ? It means that it is essential too, as much for physics than for mathematics, to understand more subtle spaces which are noncommutative spaces. So if it was only... if noncommutative geometry was only a generalization of geometry to spaces in which the coordinates do not commute, that would not be very interesting.

What I had discovered in my thesis was that, precisely, a noncommutative algebra by the simple non commutativity generates its own time, i.e. evolves over time. It's something which is difficult to explain, but which has a depth, that is to say that basically, we can summarize it in the following form. We can say, if you want, that commutative algebra is static, it doesn't move and noncommutative algebra evolves. So you have to understand that when we talk about noncommutative algebra, non-commuting coordinates, etc., you might think, a little simplistically if you like, that it's a mathematical abstraction that has nothing to do with our habits, etc. In fact, this is not at all the case because, precisely, the non-commutativity, we are extremely familiar with This is because when we write, with letters, when we write words, sentences, etc., we must of course pay attention to the order of the letters. In written language, you cannot swap letters.

If you have fun swapping letters, you get what is called an anagram. And obviously, at that time, we can have two sets of letters which are the same in the commutative frame, but which have completely different meanings in the noncommutative case. The example that was the occasion of a book that we wrote recently with Jacques Dixmier and my wife, it's if you want, this magnificent anagram which is due to Jacques Perry-Salkow and which is, precisely, *"L'horloge des anges ici-bas"*[1], which has to do with time and the anagram of that is *"Le boson scalaire de Higgs"*[2].

So, we can see that if we only look at the commutative part, if you will, of these two sentences, they are the same, but on the other hand, they are not at all the same, they have not at all the same meaning. The quantum, the great discovery of Heisenberg, is that, precisely, it is necessary to be careful. The way this noncommutative geometry has evolved, and that's why it is very, very far from being exhausted is that, on the one hand, there is a very, very strong link with physics. So here I have it developed during numerous courses, with my collaborators, etc. So there is a link with the fact that it is precisely the formalism of quantum mechanics that allows us to understand how continuous variables can coexist with discrete variables, and how we can reformulate geometry, Riemannian geometry, in a form which is much more compatible with quantum than is general relativity.

So that is a whole area, it is a whole area which is still open, which is far from being exhausted. There has been great progress. And then there was another extremely exhilarating episode that happened, is that if you want, this noncommutative geometry, precisely, allows you to encode spaces which, normally, for mathematicians, appear as very, very singular spaces. Those are quotient spaces, but these are spaces that we meet in mathematics in fact, very, very often, people don't realize it because as soon as we take what we call in mathematics an inductive limit, we will fall on a space which is of this nature, because it is defined as a quotient space. And the idea, the basic idea, is that when we take a quotient that is hard to take, we have not to look at it as a whole. But you have to look at it like a noncommutative space where non-commutativity comes from the fact that we will identify between them points which are distinct and

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Interview with Alain Connes, Full Professor of the Chair of Analysis and Geometry at the Collège de France ; Alain Connes was interviewed in March 2014 by Sophie Bécherel. Cécile Barnier and Sophie Chéron also participated in the project ; the project was funded by the Bettencourt-Schueller Foundation ; interview viewable here : https://www.college-de-france.fr/site/alain-connes/Entr Maintenance-avec-Alain-Connes.htm.

1. the Clock of the angels down here
2. The Higgs scalar boson

so we're going to have arrows, etc. And that's what makes it noncommutative.

So there was an episode completely... which is far from over of course, it's actually a space fundamental to number theory, which is actually connected to prime numbers, which is a noncommutative space. So there is a very, very long development that has taken place, which has corresponded to many of courses I have given, etc. but which continues to evolve.

And now, we found with Katia Consani, we found very, very recently that in fact, there was an object of very, very pure algebraic geometry which only involves integers with the three operations of *inf* of two numbers, of the sum of two numbers and their product, but which involves two fundamental concepts, the concept of topos, which is due to Grothendieck and the concept of algebra of characteristic 1. This is yet another story.

This object means that we have exactly the parallel with what André Weil had done in algebraic geometry, precisely to deal with a fundamental problem in finite characteristic.

What you are telling us there seems to demonstrate that mathematics generates other mathematics.

Yes, but it's not that they generate, no. What you have to understand is that I always had this long discussion with Jean-Pierre Changeux. And it's not that you beget, no, it's like if..., but let me explain why it is not that one begets. It's exactly like to say that Christopher Columbus had begotten America. Everyone would laugh. Well, well, mathematics, it's the same. The mathematician is not going to father. He will discover and he will discover a new section of mathematics. Or because if you want, this math comes from physics. And of course, I think it was Hadamard who spoke about it the best, if you will : mathematics comes out of physics because they have to do with external reality, they have a particular taste. They have a particular force, but I don't know anything that generates, no.

2 Mathematical research

We are trying to understand. We try to understand physical reality, of course, and we try to understand mathematical reality. This is something that is very, very obscure, very difficult to understand. And the way we try to understand is to work out, so here we invent concepts. These concepts are very specific. For example, I talked about the concept of topos due to Grothendieck. These are very specific concepts.

These are not vague things, they are very, very precisely defined things. And what is extraordinary, if you will, is that one of the often overlooked roles of mathematics is that of generating concepts. And these concepts, at the start, are going to be purely mathematical concepts. But gradually, they will become part of everyday life that we all share. A very striking example, it's the concept of function, you know, the concept of function is not something that is obvious to the general public, etc.

But when we talk for example of the slowdown in unemployment growth or things like that, that corresponds to very precise mathematical properties, defined on functions. And so, we have there a striking example of a concept which comes from mathematics and which, gradually, gradually, will be part of the common baggage of civilization. And one of the reasons why he can register, is that now, we don't only have printing, writing, we also have computers. And the computer is not only going to be able to transmit words, to transmit numbers.

It will be able, precisely, to transmit functions, that is to say that we will be able to see on its computer screen the graph of a certain function, etc. And we will be able to understand qualitatively the properties of these functions and the relevance of mathematical concepts.

3 Between physical reality and pure mathematics

There is always a balance and precisely, my balance... if you want, we can only move forward if we walk on two feet. My balance is between physics, of course, that I never gave up, never, because there is this essence of quantum physics, precisely, which, as I said if you want, allows this coexistence of the continuous and the discrete which is magnificent and on the other hand, there is the algebraic geometry, number theory, etc.

I was talking for example about topos. Grothendieck wrote on the topos that precisely, it was “the bed at two places which allows the marriage between discrete and continuous”. So although this is a very different approach, it is not totally disjoint.

So there is this balance between the two and hey, physics, of course, comes up against experimentation. The Mathematics also collides, in a way, with experimentation. I use a lot the computer, I use a lot of computer checks, even for things that seem impossible to watch on the computer. And there, we come up against a real reality. We run into something we can’t change. We want to know if something is true or not, we do tests, we look at this.

Well, well, it’s a bit like a physicist who will do experiments and see if his idea is correct or if he needs to correct it. Okay so there are these two sides of my job, if you will, and there is not one side that took precedence over the other, they have always remained very balanced.

Some people say that your noncommutative geometry is a bridge between quantum mechanics and classical physics. Why ?

Yes. If you will, what there is... it’s not really a bridge between quantum mechanics and classical physics. No, the bridge between quantum mechanics and classical physics is what we call dequantification. It’s a whole story, it relates to characteristic 1 which I was talking about earlier. But it is something else. In fact, noncommutative geometry, no, it’s more a bridge between the quantic and the geometry and the fact that, precisely, our geometry, the one to which we are accustomed, the Descartes’one, applies perfectly to classical physics, but does not apply to quantum physics.

Quantum physics means rethinking geometry. This is exactly my job. It’s exactly what I do, what we did with my collaborators, is to show that the Lagrangian [3], which is extremely complicated, and which contains both gravitation and quantum mechanics, the Lagrangian of quantum mechanics, this Lagrangian is understood in an incredibly simple and conceptual way when we have the tools of noncommutative geometry. But it’s still a Lagrangian you have to understand, which is at the classical level, that is to say that it is not yet quantified. So we know we’re on the good track because this Lagrangian who looks incredibly complicated, it takes four hours for putting it into formulas...

A Lagrangian, can we summarize it as a formula ?

A Lagrangian is a formula, but basically, if I want to explain to you what a Lagrangian is, we must understand the simplest principle of action, which is what is called the Fermat’s principle, and I can explain it to you in three words. Fermat’s principle is the principle that says that light will follow the shortest path for it. So you can do an analogy with... Suppose you are in the suburbs, and that you want to go inside Paris, for example.

Ok, well, then, if you know there are big big traffic jams in Paris, what you will do is that you will reach the point of the circumference of Paris that will be closest from the point you want to reach ? It doesn’t matter that you’re not going in a straight line. And that is what is called the principle of refraction of light. Okay ? So the Fermat’s principle tells you there is a principle of minimizing travel time, ok ? So the physicists have enlarged this principle to much more general things. They enlarged it to all physics and when it was enlarged says to all physics, the principle of action is precisely what is called the Lagrangian. Okay ? And it contains all of physics because it contains both gravitation and it also contains the Lagrangian of quantum mechanics. But we are still at the classical level, as one says. One needs also to quantify all that.

And so, what we said with my collaborators is that quantifying something that we don’t understand, it’s a bit illusory. And so, what we did was that we understood this classic Lagrangian, as being the Lagrangian of Einstein, that is to say pure gravitation, but on a space a little bit more subtle than space, which is simply the continuum we are used to and the space we have found, it is a space which is pre-

3. The Lagrangian of a dynamic system is a function of the dynamic variables which makes it possible to write in a way concise the equations of motion of the system in quantum mechanics. But we are still at the classical level, as we say. We still have to quantify that.

cisely a mixture between the continuous and the discrete, and this mixture can occur only through the noncommutative.

And what does this give you, to have found all that ?

It gives us first a great aesthetic pleasure, the fact that a Lagrangian, who normally takes four hours to be transcribed into LaTex on a file, can be written as a very small formula. And this tiny formula is even simpler than Einstein's formula, since it's a formula which only counts the number of eigenvalues of the length element in noncommutative geometry, which are greater than a given length. So, this is something incredibly simple and that's something if you like, which rightly says that the length element in non commutative geometry is something totally different from the conventional length element. The element of classic length, you know, it was the standard meter that we were told about when we were students and we told ourselves "The length element... The standard meter is deposited at the Pavillon de Breteuil", etc. And there was a whole story that explained the creation of this standard meter with Delambre and Méchain who had been sent, the surveyors who had been sent between Dunkirk and Barcelona to measure, etc.

So what ? An extraordinary episode happened in the 1920s, which is exactly the same in physics as the paradigm shift we are proposing for noncommutative geometry. This episode is the following. There was a congress, not of surveyors, but of the members of metric system. So people were gathered and among them, there was one who raised his finger during the meeting and he said "I'm sorry to tell you some bad news, but the unit of length changes from length.". Imagine, the meter changes length. So it's very, very annoying, if you want, we have a unit of length that changes in length.", then the others asked him. "Okay, well, it's very well, but how did you know that ?". He says "Look, I took the yardstick which is at the Pavillon de Breteuil, etc. And I measured it against the wavelength of the krypton and I realized that it has changed length".

So, disaster, etc. We cannot take an element of length that changes length, then gradually the physicists reflected, etc.

And they understood that in fact, it was necessary to take as a unit of length, which had made it possible to see that the old unit of length had changed. So they took a unit of length which is spectral. Then they replaced krypton by cesium.

And it is quite obvious that if we want to unify the metric system, let's admit in the galaxy, it will be necessary to give something convincing.

If we tell people "if you want to measure your bed. You must come to the Pavillon de Breteuil, etc." Well, it won't be very convincing. If, on the other hand, they are told "listen, you are taking hydrogen. You take the spectrum of hydrogen. There is a certain spectral line which has a certain form and you take its wavelength as the unit of length". It's amazing. Well the change that allows to go from classical geometry to noncommutative geometry is exactly the same, i.e. that in noncommutative geometry, the element of length is spectral, it is given by the inverse of what is called the Dirac operator and it is given by what physicists call the propagator of fermions, that is, something they always write as an infinitesimal. So there is there, if you want, a coincidence which is extremely, extremely strong, and which says that there is an evolution of geometry which goes from an entirely classical formality, entirely commutative to a formalism which frames with the noncommutative, but which is also spectral, which becomes spectral.

Why do you say you've not finished... ?

We're not done, but no. We are at the very beginning, if you like, we are at the very beginning. First because well, actually, we would have to go to the quantum level for the geometry of space-time, i.e. quantify this Lagrangian I was talking about, but also in understanding, for example, the geometry which underlies the prime numbers, we are still far from the count. We recently found therefore, the object we have been looking for fifteen years.

That's what I said. It is an object of algebraic geometry, but which uses very sophisticated concepts. because it uses both topos and characteristic 1.

But on the other hand, when we give the definition, the definition is overwhelmingly simple, if you want, therefore, it is surely the right definition. But then you have to develop the analog of the algebraic geometry which had been developed in finite characteristic, it must be developed in characteristic 1, it is necessary to develop a cohomology which replaces Weil's cohomology, etc. So you see, there is a whole program there, which is in front of us.

4 The tools of the mathematician

We have an incredible chance in mathematics, it is that a mathematician faced with a very difficult problem, what does he do ? In general, the problem is too difficult to attack it head on. So there is a method. You should know, for example, that if I tell you "we take a bar of chocolate that has 4 on one side, 8 on the other. How many times does it take to cut it in half so that it is finally reduced in small tiles ?". You'll tell me it's very, very complicated, etc., okay.

Well, the mathematician's idea is immediately to generalize the problem. That is to say that instead of saying a chocolate bar of 4 times 8, he will say a chocolate bar of n times m, where n and m are two integers. And then after, he will take the smallest values of n and m. For example, it will take $n = 1$, $m = 2$. It will take 2 tiles. Ok, we cut in one go, it works. Okay. And then after, he will have fun looking at simpler, but more and more complicated cases. And after a while, because he will have solved the simplest cases which are easy, the difficulty will increase like a staircase. And through this staircase, at a given moment, he will say "Ah here, that's it, I understand !" and he will have understood the general idea. Okay.

So that's the essence of mathematics. And if you want, there's a great thing, that in general, when we look at the small cases, the simpler cases, well, then, we will be able to proceed by analogy.

And analogy is a tool of mathematicians who, for the moment, completely escapes the computer because the analogy is never exact. Analogy is something...

Is it intuition ?

No, it's not intuition. Intuition is something else. Analogy is something that consists in saying that we are going to transplant methods that have worked in one case, in another case. And of course, that will not be exactly the same. You'll have to take... it's like taking a small flower, you transplant it elsewhere, if you want, it must remain alive, but the earth will be different, it will be in a different context, etc.

This idea of transplanting it, is that intuition, saying to yourself "well, I'm going to take that tool, and I'm going to bring it there" ?

Yes, well, it's true, if you want in mathematics, there is a significant part of intuition which is very, very difficult to define.

5 Intuition

It's true, it's perfectly true that in certain situations, certain situations where we have a very difficult problem, etc., you get to have an intuition. But this intuition, if someone asked you "Can you tell me,... there, what do you want ? What are you doing ?". Etc. We would be unable to say it, we would be unable of saying it, because it is an intuition which is not yet rationalized and which, if we tried artificially to rationalize it, would evaporate.

And that is something very, very difficult to understand. This is something that is very difficult to formalize and which makes the mathematician's work very difficult, it is that his entirely work, is purely rational, if you will...

Neither linear.

Neither linear, absolutely not linear, that is to say that there are periods, often quite long, in which there is a kind of incubation. Hadamard spoke about it perfectly, I will not repeat what he said, but there is an incubation period which is often long, and which precisely requires not to be too fast intellectually, because if we are too fast intellectually, we will easily find reasons why it is not going to work.

But it is often a mistake to believe that because you have to let things evolve slowly. It is necessary to be extremely patient, but at the same time you have to be exactly like a wild beast with watch out, that is to say while being patient, stay fully awake and be able, precisely, if you see something that seems to work, there, you have to jump. Of course, you shouldn't be asleep. It is not the good way to wait for it to fall from the sky like that.

At the moment, what is the beast you are tracking ?

Well, the beast we're hunting down there was what I told you, it's about number theory, etc. This happened with Katia Consani, precisely, it was that we had the impression that there were a certain number of pieces of the puzzle, this huge puzzle that underlies the prime numbers that have fallen into place. So hey, it's a breakthrough, it's a breakthrough, but it's a breakthrough that may seem too naive, etc., on a certain side, but for us who understand most of the elements that make up this puzzle, it was truly an eye opener.

Of course, we are very, very far from the goal. We are still very, very far from the goal, but that allows, if you want to hang on to the whole panoply of tools, concepts that have been developed by surveyors algebraists, so first André Weil, then Serre, then Grothendieck. And so, it gives us a species of work program. And that's great. That's great. You mean that ultimately I would say that it is more fun to have a work program, namely that we will embark in this work program, it is a bit like the sailor who embarks on long journeys, etc., there is more fun to this than to finish something, because it is open, and that, it opens something.

6 Mathematical reality

Many times you talk about mathematical reality, but explain it to us, because it is foreign to us

The problem, if you will, is that mathematics is not something you can understand, or that you learn by reading, without doing it. In this, mathematics are very different from other subjects. But mathematical reality is something incredibly concrete, it's as concrete as chair, if you want, that you can touch. But I will not try to give you examples of arithmetic, because it's too simple. But for example, let's take geometry, if you want, if you take ordinary Euclidean geometry, I can give you a statement. And then you can, after, seek to understand, seek to see if it is true or not. And I give you an example. That's what we calls Morley's theorem. It is a magnificent theorem. It is a theorem that tells you that every triangle generates an equilateral triangle. So how does this equilateral triangle emerge ? It emerges as follows : you take the triangle and you take each angle of the triangle and you cut it into three, three equal parts, okay ? So you get lines like that. You intersect these lines, it will give you three points. Well, these points are the three vertices of an equilateral triangle, regardless of the triangle you're talking about. So it's amazing, it's amazing. So you can tell me "but no, it's not true !". And me, I will give you the demonstration that it is true and we touch the mathematical reality.

So in fact, we also touch it in an extremely concrete way with computers, that is to say that well, we can... we can ask ourselves the question from when we convince ourselves that something is true, but one convinces one thing is true in two different ways. There is an experimental way. It's exactly like in physics, that is to say, well, there can be a statement, I don't know, on modular forms.

But the computer is so powerful, so strong, if you will, that it is capable, precisely, to calculate examples. And if we check on enough examples that it works, we are convinced that it's true. It is not at all the same thing as finding a demonstration. But you have to understand that it's a bit like physical reality and it's a reality that is there, that is tangible and that we can explore. We can explore it directly. We can explore it by thought, it's much better and you can also explore it through the computer. And this reality is there all the time. It is not, how to say, that you can touch itas you touch physical reality. But whatever, it is just as real. It is just as fundamental as that. I'm going back to the example of Morley's theorem, if you want, what will remove the doubt completely, is to give an algebraic demonstration. Such a demonstration exists. There is a purely algebraic demonstration of Morley's theorem.

And then, once we find this demonstration, it's great, that means that first, it works in any case, of course, because it's purely algebraic and besides, not only that it works in all cases, but the geometric figure we have, it will use what is called the field of complex numbers. But the algebraic demonstration will work for any field, so it's a great thing because it means that starting from a geometrical image and a geometric intuition, well, we found an algebraic formulation which is much more general. And so, we have taken a step, we have taken a step precisely by this kind of communication, if you want, between on one side a geometric intuition and on the other side, an algebraic intuition, which is an algebraic formulation of things and that is much more powerful in a way. But there is always a round trip, that is to say that some mathematicians have a geometric view of things, have images, mental images, others have an algebraic understanding of things, that is to say manipulation over time, unlike geometric manipulation, geometric understanding.

7 noncommutative geometry generates its own time... (?)

I would like to dig a sentence that I did not understand : "geometry generates its time".

Basically, the way time appears is that in geometry, when things don't switch, when ab is different from ba, okay, but there is an equation that occurs that I'm going to over-simplify, of course, which is that ab is not equal to ba, but is equal to b, multiplied by a transformed by time, but not by the real time we know, but by imaginary time. So the bottom line is that ab is not equal to ba, but is equal to b times a transformed by imaginary time and then, by what we call analytic extension, well, we come to a real time. So that's the basic mental idea, if you will. So at the start, it was an idea that was developed by the physicists, called Kubo-Martin-Schwinger (KMS), then by other physicists Hugenholtz-Winnink-Haag, and then by Tomita, a Japanese mathematician, and Takesaki.

And then I worked in my thesis with Jacques Dixmier on it and I did precisely, at a moment, a really fundamental find, which was that when we had the impression that this evolution over time depended on a particular choice, a state, I had found that it did not depend on it, by what we call "modulo the inner automorphisms". So it gave all a multitude of invariants of the algebras in question, and it allowed to classify them, it allowed to move forward considerably, but there was how to say... There was a philosophical message that I had felt through my intuition, of course, in this discovery and which I had for years been unable to link to physics, until the day when I met a physicist a little by chance. And he was Carlo Rovelli.

And talking to him, if you want, I realized that... we both realized that he had a similar idea, but he didn't have the mathematical tools to put his frame on feet. And in addition, he did it in what is called the semi-classical framework, that is to say, not still quantum. And so, putting it all together, well, we understood, if you will, that it's this equation, this generation of time by the noncommutative, that probably had a fundamental role in physics, role which is not yet fully established.

8 Make others admit the scope of a discovery

One of the pretexts, how to say, of a book that we have just written, which is called *Le théâtre quantique*, precisely with my thesis professor Jacques Dixmier and with my wife Danye Chéreau, definitely, if you want, to get this idea across to the general public, because well, finally, I hold very much on to this idea and get it across to a message that is fun, if you will, say a message that recounts the episodes in the life of a researcher, the heroin, etc.

And you would say that there, at this moment, we are at a turning point, precisely, on this concept of time.

You know, I don't know, because you have to understand that in the field in which I work if you want, there are two levels. There is a level which is the level of research, how things are going, how they are evolving. And then, there is a sociological level. And the sociological level, it is to what extent these things will pass in the scientific community. And obviously, these are two things that are disjoint, of course, okay. But hey, I cannot say that I have been very concerned during my career with the sociological level. So hey, then it's still a little behind...

You can pronounce yourself.

No, not really, no, let's say it's a little behind, it's true. Sometimes it's exasperating because that we see people who don't understand, etc. We see other theories that occupy the front of the scene a bit indecently, but that's how it is. So, I mean, we often did.

When we work, we have a choice. Either we really work, or we spread to say, or we communicate. Well, it's difficult, it's very, very difficult to do both things at the same time. This is almost contradictory. Why ? Because when you really work, it's an occupation of all moments and we have very, very little free time, actually. For the rest.

One works alone ?

No no no. We absolutely need others. I'm going to say, I've always had collaborators, either in physics, especially, for example, my collaborator Chamseddine and good, right now for number theory, Katia Consani. So, it is essential to have collaborators, precisely, because not only, they bring new ideas, of course, but above all also, I would say, not to be completely alone, because there are obviously very difficult times. There are obviously moments, long, long periods in which nothing happens. And if we were alone, we would have maybe more chances to get discouraged than if you work harder. That plays a very, very important role.

9 The intensity of the courses at the Collège de France

I have an audience, quite varied, very varied, who regularly comes to my classes and there is a very strong contact which is established with the public.

So it's really, with you, the research being done...

Ah yes ! And I must say that, how to say, the leitmotif of the Collège which makes its originality, that creates precisely during the months which are the course months at the Collège, an extraordinary period. Why ? Because good years don't happen all the time. The good years, basically, two weeks before the class, I know what I'm going to talk about. Basically, I have the subject, okay, but I have absolutely no details. And basically what I do is make sure I have two weeks in advance, that is to say that I know roughly what I will do in the next course. And then I work on the next lesson. So I'm looking for the next lesson and it's a period of absolutely incredibly intense research. Why ? Because we often underestimate the fact that in mathematics, if you want to understand things, you can feel like if you have understood. But there is always an extraordinary benefit to go into the smallest detail and, how to say, to exhibit all the facets of a concept, etc. Because...

You are perfectionist

Be a perfectionist, because it will generate things. And I saw thie in the course again this year because it is precisely through all the development of the course, etc., that the object of geometry algebraic appeared. So what I mean is that it's a very intense period, very hard, very hard, because it's physically hard, for example. I often have physical manifestations, if you want tension, during the lesson period, at the start, etc. But it's extremely productive because it's the period when I work the most, etc.

But it's concentrated, you can't do this all the time, obviously. If we did that all the time, we would be completely washed out, raplapla, etc.

Perhaps it is the secret of your profusion...

Yes, absolutely. Absolutely. Because if you want to often, I got myself. I thought to myself "well, maybe if I would have stayed at the CNRS", etc. But it depends. Of course, each mathematician is a special case and if you want, you have to find the equation that will allow him to work. But finally, in my case in particular, the motivation that there is in Collège which consists in saying that each course must be a course on something original that is being done, at the limit which is not published, but is really in the making. But this motivation, I find it great. I find it great. Years ago, indeed, when it was much more difficult or much more annoying, but that's how to say, it's very, very well dosed. That is to say that if you want, it's not as if I had 6 months of lessons, that would be too much But this course period which lasts between 2 and 3 months, if you want, it's perfect because it's very well dosed and it requires a certain discipline. It forces precisely to stay all the time, all the time a little worried, a little on the wire. And never, never say to yourself "Good, bah, ok, now I'm old, etc.". We can't think this, we can't. Because, well, of course, I will have to stop at some point. But we absolutely must not do that because if we do that, we will not be able to do the course.

10 Doubt

Doubt, you know, in the book, the book with Danye Chéreau and Jacques Dixmier, begins with this exhibition in Venice which is called *The Praise of Doubt*. Doubt is present all the time. Doubt is present at four in the morning, when we wake up at night and say to ourselves "Couldn't I have make one error there, etc.". And let's start checking. So the doubt is present all the time, but I will add nevertheless that from time to time, from time to time, it seldom happens, but it happens, fortunately, there are times when doubt evaporates. And precisely, I doubted for years and years that the space I had found in 1996 for prime numbers was the right one. And what we found recently with Katia Consani dispels this doubt. It is surely the right space, therefore, it is wonderful that from time to time, there is, if you see like that a way to remove the doubt. So of course we can say, "Okay, well, you have raised the doubt for yourself, raise it for the others.".

Like the goal of all these developments, if you will, it's very, very difficult. Of course, this is not immediate, but for us it is very, very important to remove the doubt, even punctually, if you want, like that, on such an important notion.

An interview of Alain Connes
by Lucia Dora Simonelli, physicist
at ICTP, Trieste, Italy
2017.03.02

I'm here with the Professor Alain Connes at ICTP ; he's visiting in conjunction with the workshop on Non-commutative geometry.
So the first question would be a sort of an ICTP specific question in this : if you had advice to give young students who want to study math, in particular, students from developing countries, what advice would you give to them ?

It's difficult : you know, to study maths, of course, they are challenges and somehow, I mean, I have always thought that the key step in studying mathematics is to understand that, you know, you don't learn mathematics. You make it, you do it and until you are really able to take a problem and solve it by yourself or try to solve it by yourself, you are not doing mathematics. Because learning maths, they are not topics that you can learn, there are some scientific topics that you can learn, but this is not the case for mathematics. For mathematics, you have to do it yourself. So that would be the best I could say, in a very short time. So it's really like ..., for instance, you know, to give a comparison, if you try to become a pianist by reading books, it's the same story, you have to practise. The practise is far more important than whatever, reading books, and all that. In that way, it's a very democratic subject and there is a key step also, the magnificience of this key step is that when a student finds a mistake of the teacher, because he is able to think by himself, and find out that he is right, and the teacher is wrong, this is something which is very important in mathematics, and which is different from other topics. Because other topics require so much knowledge that, somehow this will not be possible to do it for a beginner, but in mathematics, it's possible.

So next question is thinking about this quest to find a unified theory for the universe, I think it's interesting the impact that maybe this has had on the interaction between maths and physics. And so maybe some have prospective that at one time one field set another, but it seems that now there's sort of a symbiotic relationship and I wanted your perspective on the evolution of relationship between maths and physics.

Yeah, I think it is a very delicate issue in the sense that there is one Graal, one problem which people are trying to solve which is the Quantum Gravity, so we know that quantum gravity (?) exists, we know it's quite difficult. But in a way, for mathematicians, at least as far as they are concerned, the issue is even more important for mathematics in the following sense : for instance when Riemann gave his inaugural

Transcript of a video that can be watched here : https://www.youtube.com/watch?v=XxtnTtvlhMw talk, he was very clear on the fact that the hypothesis that he had for Riemannian geometry, the hypothesis for geometry, would not hold at the very very small scale. And he was so lucid and precise, that he had already foreseen developments that would come much later, and in particular in non-commutative geometry, because of the fact he wanted the notion of (...) or the notion of light-ray no longer make sense in the very very small whereas there were these notions that were crucial in his definition of geometry. So there is a symbiosis, but there are also, I would say, deviations, and what I hear for instance in some talks, I hear deviations because some people just want to change the rules of physics (to do what they want ?...).

So I think we must be very careful. And at the same time, what I would say, is that there is an intermediate goal to complete geometry, and that goal is very precise, it's to understand the effect, the impact on the notion of geometry that the experimental physics has provided for us in one century, where the inward-band trip, that began at the end of the XIX$^{\text{th}}$, with the discovery of the electron and of radioactivity. And there has increased our perception of the small structure of space-time by a factor of ten to the power eight in the century and that has implications on the geometric model we have of space-time and that application will be fully understood in non-commutative geometry and what happens is that space-time is no longer a purely continuum but it's a mixture of the continuum and the discrete. And so this is a lesson that was understood, it's a lesson which is very strange, that forced to change the riemannian paradigm. But this change in riemannian paradigm, of course, Riemann couldn't foresee it because it involves the quantum mechanics. So the new paradigm on geometry is very close to the riemannian one but there are nuances, and those nuances come from the quantum, they come from the formalism of quantum mechanics that has been discovered by von Neumann in 1960s. And it turned out then that the idea or the notion we want for geometric space becomes more natural, and is more easy to understand in the quantum formalism.

So you describe that there is not just the immensity of the universe but there are also these very small scales. How would you define a point ?

Okay, that's an interesting question because we can ask within a primary approach to that how do we define a point, and that question that is simple is how do we communicate with extraterrestrians or possible other civilizations the place where we are. Well, if I tell you that we are in Trieste, and so on, well, that don't tell because first of all, these people don't understand what Trieste, and then, there will have people who know general relativity, to whom we just have to give our coordinates in a coordinates system but that's also foolish because which coordinates system do we take, which invariant way do we have, and it turns out that what I was talking about before, this reunderstanding of duality has its answer that is exactly provided by the formalism. So the first question becomes how do we communicate this space in which we are, just global, not by giving a picture, how do we communicate the space in which we are, and second question, how do we define where we are in it. For communicating the space in which we are, it turns out that the best way is to give the music of the space. So if you take a shape, this is a well known metaphor if you want, which goes back to Mark Kac, so if you give a shape, like a drum for instance, of drums of various shapes and so on, it turns out that each drum has its shape as a special scale, a musical scale, and from which frequencies are depending of the shape.

And it turns out that if you want to give invariantly the space, you have to give a list of quantities which are assigned to this space in an invariant manner. Now the scale of the space is invariantly defined ; you can rotate the space, you can do whatever you want, and you will not change its scale. So this an invariant of the space. And it turns out that Helmholtz found in which is called Helmholtz's equation because he use the scale of the space, it turns out that there is a small refinement in this equation, Helmholtz was taking for line element the squareroot of the Laplacian, and you have to divide this basis by the Dirac operator, but this is small nuance. And when you know this small nuance, then you can actually reconstruct the space but you need to know a little more. You need a little more than the scale of the space, you need to know precisely what are the points. And what are the points ? Each point is defined by a chord on the scale. A point in a space, technically speaking, how do you specify the point ? So technically speaking, what you do, you take what are called the eigen vectors for the Dirac operator, their inceptions are bounded in the space (...) and you evaluate them at the point. When you evaluate them, you cannot just give a number, so you get a number in terms of the metrics, which are scalar products of those various cells, at the point, you use the metrics and it turns out that modulo the invariance, this metrics is exactly what you need to know the point.

So the picture, the mental picture is that by a misunderstanding, the space is understood by a musical scale and possible chords. And the possible chords are the points. So in a way, what happens is that you reconstitute the space by a kind of Fourier transform. And I believe that this is exactly what the brain does when you see, because when you see, there are photons which are coming in a moment because when you see, you have the photons at space eigen scale and the brain reconstitutes space like we are used to see it but what is even more important is that this is exactly the way we perceive the distant universe. Because we perceive the distant universe by looking at spectra of galaxies, spectra of stars, spectra of nebulae, and it is in this sense that the spectra can calibrate information that we receive from far stars.

In this formalism, we find out that not only, it's useful for microscopic distances but it is also reshuffles and changes the point of view on the large distances, but in a way which is perfectly coherent with our perceptions of the universe. For instance, typically, what happens is that we know that things are very very distant, you have to remember that there was some time where people didn't even know that there were things outside our galaxy, ok, it took very brave astronomers to find that. But now, we know that things are very very very distant, just because of the redshift. And this is again the spectral prover.

And here there's a concept of distance or unit of length in term of the wavelength

Sure. That's also a very important step, which is so much fun to explain because it relates to very concrete stuff. So the story starts in France more or less, during the French Revolution. You see, there were, more or less, there was a unit of length per city. There were on thousand of unit of length. That mean that where people were selling for instance tissu, travelling from one place to another, they had to measure with respect to the unit that was at the entrance of the village, of course *(laughs from Lucia-Dora Simonelli)*. Revolution was an idea of course to unify things, and they had grand purposes and all that, so they decided to... they had very good scientists, they decided to try to unify the system, by defining a unit of length. So what did they do. They took the largest available object, which is the Earth, and they defined the unit of length so that, when you multiply this unit of length by 40,000,000, you obtain the circumference of the Earth. So this is what they tried.

And in order to... Of course, they couldn't go to the pole, to measure the entire meridian, they measure angles between the stars they pointed with their instruments, so they only necessitated to measure some angular portion of the meridian. And they choose the angular portion which was between Dunkirk that is at the North of France and Barcelone which is in Spain. And in 1792, so this was during the fool period of Revolution, they sent two people, Delambre and Méchain, were sent out, to do the following ; the idea was that they would first of all have a base, what we call a base. So they had lined down on a sufficiently long distance some bars, if you want, and they had taken that as a base. They were only measuring angles, which is a very smart idea. They were putting telescopes on top of hills and measuring angles, and by doing triangulations, they were comparing the base with the distance between Barcelone and Dunkirk. And out of that was defined the unit of length, which was actually a metal bar. It was a very interesting story because there were all sources of development behind this story because one of the guys, I think it was Méchain, had to make measurements in Spain and of course, so he was measuring angles, by putting the telescope on top of the hill, and of course, he had a lot of troubles because there was a war, between France and Spain, at that time, and he had to explain to the Spanish army that, by putting his telescope on top of the hill and looking in his telescope, he was not a spy, he was trying to define the unit of length *(laughs from both)*.

So there were a lot of anecdotes interesting to develop, I love the details of this stories, I don't know why. And then what happened was the following. This unit of length was actually deposed near Paris, and when I was kid, I learned that "The unit of length is the meter which is deposit in Pavillon de Breteuil near Paris.", and so on.

So I was thinking and I'm sure many people were thinking "This is not very practical" because if you want to measure your bed, of course, they made duplicates of this. So that was the situation at the time. But then, some very interesting event happened. So there were periodic meetings of the metric system people. And this meeting has been carried on very periodically for years. I'm not sure the period was one year. But around the 1930s, they noticed that actually, the platinium bar, defining the unit of length was changing length. And how do they notice that. They noticed that wy actually measuring its length very precisely, and by comparing it with the krypton wavelength for a specific transition. So that was very bad, and gradually, they took the right step. And the right step of course was to take this wavelength, as the unit of length. That took some time. But what is very interesting to know is that now, there are instruments which are so common, you can buy them in a shop, and these instruments are based again on the wavelength. The element that is used is no longer krypton. It is cesium because cesium is very easily available and sold, and moreover, the wavelength of cesium which is used is a microwave. So it's like when you put something in the microwave oven, it's a wavelenth that is of the order of 3 centimeters. And it's an instrument which allows you to measure length up to 12 decimals, so I mean, it's abolutely incredible. And this is now what is taken as the unit of length. Of course, people will tell you it's not a unit of length it'a a unit of time but because of the constancy of speed of light, speed of light has been fixed to a very specific number. So things have evolved and now, what you see from that is that there was a complete change in the paradigm because the unit of length is no longer a localized object, which is somewhere, but it's a spectral data. And it turns out that the new paradigm which comes from quantum mechanics, which is the paradigm of non-commutative geometry, which is called spectral geometry if you want, is exactly parallel to this change of paradigm in physics. So it's very concrete. It's something which is very very concrete and the enormous advantage is that if we had, for instance, to unify the metric system, not on Earth, but in the galaxy, for instance, if you tell to people, "ok, come to Paris and compare your unit of length to the unit of length we have defined there..." *(laughs)*, they would laugh at you, they would roar, because they would say "we have our unit of length", whereas if you tell to people "take a chemical element". Of course, cesium is a little bit complicated because...

For your business, it... maybe, you need something very common...

Yeah, exactly, like helium or hydrogen. I would vote for hydrogen, because hydrogen is essentially present anywhere, whereas cesium or heavy elements of that kind, in fact, one has to know that they only come from, not only from supernovae, but from very, very exceptional supernovae. So their abondance in the universe is not so clear, but if you take hydrogen for instance, there are spectral rays of hydrogen which are very precisely defined, there would be very specific patterns, but then one would have to find hyperfine splitting, because the advantage of hyperfine splitting which is used for cesium is that an hyperfine splitting is a difference of energy which is very,

very small, and that would in the inverse law, when you pass to the wavelength, it will generate microwaves, which is much more practical, whereas if you take a huge difference of frequency, like for a transition, you will get a very, very tiny unit of length and that would not (?) be good. What I am saying is that when you communicate with people, by sending a probe, and if you are able to tell them what is your unit of length, this is marvellous. And you just send a copy of the spectral rays of hydrogen and you explain which one you want to find out. I mean this is very simple, and if they are smart, they will understand, whereas if you do otherwise, it would never work.

In this description of the fine structure of space-time, you describe it in term of the spectrum of an operator, which allows...

It's a little bit more complicated, as I said, you know, of course, the spectrum of the operator gives you the unit of length,...

Does this allow you, in a way, to combine a discrete concept with a continuous concept ?

Well, what allows to combine the discrete and the continuum is the fact that, essentially, it's a mixture of the discrete and the continuum and what the discovery of the experimental physics have unvailed, over the century, is exactly what is the structure of the discrete space. So at first, the discrete space, with my collaborators, Chamseddine, and Walter Van Suijlekom and Mukhanov, what we found, at first, we were proceeding with a bottom-up approach, namely, we are taking from experiments, and trying to fit with what was going on, etc., and gradually we found what the finite space should be, but in a recent work about 2 or 3 years ago, with Chamseddine and Mukhanov, we were very amazed because we were asking a purely geometric problem, which was motivated of course by non-commutative geometry, but which was totally disjoint from the physics and the Standard model, and so on, and by developing this problem in dimension 4, we found exactly the same finite space in the same algebra, that was put in by hand before. So, we believe that we have a piece of the truth.

But naively, why is it important to include this discrete concept ?

Why naively, is it important, this is easy to explain somehow, but I need a piece of paper, *(He takes one)*. It's very easy to understand. You see, why is it important to have this discrete piece. It is that the most obvious problem you have, if you don't have this discrete piece, is that the Higgs boson, the Brout-Englert-Higgs, I knew Brout very much, I mean, he died just one year before the particle was discovered. The particle was discovered, we know it's there, but it doesn't fit with standard geometry, why ? Because in standard geometry, if you take a function on a space, you will differentiate it, and you will get what is called a gauge potential, a one-form, okay.

Why ? Because the differentiation depends on the direction in which you differentiate, so this is why you get something which is called spin 1 if you want, which depends of the direction. But the Brout-Englert-Higgs particle is a particle which is spin 0. So it doesn't depend of direction. So you wonder how you can obtain geometrically a particle of spin 0. Now imagine that instead of having just this manifold, okay, there is a discrete element, the discrete element is just an element that tells "am I on the top or am I on the bottom ?", so now I have more information, I know if I am on the top or on the bottom. And I take a function. This function will have a development here, and it will have a development under *(showing with his hand the two faces of the sheet of paper)*. They don't have to be the same, so I can differentiate my function up *(AC makes his hand turning against the top of the sheet)*, and I can differentiate it down *(AC makes his hand turning against the bottom of the sheet)*, but I can also take the finite difference across *(showing the difference on the slice of the sheet)*. And the finite difference across, it does not depend on which direction I am taking. That's the boson of spin 0, and that corresponds to the Brout-Englert-Higgs boson. So the Brout-Englert-Higgs boson was a completely clear unmistakable sign, on a discrete structure, which was present.

And I knew Robert Brout very much and he was very interested, of course, by this understanding, which is the understanding of why, if you want, the experimental fights that physicists had, because the Brout-Englert-Higgs mechanism, it was obtain after years and years and years of thoughts, of how to give masses to particles. So all the masses of the particles actually come from this mechanism and what you find out in this model we have developed is that in fact, the main ingredient which is the metrics of masses and mixing angles, and so on, of the particles, is in fact exactly the line element for the finite structure. So the line element for the finite structure contains exactly this information, which means if you want that in this model, you have a mixture of the continuum and the discrete. But the discrete contains the information about the mass and the mixing angles part.

Thank you very much for your time and the exposure.

Transcription of a video by Alain Connes at the Collège de France, Language and Thought Symposium (October 2018) (Denise Vella-Chemla, 19.1.2019)

So that's it... thank you. So I'm going to try to lighten the mood, because... so I'm going to bounce on the introduction of Jean-Noël Robert, to point out a variant of "I think therefore I am" which I think, if you will, is actually the best graffiti I have ever seen, it was in Jussieu's restroom *(laughs)* and there was a sign that was put... if you want, which said "Please leave these toilets as clean when you go out as they were when you came in" and there was a clever little guy who wrote below : "I think therefore I wipe" [1] *(Laughs)*... Okay !... So I'm going to talk to you about mathematical language and so if you want, my talk will be in fact both an introduction to mathematical language and at the same time, a reflection on mathematical language. So this language, the mathematical language is a coded language, it is a coded language at the outset which means that there are certain habits. For example, in mathematics, if you like, it's traditional to call the unknown x. This is how in a school, a professor had posed the following exercise : "we give a triangle, the triangle is an equilateral triangle and, well, what do we know ?"... We have Pythagoras' theorem : Pythagoras' theorem says that the square of the hypotenuse is equal to the sum of the squares of the 2 sides, so you have to understand square as really a square, so, here, what do we write ? We write that we have 3 squared, it's worth 9, 4 squared it's worth 16, 9 plus 16 is 25, so we guess... the answer. Hence if you want the teacher's embarrassment when he saw the answer given to him by a student *(AC shows a comic drawing, "Find x", Pupil : "it's here", with an arrow on the x letter, bursts of laughter)*. Okay ? So obviously, it was... it was an unstoppable answer, eh, it is unstoppable, the professor cannot say "it's false" so... so, to get back to the serious stuff, because we're going to get back to the serious stuff now, so my presentation will be divided into 4 parts.

In the first part, I will give you elements of language. So I'll talk to you about geometry, about theorems, I will explain to you what a lemma, a demonstration, a counterexample, a conjecture, I will speak to you about algebra and the great difficulty of this part of the presentation, it will be not let myself be overwhelmed by the demonstration, therefore, because what interests us is language, not, if you will, the semantic content, it is the language which will interest us therefore... And yet, I can't explain it all to you in abstract, I have to explain it to you on an example because that, I will say, we will see that it is extremely important in fact to have a support which is a specific example.

In the second part, I will tell you about a book, which is a book by Hans Freudenthal which is a mathematician, this is the book *(AC shows the Lincos book)*, and it was a mathematician who understood that in fact, there was a particularity, a specificity of mathematical language, which I will not return to, which is that it is without doubt one of the only languages that is not entirely self-referential. You see, when you take a dictionary, the dictionary defines words in relation to other words. But in mathematics, it is impossible, and this is what Hans Freudenthal did in the Lincos, it is possible to build mathematics, gradually, starting if you want to... one impulse, two impulses, all that, it will mean integers, etc., etc. And he understood that, in fact, if there was a possibility of communiculating with an extraterrestrial intelligence, well, we would have to build the language entirely, it is because there, it is really a language, starting from mathematics. And that's what he did. So there are some extremely interesting reflections, but what we will see, what we will see, which is also quite extraordinary, is that in fact, the universe communicates with us, it communicates in mathematical language and I will explain to you in what form it communicates with us.

So the third part will be, it will come from the fact that the mathematical language of course evolves and very gradually, the paradigm that was at the center of mathematics during..., until last fifty years, which was the sets paradigm, has been replaced, very gradually for the past fifty years, by a new paradigm which is thecategories paradigm. And this replacement in fact, at the level of language is extremely important and in fact, it is thanks to this replacement that an extraordinary concept was born in the hands of Alexandre Grothendieck. This concept, this is the concept of topos, and it is a concept that perfectly illustrates the fact that precisely mathematics are not at all, if you like, confined to language, confined to calculations or things like that. In fact, mathematics is a factory that manufactures new concepts and for showing you the richness, the variety if you want of these concepts, the concept of topos is so... powerful that in fact it gives nuances to the notion of truth and we will see at the end of my talk that the concept of topos allows for example to define in a perfectly rigorous way what it is to be at three steps from the truth, four steps from the truth, etc. So it's something extraordinary because that it shows that mathematical language if you will, and not just mathematics but the mathematical language touches... in fact... has a philosophical scope which is far beyond what is normally accepted by the general public. The general

1. *French pun : Je suis, I am, j'essuie, I wipe.*

public confines mathematics to calculations or geometric figures but it is very far from being there. In fact, if you will, most of the concepts really important have a mathematical origin and have a precise mathematical formulation.

So, so, this is the plan, this is the plan, so what I chose to do, like I said, is to go of a concrete example. So this concrete example is a theorem, it is a theorem that has been somewhat found by chance, by Franck Morley. Why was it found by chance ? It was found by chance, it was found in 1899, Franck Morley was not concerned with the question that we are going to see, with the statement that is here. Actually, he was looking for a geometric problem that was much more complicated on cardioid, etc., but basically, he fell in his path, he fell on a fact, that we are going to consider as a fact, and that is a very striking fact, quite extraordinary, which says the next thing : it says that when you take a triangle, any triangle, the triangle is absolutely whatever here, and what you do is, you divide each of the angles of the triangle into three equal parts, therefore, these are called the angle bisectors, and you intersect these bisectors 2 to 2. What the fact says, what Morley's theorem says if you like, is that you always have an equilateral triangle here. So this is something extremely simple, extremely striking, and the reaction of a surveyor, it is doubt. That is to say if you are a surveyor and you are given a similar statement, well, the first thing you have to do is to doubt it. So you doubt, you say, "Boah, I say "it can't be true !". I'm going to make you a triangle that won't work. Okay ?". Well So there, what will start in the brain, maybe Stanislas will tell us things much more precise, but it is the visual areas, that is to say that will make the surveyor doubt, and he will try to build other triangles and he will look at what is going on. So in fact, we're just going to look, we're going to look, we're going to build other triangles, you see. We build another triangle, well, it still seems to be true, what *(laughs)*.

Okay ? So we continue, we continue like this, then you see what I'm doing here, I'm trying to do an extremely weird operation on the triangle, I'm going down the top that's there without moving the vertexes which are there, and normally when we do that, it is an affine transformation, so that does not preserve the lengths at all, so it is very surprising that despite this, the triangle between remains equilateral. So we continue like this, we continue, we look for all kinds of examples, we try to flatten the triangle, to the..., we do the same operation, well, it seems to always work. So at the end, after a while, when we have done this enough times, well, the doubt begins to dissipate, and there, we have to set off because it is not enough to have taken examples, of course, examples never demonstrated a theorem, so the doubt started to dissipate and we're going to leave, we're going to set off, we're going to set out for the demonstration. And then, in a demonstration in general, the demonstration is preceded by stages, these stages are called lemmas, okay ? So it's no coincidence that the name of lemma was used when in fact the astronauts went to the moon, they used the term lemma, and it's exactly in the same sense, that is to say that the lemma is not, if you want the result, but it is what allows you to move forward and to go to the result. So in mathematical language, the lemma has an extremely precise meaning : in general, the statement of the lemma in itself is not enough, how to say, convincing, so that it makes a theorem, it's just a small result that allows you to move forward, it's not really a theorem, Okay ? So here, we are going to see a first lemma and we are going to see how, precisely, this lemma is going to allow to better illustrate, better, if you want, to initiate in mathematical language.

So what does the lemma say ? The lemma says the following : I must not get lost in the demonstration so I will go very fast, he says if you look at the rotations, relative to the vertices, so the center of the rotation is the top, but the angles are double, and you make this product, you make the product $R_A.R_B.R_C$. So you will see, there, there is a peculiarity of mathematical language, is that when the product is well made, one starts by applying R_C. So it will seem weird but it comes because of the notation in mathematical language, because when we take a function of x, by example $\sin x$, the x appears afterwards, so that's why when we apply $R_A.R_B.R_C$, we will first apply R_C and R_B and R_A. Well. So, then, what does the lemma, the little lemma say ? It says that if I make the product of these 3 rotations, I get what is called the identical transformation. So now, what is absolutely amazing is that now we will see that, we will act. At the beginning, if you want, there was a theorem, there was a fact, we were exposed to this fact, but now we're going to start acting, to prove this lemma. How do we act ? Well, that's what we calls a group that acts on a whole. But what does that mean here ? What does that want to say here ? It means something incredibly simple, it means that unlike current language, when we will demonstrate that the product of the 3 rotations R_A, R_B, R_C is identity, well, we're just going to have a very small operation to do, which is the operation to play with parentheses. So you see the group that is going to be in action here, it will contain the rotations, but it's going to be in fact the group of transformations of the plane which preserves the lengths. And then, the thing essential is that in fact, if you take the rotation around point B which is here, well actually, we can break it down into two things : symmetry around 1, then symmetry around 3. You see, if you make the symmetry around 1, for example, if I take the point which is here, and the symmetry around 3, that will turn well from the double angle, which is there. And that works in general. So what is the result ?

Why is the demonstration so simple ? Well, because the R_B that is there, I can, as I told you, as I showed it, I can write it as s_3s_1. The others are s_1s_2 and s_2s_3. And now I can play with parentheses. It's not at all the same as in everyday language, of course. So the role of parentheses is not at all the same. In a group, we can manage with the parentheses as we want, we can do a set of parentheses, and with this set of parentheses, we immediately get the result since you see, now, I'm going to replace this product with a product where s_3, s_3 will be contiguous, s_1, s_1 will be contiguous but since s_3 was a symmetry with respect to a straight line, s_3s_3, it is worth 1, s_1s_1 it's worth 1, there are only s_2s_2 and s_2s_2, it's worth 1. So we're done, okay ? So here is the demonstration.
....
So you see the role of language, the role of writing, there, in the demonstration. Then another little lemma, okay, but that will show you how the same type of action is absolutely crucial if you want to demonstrate, it is a lemma which will also make us progress a lot, and which is the following, and which says how we will find the 3 vertices of the Morley triangle, for example, the point P which is here which is at the intersection of these 2 trissectrices. Well, what does the lemma say ? We will have these 2 little ones lemmas. What does the lemma say ? It simply says that the vertex P of the Morley triangle is simply the fixed point of the same thing as earlier, but I put powers 1 / 3. The fact of having put powers 1/3, that means that the angle by which I move is going to be a third, a third of double the angle in C. Well, that's exactly the angle that is here. What does point P do ? That's the action now, that's it, now, we're going to be in action. Well, if we do $R_C^{1/3}$, converting P to P'. And then if we do $R_B^{1/3}$, afterwards, well, we bring P' back to P. So we have the fixed point. So now we start to control things. We start by controlling things because we have 3 symbols, R_A, R_B, R_C, and we control the vertices of Morley's triangle thanks to these 3 things, okay ?. But we are far from having finished and when we are in this situation now, what do we see ? Well, with a little thought, we realize that the two lemmas that I have explained to you, the two lemmas that I have explained to you, in reality, these are really geometric lemmas, that is to say they are lemmas that are going to be still true in non-Euclidean geometry.

What is non-Euclidean geometry ? It may seem to you very complicated, but in fact, in general, if a result is true in non-Euclidean geometry, it is going to be true for spherical geometry. What is spherical geometry, well, it's geometry of the Earth, if you will, but made perfect, a perfect sphere, and in which the lines are the large circles. So, the guess we can do at that time, well, we can say "well, maybe that Morley's theorem is true for spherical geometry. "So it's the same, we're going to use our visual areas this time, and it's going to be more difficult, because now we can't take a straight line, I mean we can no longer draw on a sheet of paper, we have to show you a figure which is more complex. So this is what we do, we try, we do like a while ago, we look, "Ah, it still seems to be true ? !". You see ? I took an extremely special triangle, a extremely particular type of triangle, I took a triangle which had a vertex at the North Pole and whose the base was on the equator. It is a very special triangle because it has 2 right angles, this angle is right, that angle is right and the sum of the angles obviously does not equal $\pi = 180°$. So you see, we try, we try, we look, we look, well, it still seems quite true, I mean, it's really extremely amazing, really extremely amazing, so we use our visual areas, etc., we try, and then there, then there, we can ask the question on the computer.

So here we go... we see the question that really arises "is it true ? Is it true ?". We talk to the computer, Gérard will talk about it much more precisely than I do, we communicate with the computer, it's not very difficult for spherical geometry, it's very very simple, to tell the truth, the spherical geometry is even simpler than ordinary geometry, so we talk to the computer, we asks it the question, and after a moment, the computer tells us "no, that's not true !". Unbelievable ! The computer calculates the difference between the length of the side that is there, since the triangle is isosceles by definition, so these two lengths are equal, but this length is not a priori the same. If it was the same, it would be an equilateral triangle, equilateral, that means that there are the 3 sides which have the same length, so in fact, we look and amazing ! That's the curve of all, all the differences. At first, this difference is almost 0, but you see, it's 0 to 4 decimal places. So it means to say that if we trusted our view, we would have believed that the theorem was true, okay ? In fact, it is not true. It is not true, this is called a counterexample, we made a conjecture, we have a counter example to this guess, and now what is happening ? Because we have a counterexample to our conjecture, that forces us to completely change our strategy for the demonstration.

Why ?

Because if the conjecture had been true, if the theorem had been true in non-Euclidean geometry, the evidence would have been entirely geometric. The fact that the theorem is wrong in non-Euclidean geometry tells us that there will now be a total change of scenery and this total change of scenery, that is going to be the transition to algebra. Algebra was born from a... how to say ? was born from an heresy. Algebra was born out of a heresy a very, very long time ago, long before Jesus-Christ, heretics made the next thing : they added lengths with surfaces... and then surfaces with volumes. This is completely crazy ! You see when at the start, I told you about this little student triangle, when we write the rule of Pythagoras, I told you "the square on one side is equal to the sum of the squares...", that, the dimensions are preserved, because we add between them surfaces, we add between them squares, but there is a very old mathematician, very old, who had the idea of posing a problem, I believe it was among the Babylonians, in which he posed a relationship between surface and length, it's amazing, so, at that time, algebra was born. So it was born out of this heresy, which was to add between them quantities which absolutely do not have the same dimension, which a physicist never would do of course, which is to add lengths, areas and volumes. So then, now... So, here is the demonstration. What does the demonstration say ? It says that now Morley's theorem has nothing to do with a geometric statement for non-Euclidean geometry, it has to do with a statement of algebra, a statement of algebra that is so self-explanatory that it can be given to a computer. When, if you will, a fundamental difference, between Morley's theorem at the outset and the statement that is here, no matter what, I said I don't want to get lost in the technical details, etc., what is the fundamental difference ? The fundamental difference is that when you are in front of a geometric problem, we can dry, we can dry indefinitely. When we are faced with a problem of algebra like that, not only we are not allowed to dry because we have to do maths, but we can delegate the problem to the computer, this is called formal calculus, and this formal calculus is so powerful if you will, that in fact it can do infinitely more complicated calculations than that one, I remember once, a long time ago, in the early 2000s, I had delegated a calculation to the computer that took a whole night, I wanted to demonstrate that a certain product was associative, and well, the next morning, the computer had done the calculation, so it's absolutely phenomenal the power that computer have for making extremely complicated calculations. So when we did that, if you want, we can say, "Okay, well, we gave a proof of Morley's theorem, so if you want, we arrived. The lemma arrived on the moon, okay ? The lemma has arrived on the moon, okay ?. It was the third lemma".

But... now what is the difference ? There is a difference which is absolutely crucial, the crucial difference is the generative power of mathematical language. Because now, once we formulate the result in the form of the lemma, the last, the last lemma, which is purely algebraic, well, Morley's theorem has meaning on any field... Of course, I didn't tell you that there was a secret behind the lemma that I gave here, which was that this lemma, it applies to a field, what is called a field which is the field of complex numbers. The field of complex numbers, it's a miracle, it's the fact that you can, we're used to rational numbers, for example, that you can add, multiply, etc. We are used to real numbers, well, there are an extension of real numbers, which is wonderful, and which makes it sufficient to actually add a root of an equation which is the equation $x^2 + 1 = 0$ so that we can solve all the other equations. This is what we call the field of complex numbers, and this field of complex numbers, well, we learned it in... I think we learned it in the second, I learned it in the second, in physics, to do electromagnetism, and in fact, what is extraordinary in the field of complexes is that a triangle equilateral is characterized by the property that 3 points form the vertices of an equilateral triangle if and only if $a + jb + j^2c = 0$ where j is a cubic root of unity. So now, what's wonderful, is that we have a Morley theorem for any field that contains a cubic root of unity. So these fields abound, for example, you can take the integers modulo 7, the integers modulo 13 or the integers modulo 31, these are fields, this is called Galois fields, and they all have a cubic root of unity, so there is, for all of them, a Morley theorem that applies ; we would not have been able to imagine, if we had not had this path, through geometry and algebra, which led us here. Okay ? So, this is where we are, this is where we are. So now, yes, of course, you have to let me tell you, yes, anyway, that there is, in mathematical language, a constant use of words of ordinary language. We use, in mathematical language, words of ordinary language, and there is a rather provocative sentence, which is made through the words of ordinary language, and which is called "Disintegration of atomic measurements on nuclear spaces". So if you want, this sentence is all the more ironic as the inventor of the nuclear space denomination, first of all, it is ironic because a mathematician who knows the meaning of these words will tell you "it's trivial." So triviality, it's common, it's a kind of mathematical jargon, which is for something which is true, but which is irrelevant, okay ? So... But what is really ironic in this sentence that I have given is that the inventor, the inventor of the concept of nuclear space is a formidable mathematician, well known, Alexandre Grothendieck, but who spent most of his life being an activist anti-nuclear. *(laughs)* So, then, he is... there is always, he had an absolutely extraordinary gift for finding the right terminology. So, so, I'm coming to Lincos now, so, I'm not going to paraphrase what Hans Freudenthal says, his book is extremely precise, etc., he wrote an introduction which I recommend that you read, in which he makes reflections on mathematical language and on language in general and it explains a point, which is a delicate point, which is a point which is not at all obvious, which is that in mathematics, we use variables, well I told you about x at all the time, and it's a bit the same when we talk about everyday language. But, in mathematics, the variables have a specific meaning, and in addition, for example, there is a notion of a generic object, that is to say in mathematics, we can speak for example of a generic triangle, etc., there is what is called a generic point, when we talk about a mathematical set. So what's really amazing is that if.... So... Hans Freudenthal discusses in great detail what is happening in ordinary language, and the fact that in ordinary language, of course, there are variables : for example, we say a door, we said a goat. You will see that I am not taking the example of the goat at random.

Okay, but then of course, if we were to ask the question, it would be extremely difficult to describe, a generic door, or describe a generic goat. Only a great artist got there for the goat, and I guess you know the work in question, it's called, it's Picasso's goat, and it has this property, this goat, this incarnation of the goat has this extraordinary property, how to say, that it concretely abstract the properties of the goat through an artistic work. So, of course, in mathematics, all that, it has a precise meaning, it has a much more precise meaning, and good... to return to communication with an extraterrestrial intelligence, we tried, we tried, we tried to communicate with a possible extraterrestrial intelligence, by sending the probe, for example, by sending the Pioneer probe in space, and on this probe, we gave a certain amount of information, well, for example, we gave the Sun, finally, we gave the position of the planets, etc., etc., but, hey, is it included in what we gave in this probe, that we gave the aliens the possibility of replying us ? I pretend there is another way, it would have been to send them a little triangle from Morley *(bursts of laughter)*, okay ?... And imagine that they respond to us like that... Well here, there, you would know that... not only they did understand us, but also that they are intelligent people, etc., etc. So in fact, well, in fact, this mean of communication I am talking about is not very practical because it means that they should have received the probe as a physical object, and that they should return us a physical object. This is not how we communicate in the Universe. We know well that the Universe is written in mathematical language, but how does the Universe communicate with us in mathematical language ? So there, you are going to be really surprised, because the way the universe communicate with us, we'll see, it's... you know, we've looked a lot... we've looked a lot at... how to say... at how to label, label objects, etc. And while trying to label the objects, we felt gradually on the right idea. And the good idea of labeling objects is called barcodes. What I'm going to explain to you now is that the Universe communicates with us through barcodes. And in addition, these barcodes are mathematical objects. And I will tell you, I don't know how much time I have left, well, but I'm going to... And I need to tell you about the topos as well. So... I'll tell you about it. Why ? Because what is amazing is how it has been discovered. First, it was discovered in physics. In physics, there is a German optician, who was called Fraunhofer and who had the absolutely brilliant idea of looking at the sunlight, the rainbow which comes to us from the sun when we pass the sunlight through a prism, to look at it with a microscope. And he noticed that there were black lines. There were a number of black stripes. So at the start, he must have thought his lens was dirty, etc. Then, well... And finally, during his existence, he found 500 black rays. Then there were physicists, I think it was Bunsen and Kirchhoff who succeeded in heating bodies like sodium to obtain rays which this time were bright lines, not black lines, on a black background, which corresponded to the lines of the spectrum of Fraunhofer, that came from the sun. And they realized, if you will, that each chemical element, had a barcode. So each chemical element, for example hydrogen, etc., had a barcode. Except that... there was a barcode they couldn't find on Earth. There was a barcode missing on Earth. And so, as good physicists, they gave it a name, they called it helium.

They said "there is a chemical body, which is not present on Earth, which is in the heliosphere and which is called helium." Miracle ! There was an eruption of Vesuvius. They did the spectrography of the lavas of Vesuvius and they found helium in it. So this is extraordinary ! What does it mean ? It was the first step. It was the first step in barcodes in physics. The second step which is absolutely staggering, it's at the level of quantum mechanics, it's Schrödinger, when he had the idea of his equation, he had the idea if you like, that he had an operator and this operator, in fact, this is what we call spectra of course. So he went to mathematicians and asked them, "What is it a spectrum in mathematics ?". So his friends told him, "Well, go and see Hermann Weyl, and he will tell you right away what it is.". And Schrödinger said, "Especially not, he would calculate the spectrum before me.". So what Schrödinger did which was extraordinary is that he kicked off the fact that all these spectra, all these barcodes that come to us from the Universe, in fact, they have a reason to be mathematical, in fact, they are mathematical beings. These are the spectra of operators in a Hilbert space.

And when you see them, they seem very complicated, but the operator is a lot simpler. And it is the operator who gives us the key, for example, the key which allows us to understand the elements table of Mendeleïev. So in fact... the Universe speaks to us, it speaks to us, but it speaks to us in a spectral manner, it speaks to us by sending us barcodes, these barcodes are shifted towards the red, it is which allowed us to understand the expansion of the Universe, etc. But you see the phenomenal role, there, of writing. There, I'm not talking about mathematical language, I'm talking about writing. So the Universe sends information in writing things to us, and its writing is spectral. So now I come to the evolution of mathematical language, okay ? So the mathematical language has evolved of course. He has evolved over the past 50 years, and it has evolved in ways that will be quite difficult to get across teaching. The reason why I think it will be quite difficult to pass in teaching, this is what happened when Lichnerowicz promulgated the teaching of set theory, in secondary classes, even in middle school. And I remember an exercise I saw, I mean I had witnessed an exercise. Exercise is the following : we take three sets A, B, C. We trace Venn diagrams, okay, *(laughs)*, and the subject of the exercise was : suppose $A \cup B = C$ and the subject of the exercise was "Hatch the empty set." *(He shows the exercise with the 3 sets.)*. So there was only one student who found it, and he put "I couldn't, it's empty.". *(laughs)*. Okay, so it is clear that we will have problems for categories, well, categories is already the level above set theory. But as I said in my introduction, the categories allowed, they allowed precisely, thanks to their flexibility compared to set theory of inventing a language, if you will, of developing a language, that was, how to say, that was formulated, initiated etc. by Alexandre Grothendieck and that is the concept of topos.

So the concept of topos, it's good that I simply give you an idea. And this idea is that instead of focusing on a space, like the space of earlier, the sphere, or something like that, instead of the space I'm showing you, which is front and center, the space will play a completely different role. The space will disappear, will be behind the stage, but it will play the role of a Deus ex machina, that is to say that in fact, we are going to do set theory as we usually done, but the space in question will serve as a parameter. That is, the space in question will never be at the front of the stage, it will be behind, and then, what we realize when we do this, is that all the properties of set theory, which are ordinary properties, for example the demonstration that I gave you of Morley's theorem, will continue to take place, provided that you never used the absurd reasoning, provided you never used the excluded middle rule. All these reasonings will continue to work, it's extraordinary because it gives you reasonings that will work with parameters, that will work with this hazard, if you want.

The topos introduces a hazard, okay ? And so now, what's amazing is that, from this fact alone, the notion of truth will become more subtle. And instead of only having the true or the false, which allowed us to reason by the absurd, we are going to have a notion of truth which is going to be a lot more nuanced, much more subtle, than the ordinary notion, we will continue to work as if we was working in set theory, and what seems likely to me is that gradually this notion will allow us to formalize situations in which to say that one is right, that one is wrong is completely naive, you see, when for instance you see a chat on TV or a stuff like that and where the notion of truth will become much more interesting and much more subtle, and adapted to a given situation. So that was done by Grothendieck, so there is the notion of truth in a topos, and there is as I told you the possibility of having nuances. So I will finish showing you a sentence, sorry, a text, a page of math language text, but to show you the richness, the variety of mathematical language, in all its splendor, this page of text, this is probably the last letter that Grothendieck wrote to a mathematician and it is about of... it is a question concerning original paradise, topological algebra, perennial semi-simplicial category, the eye of the surveyor, bundles of sets, from having felt, etc., categorical topos, etc. So you see the immense richness of mathematical language, and to what extent, precisely, the philosophical significance of this language is something which is often overlooked, but which is of an incomparable power. Here, so, I will finish on that, thanks.

Mathematics and thinking in motion
Alain Connes

The purpose of my talk is to make you feel two things : the first one is that I am going to tell you a certain number of stories, on mathematicians, and the second is to make you understand that mathematics are a factory of concepts, but of concepts absolutely fundamentals and concepts that relate, if you will, to life, and that are not at all confined to calculations with numbers, or things like it ; too often we get the impression that the mathematician is someone who does calculations ; of course, he happens to do calculations, but what I'm going to try to make you understand, precisely, in this talk, is that the mathematical technique emerges from time to time on fundamental concepts, on fundamental ideas, and these are ideas that can be explained simply and that relate to life, that is to say that they are also important, I think, to people who deal with humanities and not only for people who are going to do hard sciences. So, there will be a portrait gallery. We will start with Galois.

And if you want, Galois, it's the prototype of the mathematician who had an absolutely incredible life : he was born in 1811, and he was 17 years old when he found its most important things. And what happened then, there was a succession of misunderstandings, in fact. If you like, in 1829, Abel died. And basically, it's Galois who takes up the torch of Abel's ideas. But in fact, I learned this from specialists of Abel. Abel had come in Paris, but it is absolutely impossible that he met Galois : Galois was too young when Abel came to Paris ; I always imagined that they had met in a Parisian cafe, where they had both discussed. Apparently this is not possible. So when he was 17, Cauchy who was an academician, had already done in 1829 two talks on the works de Galois, at the Academy, in May and June. That was so in 1829. And in July 1829, Galois' father committed suicide because he had been the victim of a slander campaign which had been carried out against him, and in addition, Galois fails for the second time at the Polytechnic School. So

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it was the second time that he appeared at the Polytechnic School. At that time, the Polytechnic School was at the top of the Grandes Écoles, it was the second time he failed. This is where there was the scene apparently where he threw the chalk cloth over the head of the math examiner, because the examiner did not understand Galois'explanations for the logarithm.

Fortunately, Galois was received at the teacher training college. And in January 1830, there is a letter from Cauchy to the Academy which says that he will speak to Galois. So it would be the third time, and then finally Cauchy gives up, and I think, well, we think, and historians think, if you like, that it was agreed with Galois because there was the Big Price of the Academy which was to be given in 1830 and Cauchy had convinced Galois to rewrite his article, and to run for this grand prize. Then, what happened was absolutely dramatic because the academician who was to report on Galois' article, it was Joseph Fourier. It is a very very great mathematician and Fourier apparently he was at the top of his stairs at home, he got his feet in his dressing gown and he tumbled the stairs, he died. So big problem, big problem, and if you want, there was such a mess at that time that the Galois manuscript was lost. So not only Galois did not get the price he might have deserved, the prize was given to Jacobi and Abel, of course two immense mathematicians. Jacobi was a German mathematician, Abel was dead, he died in 1829, the prize was given to Abel posthumously.

But if you want, Galois couldn't complain about not having had its great price ; on the other hand, he could complain, at the time, there was no photocopier. So he had written his manuscript ; at the time, you were writing the manuscript and then it was over. He gave it to the Academy but handwritten lost. So he had complained several times to the Academy, but manuscript lost. And so in 1830, the grand prize was given in June 1830 and in July 1830, there were the Three Glorious Years. It's Les Trois Glorieuses, and Galois was at Normal School and there he was bitching because, in Normal School, the students were confined, they could not go on the barricades. On the other hand, the students of the Polytechnic School, them, they could, so there, Galois started to really revolt. It's very, very weird, if you will, well, he had barely 18 years old. So he started to rebel and he rebelled against the director of the teacher training college. And after the summer, therefore, he started to campaign more or less, and little by little, he managed to get kicked out of Normal School. So he was expelled from the Normal School in January 1831, and there are something incredibly ironic, which is that Galois was on the street, if you want, he didn't have a salary anymore because at the time, and that's still the case now, the students of the Normal School received a small salary. So he was on the street and then to earn some money, he had created an algebra course, a course which brought together a number of people who came to listen to him because he was a magnificent mathematician despite his very young age. And the total irony is that its course of algebra, it gave it in the street which now is a street adjoining the Sorbonne, which is called rue Victor Cousin. Why is it ironic ? It is ironic because the person who signed the dismissal from the Normal School of Galois is called Victor Cousin. So a few years ago, for the 200 years from the birth of Galois, I had to give the talk at the Academy of Sciences on Galois. And at that time, I wanted everyone to agree to rename rue Victor Cousin rue Galois. Well, it was not possible, but it should still be incredible.

So this is what happened. So after, therefore, it must be said that Galois died at 20 years old. And the last two years of his life, he didn't a lot of math. It's incredible, it's absolutely incredible. And what happened is that once he was expelled from the Normal School, there still had been another academician who wished him well, his name was Poisson. And in mathematics, there is a well-known formula called the Poisson formula. And if you want, Poisson had convinced him to rewrite his manuscript and to present it to the Academy. So Galois had executed. He had rewritten his manuscript. He had worked, etc. And in the meantime, of course, after The Three Glorious, everyone started to be extremely disappointed with the new power. And Galois was one of those people. So the first thing he did was not very, not very clever, well. It was at a banquet celebrating the release of opponents of power. So what, he was at this banquet, and he had raised his glass to Louis-Philippe. So, all the people were saying "He's completely crazy !" : he was at a banquet against Louis-Philippe and he raised his glass to Louis-Philippe. And in the hand, he had a knife. First, people didn't understand why he was raising his glass at Louis-Philippe ; secondly, there was a spy who was there and who saw that he had a knife in his hand. He had been arrested, that was in the month of May 1831 he had been arrested, and had been judged fairly quickly. He had been judged by a popular jury. But as he had been judged by a popular jury, the people had seen it was a little weird, well, well, I mean, it didn't defend himself, basically, he said... So they acquitted him. I believe he had been acquitted in June 1831. And a month later, he received

Poisson's report on his article. So there, disaster because Poisson said it was a very very nice safe-lying theory, but that there weren't enough details in demonstrations, etc. So he couldn't accept the article. And Galois, when he received this report, he wrote by hand, in the margin of the report, he writes "Oh, cherubs !". It means he saw that people didn't understand nothing he did. At that time, he defended himself a little, that is to say, there, he got arrested. It was on July 4 that he received Poisson's report, he was arrested on July 14 at the head of a demonstration against Louis-Philippe. And there he was put in prison for good ; he was put in a prison which is called Sainte-Pélagie ; and there are a lot of you probably, who imagine that if they were in prison, they could at least think quietly with books ; in fact, it was not like that at all, because Galois, he was among the condemned and it was absolutely terrible because that the other condemned forced him to drink very strong liquor, etc. ; I mean it was absolutely orthogonal to his... what he was doing and in fact there, he met Nerval. Nerval met him while he was in prison.

And then, it's terrible, it's terrible, because if you want, Galois remained in prison until March of the following year, 1832. He was not 20 years old. And in March 1832, the reason for which he was released is that there was cholera in Paris. And they emptied prisons so that there's not too much damage. So he was put in a nursing home and in this nursing home, he more or less fell in love with a girl who was there, without realizing that she was already with someone else.

Well, it all ended in a duel, okay. And then there, it's the same, if you want, I guess each of you would imagine that if he were on duty to fight in a duel, he would have more skill than the opponent, so it would be fine, he would get out of it. Unfortunately, the duel in which Galois was caught, he tried to get out of it before. He tried to say that... But unfortunately, it was an absolutely terrible duel, it was like russian roulette, it was a duel in which there were two revolvers of which only one of the two was charge. And they had to put them on their stomachs. So he had, of course, a bullet in the stomach. Back then, and even now, it was deadly, and the others left him there.

He was found by a peasant on the spot, who took him to the hospital and he died the day after. Well. And he left a wad of papers, that's what he says, that's what he says in his stuff, so it was... So there are people who will make you believe that he found all his results the day before his duel. It is absolutely not true, I mean, obviously he had kept thinking and it was to the point... he had to force himself so much to keep doing math while he was in awful circumstances, that people who saw him when he got out of prison said he looked 50 years old then that he was 20 years old. Okay, so that's to tell you a little bit what the passion that inhabits him was, and it's a miracle, finally, it's a miracle that we had his work.

It is an absolute miracle that we had his work. So that's what he wrote and that he left in his letter-will. It's his testament that he had left to his brother, and to his friend, he had a friend too.

And what has happened, then, is that 10 years have passed. And by an extraordinary hazard, Liouville, who was a contemporary of Galois, who was just two years older than Galois, found the Galois papers. And he understood that these were absolutely great things. And he talked to the Academy. So if you want, 10 years after Galois'death, it's Liouville who is here. Well there, obviously, he is much older but he was a contemporary of Galois, he was someone who was born in 1809, therefore two years before Galois. And so, Liouville understood the extraordinary strength of the works of Galois if you want.

So he wrote that, but I show you that he wrote it correctly, so it's like that.

He talked about it at the Academy. And then, gradually, the work of Galois have been understood. And then what I'm going to do, I don't want to bother you with too complicated math, I'm just going to give you the essence of Galois theory. I'll give you the essence by giving you an example. What Galois says in his testament is something incredibly visionary, if you will, what he says is :

"You know my dear Auguste, (he had a friend named Auguste) that these are not the only subjects I have explored. My main meditations for some time have been directed on the application to transcendantal analysis of the ambiguity theory."

So Galois discovered this theory of ambiguity. And in this letter, at the end of his life he says that not only he applied it to polynomial equations.

But in fact, he applied it to the theory of transcendantal functions. No one knows exactly what he had in mind. It's a fact that nobody can say that we now know what Galois had in mind.

On the other hand, we know very well what he had in mind for the polynomial equations. And so, for polynomial equations, I'm going to explain what ambiguity theory. So what Galois understood, if you like, is something quite extraordinary ; it is that when you give yourself an algebraic equation, for example, I gave you an equation so, we knows how to solve it. You know now, I mean with the computer, you can control all that, you can plot the graph of a function, you can solve a polynomial equation, and all that. But the computer will never give you zeros unless with some precision, it will only give the roots except with a certain precision.

So, what Galois theory says, it says something extraordinary : it says that when you take an equation like that, which is irreducible, that is to say that we cannot factor it into a product of 2 factors with rational coefficients for example. So when an equation is irreducible, what Galois theory says is that there is a group which operates on the roots, here on the 5 roots, and which means that we cannot, if you want, isolate a root. In other words, there is an ambiguity between roots ; this group turns the roots. And any relationship that is checked between the roots, any rational relation which is checked between the roots, for example, with the computer, you can see that this relationship, is almost verified, the fact that $E = 4C^2 + 2D^2$, you can verify it. In fact, what Galois theory says is that there is a group that swaps these roots, that is, they can move from one to the other. And so that if a relationship like that takes place, it will take place for them permuted roots. And what Galois theory says is that by this group, you can transform any root into any other. So what Galois says after, actually what I'm telling you in particular here is that it's impossible to have this relationship. Why is it impossible to have that relation ? Because if you have this relationship, you can see that the 5 roots are real. This is not at all difficult to demonstrate. Therefore you have 5 roots which are real. But suppose you have a relationship like the one I wrote : $E = 4C^2 + 2D^2$. Well at that time, as E can become any of the other roots, C and D will be other roots too. And you can see that all the roots should be positive, since they are sums of squares. And so it's not possible. It is not possible. So it's extraordinary !

It tells you that without calculating and without getting your hands dirty, or anything either, you know that this relationship is not possible. In other words, with the computer, it will tell you "But it's true, it's true!". It will say with decimals and all that. No! Galois says "it's not possible, this relationship is not true!". And it is not true by pure thought, it is extraordinary! This is something extraordinary! Because he understood that behind an equation, there is not only the numerical value of the roots. No. There are the relationships between the roots that can exist, and what does Galois theory is to detect exactly all the relationships between roots. And they are detected by a group. So don't believe people who will tell you that it was Galois who invented group theory. No, the people like Lagrange, etc., knew what groups were like before him. But Galois is the first modern mathematician. That is to say, he is the first mathematician who had this dazzling, if you will, who did that certain things like that are true without having to calculate or what whatever, okay. We have an abstract theory, it's the ambiguity theory and solving an equation is gradually reducing the ambiguity that there is, so that finally, on the equation, we can affirm such a root, and such root, etc. Okay. So that's the ambiguity theory. And here, in this case, we can calculate what the Galois group is. So the group of Galois, you see the 5 roots, they are indicated here. The group of Galois, he will swap them. But it swaps them if you want in a transitive way, that is if we iterate these permutations, if for example, I take the root which is above in the middle, it will go on the first; and afterwards, if I look where the first goes, it goes on the last; after, if I look at the last, she goes on the penultimate; if I look at the penultimate, it goes on the second. So you see that you've gone all around the turn.

So good, and that is always true, that is to say whatever the equation you take, Galois tells you that if it was reducible, there is a group that swaps the roots. So there are a lot of mathematicians who believe they know Galois theory, because they say that Galois succeeded to demonstrate that an equation is solvable by radicals if and only if his group of Galois is solvable. But in fact Galois, at 17, had better than that, he had... a theorem...

I'm going to scare you but don't worry, we'll move on to another subject right away. So what Galois demonstrates is that if we take a equation he calls primitive. It's a certain technical definition, for that it be solvable by radicals, it is necessary and sufficient that we can index the roots by a Finite field. It was Galois who invented the Finite fields. That's enough funny because the French are modest, because the Anglo-Saxons call those Finite fields the Galois fields. If we translate this literally into French, it gives Galois body. But in France, we don't use this terminology : we are talking about a Finite field. And then the Galois theorem, which he had when he was 17 years old, that for a primitive equation to be solvable by radicals, it is necessary and sufficient that one can index its roots by a finite field, so that the Galois group, then hold on, is either contained in the semi-direct product of the affine group of the finite field by the Frobenius, by the powers of the Frobenius. Okay, okay, good. *(laughs)*

And so when I prepared my presentation for the Academy, I have noticed that in fact, Galois knew countless things and that he knew for example, now what's called the theory of Sylow, which is a theory that was developed perhaps 50 years after the death of Galois. So that's to tell you a little bit about how well he managed to see so far. And at the end of my talk, I will show you a text by Grothendieck and this is a text that is fundamental because it applies marvelously to the case of Galois, and this is a text on creativity, on the discovery and the fact that true creativity, it asks precisely if you want to return to that spirit of the child which is both free but also who does not accept if you want the weight of knowledge that we put on him. So we'll come back to that, okay.

So now I come to another subject, because I don't want to neglect physics. And another subject that is so close to my heart - which is, if you like, another great discoverer, in the XX$^{\text{th}}$ century, if you like, is the discovery of quantum mechanics. Now let's move on to Heisenberg and quantum mechanics. So we take a break if you want. What I tried to do was to choose subjects that each show you a new concept that has been discovered, either by doing mathematical research or by doing research on nature, on physics. But each of these notions is a notion which has a meaning, which has an absolutely fundamental meaning.

So the story of Heisenberg, it's actually connected to a place, and that place is an island which in German is called Helgoland ; in French, we translate Heligoland. It is an island in the Nordic countries. And it's an island that has a particularity, I no longer know if this particularity is still true today ; in any case, it had a particularity in the 1925s, which was that it had no pollen. There were no trees, there were no sources of pollen. So what is the connection with Heisenberg ? The link is that Heisenberg was a physics student, finally a student, he already had a lot of experience... He was in Göttingen, I think. And at one point, it was in the month May he got a terrible allergy, with fever if you want. So his head was swollen, well all that, and so at the time, the only one remedy, we did not give antihistamines, the only remedy was to send it to Heligoland. So he was sent to this island. They told him to stop teaching, etc. and they send him to this island. And it happened on this island. He was housed by an old lady in a house, perhaps one of the barracks up there. And then at the time, he was looking for... *(little interrogative circumspect noise).* At the time, he was looking for...

He was trying to... At the time, quantum mechanics was at a prehistorical state, that is to say that we had decided on what we call certain principles, which were used to calculate energies and all that, but I mean that's was absolutely not a real theory. And Heisenberg was thinking about a problem. Basically, his problem would take too long to explain, if you want, the idea, basically, at the time, we conceived the atom as a small solar system. But it didn't work. Because what's going on in a system like the solar system, is that, for example if the electron revolved around the nucleus, it emits energy, and therefore in fact, its orbit should shrivel up on the nucleus. And that is not what is actually happening. So there were things like that that didn't stick at all. And so, Heisenberg reflected on this. It is based on experimental results, this called the Ritz-Rydberg principle. And hey, he had this calculation he wanted to do and when he was on this island, he started doing this calculation. There were things he didn't understand, all that. And then one morning at 4 a.m. in the morning, everything worked ! He had this extraordinary revelation ! And instead to go to bed, he went to climb on one of the rocky peaks *(laughs)* which are on the edge of the island. He settled upstairs and waited for the sunrise. And in his memoirs he describes in an extraordinary way if you will, this enlightenment which he had and he really says, and it is true, that he had all of a blow before the eyes an immense landscape which was revealed to his eyes, but it was an intellectual landscape, of course ; this landscape was the essence of discovery he made, if you will, it's something incredible ! He discovered that when we do calculations, this is Heisenberg, let's come back to it. What he discovered is that you see, when you do physics, well for example, you write $e = mc^2$ or stuff like it. You could write $e = c^2 \times m$, it's kif-kif, these are numbers. Okay, well, I mean, it doesn't change anything. What Heisenberg found, it's something incredible. Heisenberg found that if you try to manipulate position and time, we talk about speed, but we have to talk about the moment : the moment is the product of speed by mass, okay ? So if you try to manipulate both position and time, at the very least microscopic scale of a tiny thing, an atom or something like that, well you can always do whatever you want, you won't never to fault what is called the Heisenberg uncertainty principle, okay, which is $\Delta x \Delta p$... where Δx is the uncertainty about the position, Δp is the uncertainty about the time. Well that's always bigger or equal to $\hbar/2$, what is $\hbar$, this is the constant that Planck had introduced at the beginning of the century, to explain certain physical phenomena.

So there, I have to tell you a little story, about the uncertainty principle because hey, *(whispering)* I think there is a book by the way, which is not bad, by the way... But in fact, concerning the principle of uncertainty, if you really want to feel how this principle has disturbed folks, there's a story I need to tell you. What is sure is that Einstein didn't believe it. However, Einstein is behind the quantum theory, I mean, it was Einstein who had the idea that the photon had energy levels that were quantum. So Einstein didn't believe it. So Einstein had imagined a device.

At the time therefore, Heisenberg found his principle of uncertainty towards the late 1920s. At that time, there were what were called Solvay congresses ; these were meetings of physicists, in small numbers, and of course, they were talking to each other.

So there was a Solvay congress, I believe it was in 1830, or something like that. And so Einstein had imagined the following thing ; he had imagined to fault the principle of uncertainty, but not on the position and time, but on $\Delta t \Delta E$; that is to say that... *(sigh, sigh)*... Time is the dual variable of energy, just as position is the dual variable of the moment. And the principle of uncertainty gives you something similar to $\Delta t \Delta E$. Something like $\hbar$ or $\hbar/2$, it depends units. So Einstein didn't believe that. And Einstein had imagined... Of course, he always did the same thing, that is to say that when he did not believe something, he imagined a thought experiment. An experiment of thought, what does that mean ? That means I will make you a very coarse drawing : but in fact, we can very well imagine that this experience be made more and more precise. Okay ? So the drawing very rude, it was as follows : *(drawing the device on the board)* there, we will put a small spring, and then here, we will put a box. And then we will put like a sort of cuckoo. And then with this, there will be time here, okay?

That was his system, and then there is a kind of thing. And then there are... That's the system.

So what was Einstein's idea? Einstein's idea is that Δt, well, we will control it since we have time here, okay. So this is the t therefore. And ΔE now? So what does that mean, what do I have says here *(showing a place in the drawing)*? It means there will be a moment given where the cuckoo is going to "Touc!". It will emit a photon. And we will know what time he emitted it since there is this thing that marks the hour, okay. So Δt, *(noise to express that we don't know what...)*. So now ΔE. Well the photon, that, Einstein, he knows it, the photon, it weighs $h\nu$, where ν is the frequency of the photon. So this is e if you want, it's energy. So when the photon comes out, this thing, it becomes a little lighter... *(seeing that he may seem to have lost some understanding of his audience)* Do you know the story of the truck that was carrying holes, right? You don't know it? It was in the mountains, okay, then at some point, the driver, he felt heavier, he backed up, he fell into the hole, okay. *(laughs)*.

Okay, I'll start again, okay? So here, once the photon has been emitted, okay, this thing gets a little lighter, so it's okay, if you want, the needle, it will go up a little bit, and looking at how much it went up, we're going to know ΔE so in fact, Einstein said "bah we're going to know ΔE, we are going to know Δt, with as great a precision as we want. So we will not have the principle of uncertainty". So he said that. And it made terribly scared to Bohr who was chatting with them, because Bohr of course believed in the principle of uncertainty, it frightened him terribly because... What was the reason he was afraid? The reason for which he was afraid of is that when you do the math with this system, proposed by Einstein, what will intervene is the gravitational constant because you see, the clock, when it goes up a bit, it's in the gravitational field, so when you go to get how much energy has decreased, you will use the gravitational constant, so obviously, the gravitational constant, it absolutely does not fit in the $\hbar$ by Planck etc. Planck's theory, it is completely separate from the gravitation. So Bohr said to himself, it's done!

So there is an extraordinary photo, on which we see Einstein going out very proudly of the Solvay congress hall and we see Bohr following him a little like a little dog, and who is, well... And then what happened is that it is not the end of the story. The end of the story is absolutely wonderful, because what happened was that Bohr came back to his hotel. Obviously, he didn't sleep, he didn't sleep all night long because, and he found the answer... And the answer is fantastic. The answer is absolutely fantastic, because, if you will, well, it seemed impossible, impossible ! Why ? Because, as I said there will be the gravitational constant when you go to do the math and that is impossible for it to work ! It is impossible to find the $\hbar$. Where does it come from ? What Bohr found overnight, he found the same that has laid Einstein, in fact, to general relativity. *(Alain Connes writes with the chalk formulas on the board)*. At the time ! That, you know, now, this year in november there are going to be a bunch of celebrations of the discovery of general relativity by Einstein. It's been exactly 100 years. That's why there are going to be all these celebrations. So that's exactly 100 years. And so it was ten years, or even more, fifteen years before the story in question. What does that have to do with the thing ?

What this has to do with the thing is this : what says general relativity ? It says the passage of time if you write the metric you have what's called Minkowski metric, in fact, which is due to Poincaré, so space-time if you want. When that is the space-time of special relativity, and if you look at the metric of the space-time of general relativity, as a first approximation, which happens, is that the metric does not change for the usual coordinates : we are in a Euclidean space. However, it changes for the passage of time, and the way it changes is that the coefficient dt^2 is multiplied by 1+ twice the Newtonian potential $V(x, y, z)$.

Don't worry, it's not... Good. What does it mean ? It wants to say that time passes differently depending on altitude, ok ? But the clock has changed altitude a little bit *(laughter)*. So its time has passed differently. You do the calculation and you find the principle of uncertainty of Heisenberg. It's incredible ! It means that Bohr, if Einstein hadn't not discovered general relativity *(bursts of laughter)* fifteen years before, he would have been right, okay ?... Nobody would have believed that the uncertainty principle was valid. But, because of general relativity, that he even had invented, he was beaten, he was put in default. So the next day morning, Bohr came back, triumphant, I mean, it's extraordinary ! It is really extraordinary, but if you want, all that is to try to make you feel the fact that none of these notions was accepted at first. Not at all ! Absolutely not. There is always an absolutely terrible resistance, to things that are new like that... And so, what's unbelievable in quantum, which is mind-boggling in quantum, if you want, and that I think it has not really passed into knowledge. Yes, then, good. I'll talk about it after that, I'll talk about it afterwards. I will come back to it. What's amazing in quantum, if you like, is the fact that, and that, it comes from the Heisenberg uncertainty principle, is that, unlike the classical physics, when you do an experiment in quantum, you cannot reproduce the experience. It is something fundamental. When I was trying to tell you, if you want, that I was going to explain to you concepts or concepts... These are concepts that make such a break with the classic view, if you will, that it's a huge difference. What do I mean by that ? What I mean by that is that if you do a quantum experiment, for example you send a photon, and this photon, it will go through a very small slit which is little close to the size of its wavelength. And afterwards, you will receive it on a target. Well, the fact that you receive the photon at a given location x, this experience is not reproducible. That is to say, you can redo experience with as much precision, given the same initial conditions, etc., the end result will not be the same. This is incredible ! And it won't be the same because of Heisenberg's uncertainty principle.

So you can say to me, "Okay, well, okay, well, I don't care record, there is a bit of a hazard, what ! A microscopic hazard, I don't care !". But no ! Now what happens is that the fact that there is this basic uncertainty, if you will, well, it was used to produce random numbers. That is to say that there are Swiss who made a device that works, now it's with an LED lamp, you know the little LED lights there, like that. So these little lamps send out photons on a target, that's it. We look at where the photon arrives, it hits one of the target's tiles. And from there, we make a number and as it is a quantum phenomenon, that is to say that it is a phenomenon which is not reproducible, it produces random numbers, which are so random that even if an attacker wanted to reproduce the same thing, it means if he knew all the data on the system, he would not be able to reproduce the same number. Whereas with a computer, if you fabricate random numbers, if the attacker knows your system of manufacturing, it will happen to reproduce them, the random numbers, okay ? So it's phenomenal, it's phenomenal ! So from there, if you want, this extraordinary truth, in fact, brings out an idea, which we have started to to exploit and this idea is this : you see, we are used in physics to attribute any variability to the passage of time, that is to say that well... I once remember my teacher, I had a teacher, I don't know anymore if it was in Math. Sup.. He told me to go on the board, so I go on.

He questions me. And then he does this to me *(gesture of a curve drawn in the air)* - Ouhouh ?! *(laughs)*. It looks like this... He says to me "Mr. Connes, what is the variable ?". So, I was doing kinematics. I'm thinking... And then after a while, I answer him : "it's time !". It was the right answer ! You see, normally there are a lot of things that are variable. And all physics is written by noting d/dt of something equal something else... All physics is written as a function of time. And in fact, if we think enough, at the conceptual level, we realize in fact, that quantum mechanics immediately causes paradoxes, very very violent paradoxes, very very strong, if you will, and which come precisely because we attribute variability to the passage of time.

And there is a fundamental idea that is hard to get by, but that we tried to popularize etc., and this idea is actually that the real variability is the quantum variability and that time in fact emerges from that variability. It means that time is only a secondary phenomenon, it is only an emerging phenomenon, which results from quantum variability, but which is not not at all fundamental okay.

So to try to get this idea across, in fact, I'm not going to give all the details, we wrote a book, therefore, with Danye Chéreau and Jacques Dixmier, we wrote a book together, called Le Théâtre quantique. And in this book, you'll see an introduction to that idea, which is, we hope, understandable although a bit cryptic obviously, that is to say that we don't give all the details etc. But the idea comes from another quite extraordinary mathematician Von Neumann.

That is to say after the discovery of Heisenberg, if you will, after the great discovery of Heisenberg, of course, mathematicians formalized what Heisenberg had found. It took a while. What Heisenberg had found, so I remind you, was that you can't swap them letters, variables like $e = mc^2$, you cannot write $e = c^2m$. We can't do that, okay ? Then there are people who will say to you "Wow la la la la ! What's going to be complicated all this !". But in fact, no, back to Heisenberg.

You see these two sentences, so this is an anagram that has been found by Jacques Perry-Salkow, who is quite extraordinary and who was the birth of the book I showed you. But what does an anagram ? It means that if I had the right to swap letters, I would will hold the same result : not terrible ! *(laughs)* has *a2bcd*... So you see, in the commutative, it gives you the same result. But of course, we we are all used to paying attention to the order of the letters... language ! Language is made for that. And the discovery of Heisenberg can incredibly simple saying to herself : she can say to herself by saying that Heisenberg, he found that we had to pay attention to the order of the letters, when we do calculations with microscopic variables, it's wonderful ! It's something absolutely wonderful, okay ! Okay so Von Neumann worked out on that, he found that he needed a mathematical formalism called the formalism of Hilbert spaces is a fairly complicated thing.

So, you know, in my introduction, I said that I was going to talk about Von Neumann algebras, okay. So right there I'm talking about it, okay. I don't give you too much detail, of course, not too much detail about the types and all that. But now I'm going to tell you about another mathematician, and that was the starting point, really, of my work, my thesis, etc., and which is the tool which made it possible to have, the essential tool which makes it possible to give a sense of this idea that the passage of time emerges from the hazard of quantum. So the reason I'm showing you his photo is that unfortunately he died on October 9 at the age of 91 ; his photo was taken when he came to Bures-sur-Yvette exactly 30 years ago. he spent a year in Bures-sur-Yvette 30 years ago, and why he's a person absolutely extraordinary ? He's an extraordinary character because that for example he was in the military at the time of the war between Japan and the United States, but he was deaf since the age of 2. So there was a time when all of his fellow believers were running for shelter, because there was a bombardment. Tomita did not move, and when his military friends came back to see him, they said "but are you crazy ?...". They were shaking him, and he said to them "What bombing ?". It was at that point, he was known like that. And then there was an episode where the officer who commanded them said that he wouldn't be on the expedition that they were going to do because as he was deaf, it was rather a problem. So he stayed and everyone else was dead. And apparently, but I'm not so sure, apparently, it was the next on the list of suicide bombers when the war ended.

Then he had a teacher, it must be said that shortly after the war, when he was at the university, instead of lecturing, instead of going to listen to the classes, the students were going to plant potatoes so there were starvation. They were going to plant potatoes near the university. So he had a teacher. He had a teacher to do his thesis, his teacher was called Ono, and his teacher, the first time where Tomita went to see his teacher, because he wanted to do a thesis ; his teacher takes a big big book, I don't have brought any with me. The teacher gave him a book, oh ! more than that, twice that easy, okay, he gave it to Tomita and he said "Read this book and come back and see me when you'll have understood everything". So it's okay like that. So during 2 years, they do not see each other. And then after 2 years, by chance, Tomita rmet his teacher in the corridors of the university. His teacher remembered when even : "So this book, is it going on ?". And Tomita replies "I lost it after a week..." *(laughs)* He was an absolutely great guy. He told stories that were absolutely great. He made an absolute awesome. Only, since he was deaf, if you will, it was very very very difficult to communicate with him. It was really very very difficult, most of the time, he would turn off his device. *(laughs)*. Me, that was the starting point of my work if you want. The start of my work was the fact that so Tomita and then after Takesaki, who had resumed work of Tomita, had found that, on a Von Neumann algebra, like Von Neumann had defined them, there was an evolution but which depended of a state. And then, what I demonstrated in my thesis, is that in fact, it did not depend on a state and it was enough to have non-commutativity, that is, it was enough to have an algebra, to do calculations in the which you pay attention to the order of the terms, so that there is an evolution in time, so that there is a passing time. So then, there were a lot of consequences of that, of course. And in fact, the bulk of my work was if you want to develop geometry for spaces which, unlike Descartes'spaces, because Descartes if you want, had managed to understand that there was a duality between geometry and algebra because Descartes understood that one could encode a geometric space by coordinates and then do algebraic calculations instead of do geometric calculations. The simplest example possible : if you want to demonstrate that the 3 medians of a triangle intersect. Well there are several ways of doing it, but the simplest way is to do the calculation of the barycenter. You take the coordinates then you calculate the third of the sum of the coordinates. What is the advantage of the algebraic demonstration on the geometric proof ? You can of course do a geometric demonstration of the fact that the 3 medians of a triangle meet. But suppose I ask you to demonstrate it in dimension n? *(laughs)*. While the algebraic demonstration, it is obvious, you do $1/n$ times the sum of the coordinates and then that's it, it gives you the intersection point and then it's over. So you see the power of this back and forth, between geometry on one side, and algebra on the other. So what Heisenberg discovered was that there were incredibly natural spaces in which precisely, the coordinates do not commute. And these spaces correspond to observables on a microscopic system. And so for me, the main part of my work was to develop the geometry for sorts of such spaces. So time is still pretty short, instead of talking to you about my work, I'm going to tell you about another mathematician absolutely extrordinary, whose name is Alexandre Grothendieck, and who died a few years ago, and the reason I'm going to tell you about him, it's not because that I want to describe topos theory to you, because that is a marvellous theory but it wouldn't pass, I don't want to talk about it. After maybe... But it's mainly to explain to you, to show you what Grothendieck says about creativity and this absolutely necessary need to find, when you are faced with a very very difficult problem, your soul of fant and this kind of, precisely, openness, sensitivity, etc. who is too often completely erased, completely erased by the weight of knowledge. So this is what Grothendieck says, I will read it with you, and then we'll stop there. So this is Grothendieck when he was young. He had an extremely tumultuous life too. So this is what he writes. He writes : *"In our knowledge of the things of the universe, be they mathematical or others, the renovating power in us is none other than innocence. It is the original innocence, which we all received in sharing at our birth, and which rests in each of us, object often of our contempt, of our most secret fears. It alone (therefore this innocence) unites hum militancy (of course, research is a school of humility, everyday school humility) the humility and boldness that make us eencounter the heart of things that allow us to let things get into us, and to soak up it."* (This is the first thing he says. Then he says :) *"This power there, (That's very, very important now.) This power power is by no means the privilege of extraordinary gifts."* You see, when sometimes we attend exhibitions, on mathematics, you have the impression that wow! These are aliens these people, no, you shouldn't have that fear at all, absolutely not. He is coming on the contrary too often that overly intelligent people have an immediate reaction and that this immediate reaction in fact is false. That is to say they tell you "it's not going to work for such and such a reason...". In fact, if they had thought more, they would have noticed that it works, okay. So this what

Grothendieck says is that : *"This power there is by no means the privilege of extraordinary gifts, of a cerebral power, let's say out of the ordinary, to assimilate and to handle with dexterity and ease, an impressive mass of facts, ideas and known techniques. These gifts are certainly precious and sources of envy surely for the one who, like me, was not filled like this at birth beyond measure..."*. There, it is really ironic, ironic, I don't like to say the biggest because that the biggest, what does that mean..., we cannot compare different things but it had a phenomenal influence on the mathematics of the XX$^{\text{th}}$ century. A phenomenal influence. So hear him say that... it's reassuring, let's say ! *"These gifts are certainly precious and sources of envy surely for the one who like me was not fulfilled thus at his birth beyond measure. It's not these gifts, however, or ambition even the most ardent (ambition is not enough) ambition, served by a flawless will, which make these invisible and imperious circles cross that enclose our universe. Only innocence crosses them, without knowing it, nor without worrying, at the moment when we find ourselves alone listening to things, intensely absorbed in a breeze."* So what he explains is that there is nothing more fruitful than to grasp a question and think about it, but in this way, in a way completely independent of the weight of science, etc. Okay. Good sure, well, to get to the problem, you have to know a number of things but then you have to think about it like that. And so he goes on to say : *"Discovery is the privilege of the child. I want a small child to speak, the child who is not yet afraid of making a mistake, of looking silly, of don't be serious..."*

For example, a while ago, there will be questions, so okay, beware of that, you shouldn't be afraid. There is even a Chinese proverb who says, "If I ask a question, I look silly for 5 seconds ; if I don't put it down, I look silly for the rest of my life.". So this is what he says, therefore, not to be serious, not to be like everyone else. And it's However, there is a typically French attitude that is quite in an assembly : we are afraid to ask a question, except when we know the answer *(laughs)*. *"Neither is he afraid that the things he looks at will have the bad taste to be different from what he expects from them, from what they should be, or rather from what it is understood that they are, that is to say, what the majority of people will tell him they would be ; he ignores dumb consensus and flawless, which are part of the air we breathe, that of all people sensible and well known as such. God knows if there were any, sensible people and well known as such, since the dawn of ages ; our minds are saturated of a heterogeneous knowledge, tangle of fears and laziness, cravings* *and bans, general information and push button explanations...".*

A typical example is what is called the butterfly effect, the number of people who brooded over it without knowing that it was an idiocy, it's something considerable. But I mean, it went on, it went on for a long time. So I continue... *"enclosed space where information, cravings and fears come to pile up, without never let the offshore wind rush in, except for the know-how of routine. It would seem that the main role of this knowledge is to evacuate a person. living concept, a knowledge of the things of this world.".* That is what counts, it is this living perception. For example, for loving math, you have to do it, of course. And whatever the problem you are watching, but what's important is that you do it, it's not that you take like... If somebody tells you a theorem, for example, if you want, you shouldn't have too much demonstration. There must be search for it yourself, even if you can't find it. You will win. Why ? Because if you search for it, by yourself, when it is given to you will say, even if you can't find it, well you will say "but of course, that was it, and that was it !". If you don't look for it and we give it to you, it comes in through one ear and it comes out through the other, and then you forgot it after half an hour. So it's very very important to do it, okay. So therefore... Its effect is above all that of immense inertia. He talks about weight of this common knowledge, often overwhelming. The little child discovers the world as he breathes. The flow and the reflux of his breathing make him welcome the world into his delicate being and make him project into the world that welcomes it. The adult also discovers, in these rare moments he forgot his fears and his knowledge, when he looks at things or himself with wide open eyes, eager to know, with eyes new, child's eyes. I hope you feel the most important in what I said. It is that it doesn't apply at all to math ; which means you wanted to do the humanities, you wanted to do the linguistics, whether you want to do anything, maybe even art, if you will, it's crucial that you get the message. And that you have understood that, in particular in mathematics, they have a far greater scope than calculating with numbers, calculating with numbers, etc. It's not at all that, it's a kind of version of the philosophy which is much harder because indeed, to arrive to a new concept like Grothendieck's concept of topos, it took years and years of reflection... But it gives you thought tools absolutely fundamental. And I don't have time to talk about it, but the concept of topos, it's a concept that shows you that the notion of truth, when we say for example in a current way of something that it is true or that it's wrong, well, when you look in a topos, it's a universe which is different from the universe, well one thing may be partially true and partially false, it may be true from a certain point of view, it may be false for another point of view, etc. So it gives a tool of thinking which is incredibly well suited in fact to life, to politics, to 36 things, but which has not yet passed into the common domain. It's a concept which is still a concept in the mathematical field, and which is not has not yet passed into the common domain. And we would gain a lot if you want, if all of these wonderful things that have been discovered, now become part of the common domain. So my scribe was going in that direction okay, to try to make you see, to a little surreal way if you will, that there are these wonderful things but that good of course, you have to make an effort to learn them and an effort to know them. Here.

Questions to the speaker

- Thank you so much. Perhaps, therefore, we do what we said. If you have questions, clarification on what was said, therefore, questions that you shouldn't be afraid to ask, I have some, but I'm sure you have some too...

- You talked about the butterfly effect... and that it didn't exist.

- I didn't say it didn't exist. But I said it was a vast illusion. Because what I mean is like saying that there is a butterfly that will fly, then the plane that follows another plane does not go to take off ; there is a colossal damping effect. Of course we can make a mathematical system which depends on few variables and which is such that when we move a variable a little bit, it will change the results. But from there to make believe that a small butterfly which flies at a place, it will create, I don't know, me, a hurricane in another place, it's ridiculous... Well we can remember where it comes from, it comes from the fact that there are differentials equations in mathematics, which are such that if we change a whole little bit the initial conditions, it changes the result considerably that becomes exponentially larger. That is true. But it's true in a particular model. This is true in a model, in which there is no depreciation, as it occurs in nature. In nature, fortunately, depreciation occurs, because otherwise, in the nature, we would look at the butterflies everywhere, and then we would be scared *(laughs)*. Fortunately it is like that. But it's common sense, it's common sense. But we have seen perhaps I do not know how many politicians or people who repeated the butterfly effect without understanding anything, since if they had understood anything, they would have realized that it was, huh, good... This is a typical example of people repeating things without understanding them, simply because they say to themselves : "Ah yeah, he is a powerful person who said it, so it must be true, what !"

- Thank you.

- It was a question about the fact that you often said that the physicists expressed everything as a function of time, and which we often considered that it was the variable...

- fundamental.

- and you said that in fact it turns out that the real variable is the quantum variable, and I didn't understand how time flows from this variable.

- This is quite a story. Basically, this is the story of my trajectory. That is to say what actually happens, but it is explained a bit in our book, but it is above all well explained in a talk I gave at IHES in May, and which I think must be on the IHES website, go and listen to this presentation, I could say two words about it. But basically, it is that Von Neumann created the Von Neumann algebras as being systems where we have a partial knowledge of reality. And with work by Tomita, then my work during the thesis, we understood that if we had a system that has a partial knowledge of reality, at that time, there is a time that emerges. That is to say, there is an evolution over time. Since everything is quantum, and the knowledge we have of reality is actually partial, that's what, with the work I did with Carlo Rovelli, that's what should explain the passage of time, that's what we called thermodynamic time. This idea of thermodynamic time, it is well explained in our three-authors book.

- Suddenly, my question is somewhat similar to Constantin's question concerning time. So suddenly, the fundamental constant, there is no longer fundamental constant, since ultimately everything is based on quantum variability ?

- There is one thing I did not mention but I had transparencies so I can actually show them. It's important, it's so important, it's this idea of variables. Because ultimately we come back to the idea of variables. What is a variable, you see ? What are we teached in class that a real variable is... A real variable, it is an application which goes from a set X into reals. It is like that we are told what a real variable is. Now if we look at this definition of a real variable, we see in fact, with a little reasoning, we realize that we cannot have the coexistence of what we call continuous variables, variables that take for example an interval of values, and discrete variables, which take discrete values. *(He draw an interval and points on the board)*. And the reason why we cannot have this coexistence, is that if we take a continuous variable, the set X must at least have the cardinality of the continuous but if it has the linearity of the continuous, we cannot have a discrete variable, because it there will be points that will be reached too many times. We can't have that. The extraordinary value of quantum formalism, as von Neumann has developed it, is that in quantum formalism, everything is solved : to say that in quantum formalism, in fact, a variable is the spectrum of a self-attached operator in Hilbert space, it's a bit complicated but here, for operators in Hilbert space, we can have operators which have a discrete spectrum and operators which have a continuous spectrum and which coexist. And then, what is extraordinary, is that in fact, it joins exactly Newton's thought. That is to say, Newton, in his writings, when he was trying to define what an infinitesimal is for example, he write exactly the right sentence, which corresponds to the quantum. That is to say, he said a variable is infinitesimal. First he said what was a variable. Now quantum formalism gives exactly the right answer by comparison to Newton. This is the first thing. And so now what happens is that once we have this formalism, of what it is that a variable, we realize that of course, the discrete variables can coexist with continuous variables only by non-commutativity, and we see that it is this non-commutativity that creates the passage of time, okay ? So in fact, the $\hbar$ still exists in fact, Planck's constant is still there, but this is what is extremely striking, it's that time should not be considered as a gift fundamentally born, but as an emerging datum, and that if we had a absolute knowledge of everything, time would not pass. It's amazing to think that, okay, that is, the reason why we feel that time goes by, etc., it's because we have a partial knowledge of the universe, okay. That's what is great if you want with this game of physics. In the book we wrote with Danye Chéreau and Jacques Dixmier, I must tell you that Danye Chéreau is my wife *(laughs)*, what we do is we found a very striking sentence we used to express the idea I just told you. We said "The hazard of quantum is the ticking of the divine clock". You know, Einstein said "God doesn't play dice". So that's the answer. The hero of the book's response to this Einstein's joke is that "the quantum hazard is the ticking of the Divine clock". That is to say, it is because there are constantly these little tricks completely random, pop ! pop ! pop ! that happen, that time past. Nature has a phenomenal imagination. And that's what gives the grow up to measure random numbers, it's amazing, it means "It takes the pulse of nature", pop ! Come on, it's a random number, pop ! another... You can always try to reproduce them. Well well that's incredible, it's the sentence that says it's, "the quantum hazard is the ticking of the divine clock".

- To try to put that a little bit clear, suddenly, if, let's admit, well, we can always hypothesize, if for example, precisely, this nature was quantum stable, if it did not move, time would not pass ?...

- Well, no, precisely, it keeps moving !

- But let's admit that we imagine that it does not move. It means that time would not pass ?...

- Ah yes ! No, no, no, that's not it ; if we knew it completely, if we had all the knowledge, there time would not pass. Time flies because we have partial knowledge, it's thermodynamics, Thermodynamics tells us, by engineering of Boltzmann, it tells us that entropy, for example, is partial knowledge about things, okay. So the passage of time is related to that. But nature keeps moving, okay ? !... *(laughs)*

- I have a question because you said that for your thesis, you were inspired by Tomita who showed non-commutativity...

- No, no, that's not it. Well yes yes, it's a detail... ! *(frank laughter)*.

- Could you just explain what are called types ?

- Ah yes, the guys ! ! Of course, of course, absolutely, well the three types... So the three types. Where are they, the 3 guys ? The first type, that's him... *(He shows a photo, laughter.)* The 3 types, therefore : type I is a quantum system such as in fact, Hilbert space of the quantum system breaks into a tensor product of two spaces, that is to say that it is really the simplest case that we can imagine, and that was what people had imagined it would always be the case. They always imagined that when you take a subsystem from a quantum system, we could break the Hilbert space into a tensor product of two sub-systems, so that the first system corresponds to the first space of Hilbert, to the operators in the first Hilbert space, and the other to operators in the second Hilbert space. So what von Neumann and Murray discovered is that there were actually two other types. It is to say that there was a way to have quantum subsystems that weren't at all a splitting of Hilbert space into a tensorial product. So the first type, there were the real dimensions. And then the Type III that remained was the others. And before Tomita, we had no tool to attack type III, okay. So what Tomita found is that in type III, there was this group $\sigma_{t\phi}$ and then what I found in my thesis, me, afterwards, was that the group that Tomita had found, in fact, it was unique modulo the interiors, that is to say it defined a real evolution, independent of everything else. So that gave a lot of invariants, etc., it made it possible to unblock everything. Okay ! So but it's incredible because von Neumann had defined these quantum subsystems so completely, how to put it, it's in von Neumann's writings, completely abstract. And we never would have thought at the time of von Neumann that it would have been linked to time, to the passage of time, I mean, it's absolutely amazing. It means depth of quantum. Heisenberg discovered that it came from non-commutativity, von Neumann reformulated it as operators in Hilbert space, the problem of subsystems arose, and from there comes the passage of time, it is fabulous !

- Could you explain to us why the entropy principle results from the partial knowledge that we have of the world ?

- This is Boltzmann and poor Boltzmann was so misunderstood in his day he ended up committing suicide. He had an idea absolutely... He have the following formula engraved on his grave $S = klogn$. That is engraved on the tomb of Boltzmann. He committed suicide near Trieste. This formula, what does it say ? It's one of the simplest formulas but one of the most difficult to understand. What is the integer n ? It's the number of microscopic realizations of a macroscopic state. I have to tell you a little bit about history : history is during the period where people discovered the steam engine, and then there were locomotives and all that, and so what people had discovered was that they had a way to turn heat into energy, movement, everything what they want. And that's how the railway started, etc. And they asked themselves the question of what we called the yield of machines and all that. And so of course, if you will, there were amounts of heat dq so that were between two systems etc. But we noticed fast enough that if we took two different paths to go from one point to another, from one state to another, the integral of $\int dq$ if you want, it wasn't preserved, that is to say that we cannot define the amount of heat of an object. On the other hand, we realized that if we divided dq by what we call the absolute temperature, well, that quantity if you want, it was well defined, that is to say that whatever path we took, between a state and another, the whole thing gave the same result. And it's which had made it possible to define entropy. But this entropy, it was defined for macroscopic systems which were given by the temperature, the pressure, the volume, well I don't know what, if you want a number of macroscopic quantities, there had no interpretation, none, and it was called entropy. It was called entropy, S. But this entropy had no philosophical meaning. since precisely, that's what we're talking about, okay ? And the incredible Boltzmann genius, it was this formula $S = klogn$. $dq + ds = logn$, that is to say what Boltzmann understood is that each time we take a macroscopic state so a given volume etc., we can have the same macroscopic state, from totally different microscopic states. That is to say that the simplest example is to take red balls and white balls, and stack them in a tank. And you have for instance 50 red balls and 50 white balls. You can see that you can stack them in 36 different ways, okay. But the macroscopic state matching will tell you that there are half of the red balls and the half white balls and that's it. The rest, you don't care. Well, what Boltzmann understood and which is incredible, is that entropy, which was defined completely ad hoc by the people who made steam engine systems and all that, well it was just the logarithm of the number of microscopic realizations of a given macroscopic state. Of course, there had to be a constant ahead. That's what we call Boltzmann constant, it's normal that it bears its name. So this Boltzmann constant, it's not the same as Planck's constant, and it is, well, obviously it must have the dimension of an entropy etc. Okay. But this is the most incomprehensible formula, and the most brilliant whoever that is, this formula of agreement. And it is very difficult to understand. What is very difficult to understand is that the laws of physics, not particle physics, but the laws of ordinary physics are invariant when we change t to $-t$. And if you want, which is very difficult to understand is that one of the fundamental principles of thermodynamics is that entropy increases. So we say : "But the time, which way does it go ?". That haunted people for years and years. And Boltzmann, he had understood countless things simply because of this idea. This is a wonderful example, with a very simple formula, but precisely if you want, that is also a very important thing that I would not have wanted to forget to tell you, which is that there are a number of notions in mathematics or concepts in physics like that, which have an extraordinary power, and this quality is to put thought in motion. This formula this is a typical example, you look at this formula, you try to understand it, there you go, your thinking is moving now. She has an extraordinary potential to set thought into motion. Because you can say to yourself "Why is it increasing ?". Basically, the explanation of Boltzmann's reason why it is increasing is that, in general, we will go to states that have more and more microscopic realizations, that is, which are more and more likely. And then to put it on a solid foundation is another story...

- So you've told us a lot about physics, and we know that in this moment, theoretical physics becomes a bit of a mathematicians haunt, for example with string theory, and suddenly I was wondering if you, in a sense, could consider yourself more like a physicist doing math ?

- It's a good question, I had friends who, knowing my opinions on string theory, said I was a kind of machine in which we put money, then, but if we put 1 euro, I will speak for 10 minutes against string theory. So I'm going to spare you. No, but I will quote a sentence from Hadamard. I have to find it already *(laughs)*. Wait, I have to find it... So, I think I'll get there. Here, it is a sentence on the link between mathematics and physics ; what says Hadamard, to characterize the depth of mathematical concepts that come straight from physics, he says (I say it in English so, but it's very easy to translate into French) :
  "... not this short lived novelty, which can too often only influence the mathematician left to his own devices, but this infinitely fecund novelty, which springs from the nature of things"[1]

  So here's the answer. The answer is that there is no on one hand mathematics, and on the other hand physics. It's the same thing : we try all to understand, okay. And precisely, there is this extraordinary depth in certain mathematical concepts which come directly from physics. Like Heisenberg. It is inexhaustible because it came from what we see, it came from experience, it came from physics, it is nature which speaks to us, which tells us something okay. This is priceless! But this is not the case with string theory, because string theory is a deviance which came from abstract mathematics, etc., and it has no contact with experience.

- There is another mathematician, whose name is Carlo Rovelli *(precision d'Alain Connes : "He's a physicist, he's a physicist" (laughs))* and he says that for him, the beauty of physics is a simple idea, which opens us up to a completely new world and at the same time this world is real, this world is correct. And I was wondering for you, what do you think about mathematical beauty...

- That's a good question *(sigh)*. First of all, there are a lot of people who, and I think it's true, who will tell you that the notion of beauty is a

1. *"non une brÃ¨ve nouveautÃ© qui souvent influence le mathÃ©maticien rivÃ© Ã ses propres prÃ©occupations, mais une nouveautÃ© infiniment fÃ©conde qui jaillit de la nature des choses."in Jacques Hadamard, PrÃ©face Ã lâintroduction au calcul tensoriel et au calcul diffÃ©rentiel de G. Juvet, Albert Blanchard, Paris, 1922..*

very relative notion, that is to say that everyone has her own different notion etc. of course. But hey, I admit that for me, the mathematical beauty, it is when, after terrible, terribly complicated calculations, we arrive to the same thing, we arrive at the result, but by an idea of incredible simplicity, a bit like the egg of Columbus okay. For me, that's it, the mathematical beauty, for me, beauty is the simplicity of an idea, but actually, of an idea that will... well for example, I don't know... when we were talking about Galois, I will give you an example of this beauty. that said Galois ? Galois says when we take an equation, we take a polynomial equation. So the first thing we're going to do is to find a function of the roots, which when we swap the roots, will take... Well, for instance, we take an equation of degree 5. It is necessary that when we permute the roots arbitrarily, this function takes 120 different values (5 ! different values). So how does he do it, Galois, to find such a function ? It's very simple : he says "if I call the roots A, B, C, D, E, okay, I take A plus 1,000,000 times B + 1 trillion of times C etc. Obviously, when I swap them, it will only take different values. It will take 120 different values". It's the first thing. Second thing, what does Galois say ? He says "well, now let's take for equation the equation which has these 120 roots as roots. We take this equation and we break it down into irreducible factors. We can express the roots of the original equation based on these irreducible factors and we will obtain, by taking these irreducible factors, permutations of roots of the equation. Theorem, that's the mathematical beauty. Theorem : The group of permutations obtained does not depend on any of the choices we have done. That is, if, instead of taking A more than 1 000 000 times B etc., I had taken 1 000 001 or whatever, I would have obtained the same group. That, it's mathematical beauty, it's something incredibly beautiful. Why ? Because that means we gave a recipe, which looked completely arbitrary, and we arrived at an invariant, we arrived at a group, which is a characteristic of the equation, which will give all the results that we want, and that is of a biblical simplicity, at the end, that is to say that the how it is defined is biblically simple. For me, that's it, mathematical beauty. But this is an example, I mean, define it abstractly, if we gave an abstract definition, it is obvious that we could find a counterexample... In fact, if you're looking for general things about beauty in math, on things like that, read Grothendieck's *Crops and Sowing.*

Because that, it is... Grothendieck was not only a mathematician, in fact, he was a literary, and he was someone who was capable in his writings, to go very very far in the analysis of what mathematics is, of what that beauty in mathematics is, etc. So he wrote 1500 pages, these 1500 pages, you can find them on the internet, okay. And don't listen to people who will tell you he's crazy because it's not true, it's not true : he was someone who was wonderfully intelligent, and who wrote wonderfully as a literary. He has an extraordinary vocabulary, etc. I made a presentation, at the seminar of Antoine Compagnon, on Grothendieck and Proust, comparing them precisely, and I think it's available this talk, maybe on the Collège de France website or on my website. Therefore, because I mean, because it's very striking, it's very striking to see that these are two individuals who have achieved something that few people both succeed in, which is not only a work, for Grothendieck, but also if you want what Grothendieck says, what he explains is that in fact, if we want to be realized of course, well, it's good to make a real analysis but in fact the main difficulty we have, is to understand yourself, and to understand yourself, it seems silly *(laughs)*, right ? And to understand yourself, you have to basically self-analyze, that's what Grothendieck did and in a way, this is what Proust also did in his book. They are also people who from one point on, stopped living and spent the rest of their life to re-analyze and understand their past life, okay, etc. And in both cases, the result is wonderful. Therefore the best answer, I think, is to go to *Crops and Sowing*, and reading it, not leafing through it, you really have to read it, you have to read it carefully, you see, like the passages that I read to you quite time.

- Please specify : *Crops and Sowing*, this is the book that Grothendieck wrote and which we have had access to for a short time, finally...

- No, not a long time ago, we had access to it for a very long time, but well, he wrote other books. He's a character, all these characters have extraordinary lives. Grothendieck had an extraordinary life because that : in 1970, he left the Institute of High Scientific Studies (IHES) and he became again, because that was his fundamental temperament I think, a bit of an outcast if you want. And from 1991 he took refuge in a village in the Pyrenees. And no one had any news of him, but he continued to work, he continued to write. And then, not only he wrote *Crops and Sowing*, but he also wrote another wonderful text, which is called *The key to dreams*. And it's a mystical text, but I'll say it's the same, I mean, it's the same... But it's extremely interesting, but I think for example, for people who are literary women, these texts have an infinite value. There are theses to do on this, there are 36 things to do, of course...

- So I saw somewhere on the internet I think, I'm not sure of my sources, that in fact you thought that mathematics existed were without men in fact, that even it was not an invention made by men, and I have a little trouble understanding that actually, because often, we see math as something very abstract that would not exist without men inventing them, so can you explain that ?

- Okay, I can give you the answer. The answer is very simple. You take chemistry. This is a subject that I, myself, discuss when I was in Math Sup and Math Spe, so you have all this stuff. So we have the compound bodies then we have the simple bodies. Simple bodies, there is the periodic table of the elements. The periodic table of the elements, unbelievable but true, there is the Pauli exclusion principle, and a very small equation, which gives it to you. That's enough for me. Why ? Because let's imagine that there is another planetary system etc. If they are intelligent beings, they will understand chemistry, that there are simple bodies, they will be the same, they will not have... I mean they will not have chemical bodies, they will not have simple bodies different from ours, so they're going to understand simple bodies. And then, if they're really smart, they'll try to find, good, they will have the periodic table of the elements. They will try to find what is the abstract origin of the periodic table of the elements. Well if they are really smart, they will find the same thing, they will find that there is the Pauli exclusion principle. And then there is this little equation... What does it mean ? It means that, behind the apparent, how say, arbitrary of the world around us, there are some incredibly simple rules, chemistry, iteration since trees are all governed by that iteration, and that in fact, there is a way of understanding the world, which instead of being chaos, if you will, is something much more structured, and that is structured by mathematics. And there is no reason why, of course, people give the same names to the mathematical concepts they will have used, but it is quite clear that they will use the... , if they are different beings and they will have 1, 2, 3, 4, 5. They will not say it from the same way, but they will use mathematical language, this language will be in correspondence with ours, as the Chinese language is in correspondence with ours. So, it is in this sense, it is in this absolutely fundamental sense, that I say that mathematics pre-exists, why ? Because it would be incredibly pretentious to say that we invented whole numbers. So at that time, why would chemistry have already used these things to exist ? It seems completely stupid. So actually, what I'm saying, is that, when we found, when Watson and Crick found the DNA structure in double helix, they didn't invent it, nobody will believe that they invented it of course, they discovered that. It was a reality, and this reality, it pre-existed to them. Well, for math, it's the same, it's exactly the same, that is to say that we discover, a bit like an explorer will discover something. This explorer, he has free will, he can go to such and such a place. This is what makes people believe people actually it's like art. But no ! It's not art, it's exploration. And it's an exploration all the more... how to say ? real, that this reality, it resists. If it was art, we could say anything. This is not the case. This is not the case. There is a terrible resistance... And so an example, another typical example, is that if for example, I write an equation etc., and then good, for example, Galois said that the calculations we had to do to follow his method were impossible to do and in his day it was impossible to do. And for showing that it was impossible to do, when I gave my presentation to the Académie, I explained to people the Galois method and I asked them "Can you give me an idea... ?", Well, because each root is expressed as a polynomial according to the roots of the auxiliary equation. I asked them, I gave them an equation like the one that I gave you a while ago, I told them "can you give me the order of magnitude of the coefficient of order 0 of the polynomial which expresses the first root ?"... 1 million out of 1 million, or something like that. No, the answer is a 500-digit number over a 500-digit number ! The computer does it now, and the computer checks that Galois was right ! So to say that Galois invented it, I mean, it's a bit big, what ! No no no ! No no no ! We discover, we discover, but exactly as the electron microscope was necessary to find out the double helix structure of DNA, the mathematician invents tools conceptual to successfully perceive this reality. But he invents conceptual tools of course. But it's a reality that is there, it resists, it is completely tangible and it governs nature. It's more fundamental, for me, this reality is more fundamental than the nature that is around. It pre-exists that, okay. If you want, it would be the same... I think that it would not be fair to say that nature is only written in the language of mathematics ; math is more than that, it's more than that. Nature is consubstantial with mathematics. We don't realize it because you're not smart enough to give an account of the explanation which is behind all these phenomena. If we realized more, we would know much better. And it's all more true with quantum. I mean, the quantum there is obvious. The quantum, it's a reality that we only perceive through mathematics, we absolutely does not perceive it otherwise. That is to say that the people who make experiments with quantum, in quantum optics, they understand that it's that Hilbert space, they touch it like that, okay, it's incredible, really incredible !

- Thank you.

- Are you ready to answer a few more questions ?

- Yes, it's okay, it's okay.

- For us to have more facilities for example with the abstract concepts, in mathematics or in physics, or even with reasoning, what would you recommend in education and teaching, to present, in primary school or even in secondary ?

- I will answer, first not in primary school, nor in the secondary, I answer for you, because it is the most useful. So for you, what I advocate is the following. To answer a question, even a complicated math question, you leave everything in the air, you're going to do a walking tour, okay. And the question, you keep it in your head, okay, you keep it in your head, and you reflect. Obviously, it can be different depending on the individual, but for me, it's the big secret. That is to say a calculation, as complicated as it is, you can say to yourself "Oh ! I will never get there if I try like that !" No ! You go for a walk, and you think about the structure of the thing, and after when you come back, well, you will see that it improves things drastically. So now, in primary school, I don't know, me, everything I can tell you, it's my own experience, because I don't know any else. But my own experience is that when I was a kid, when I was 5 years, my father forced us to do calculations. We were in the garden with him, he was with us, and he made us do operations, and at the time, we did the four operations, that is to say we did the division, at 5 years old, we was multiplying, we hadn't waited for the sixth to learn the division, and all that, we did that. And after another experience which happened to me... And I loved it, it was probably also my relationship with my father, it scared me at the same time, but I was happy to make him happy. Anyway, I don't know, so I don't know, I don't know how to explain that : I will say that I think there was an extraordinary virtue in doing operations like that, that is, memorizing the multiplication table and then we didn't forget the multiplication table, if we made multiplications and additions all day long, we did not forget it, we knew it afterwards. And it became absolute automatism. So there was that, and I liked it a lot. Another story than I have, it is that once, that, I find it absolutely extraordinary, once, I met a friend I hadn't seen for a while, we were playing soccer together, in time, and then maybe 8 or 9 years later, I take the TGV to go to, I think it was in Rennes or something like that, and then I go to my place of TGV and then, I looked at my number, and I see someone next to it kept his number and he was my boyfriend. We started to chat etc. And then then, the usual discussion, you have children, he begins to explain to me he has a son, and that his son is weird. It must be said that my boyfriend is literary. I said "why ?". Well you know, well first, he had been sick when he was young and then once, when he was 5, we were together, we were on the beach and then he looked painful ; I was worried, I mean for an hour he was there instead of going for a swim he was a little white and then after an hour, he comes to see me, so he's my boyfriend who tells, he comes to see me, and he says to me : "Daddy there is no greater number !". I said to him "Look, your son, he's great !" *(laughs)*. He tells me. "Ah ? Yes of course !". I asked him if his son had found a demonstration and he had found a demonstration, which is not the usual demonstration, it was not adding 1, it was multiplying by 2 or something like that, whatever, he had found a demonstration. It's amazing, but afterwards, he said to me, "you know, he had problems at school" *(frank laughter)*. So he told me about his problems at school. So that will answer your question for primary school. He was in elementary school, so we had him posed the following problem : it was "a florist has 120 flowers, she makes 4 bouquets of 17 flowers, how many flowers are left ?", okay. So he had had "zero, has no sense of operations". He was not stupid, he has found 120 since she didn't give them *(laughter from all)*. When he told me that, I said "well listen, your son, he's a mathematician". And so, we organized therefore with his father a meeting at Tea Caddy in Paris, it is a charming place. And her son at the time was 12 years old. So, things have evolved luckily, it took a while, and now he's a great mathematician who is a teacher at Orsay. So incredible, incredible, incredible ! So I believe that what matters is what Grothendieck says is to go back to this childhood state and to ask the right questions, and then not to hesitate to be against the current etc. Above all I mean that the moment we become mathematician, this is the moment when we are able to tell the teacher that he is wrong and why, that means being able to resist its authority, to say that we have reflected, and that we do not agree, and then to be sure of ourselves, because that one has thought for oneself, it is hyper-important.

- No actually, I have a question, even if I think you have a little answered this question by your comments, but I would really like to know for you what is the purpose of the work of a scientist finally if you think that rather, it is to increase, to advance fundamental knowledge, or to make it accessible to its public for possible application.

- There are these two aspects, which we should not mix at all, there are these two aspects. I think the real motivation is to move forward basic knowledge. In other words, the real motivation which must be precisely independent of any authority, of any desire to knowledge, etc., the real motivation is to try to understand, understand where we are, where we landed, okay, that's it, it's very simple to understand, that's where we are.

- No, but on this question of the popularization of knowledge and in particular mathematical knowledge. To hear you, there is a risk : the effect butterfly is one. I was going to say what I had learned, for example, from entropy, what philosophers, what some philosophers can do of entropy, there is sometimes indeed a great danger of a kind of... and at the same time, you seem to be saying that there is a need. So if we basically asked "how to avoid the danger and respond to the need, would you be care... ?

- Yes, yes, that is a very good question. There was Sokal, there was Deleuze. There was Lacan, I don't know if you know, but Lacan said in a seminar that the number $\sqrt{-1}$ is the symbol of the male sex, okay. This is what we call the imaginary pure number!! *(laughs)*. You had to do it anyway, huh?! And in addition, he once did a seminar, where he had a theorem, and his theorem was that "Don Juan is compact". Someone had told him the definition of a compact space in mathematics. So that, of course, it's stupid, okay, it's absolutely stupid. And what is it? These are poorly understood math concepts that are used as a psychological authority over others, that is, they are used because people will not understand and the butterfly effect is a blatant example, as a psychological authority because people when they don't understand, they are in a position of inferiority, their understanding is blocked, and if you want, they are impressed etc. So there is this terrible way to use math which is precisely to use large words, like a kind of psychological power over crowds. So that, it is to be banished at all costs. However, I'm sorry if you want, that concepts as beautiful as Grothendieck's concept of topos, not be better known by people who would need it because like I tell you we are all now victims of scientism which consists to believe that a thing is true or false when in reality there are situations that are much more subtle than that, much more subtle than that and who ask for a thought tool that the concept of topos gives and it's a notion which is delicate, which is difficult, which requires, to know it, to understand it, mathematical knowledge. So what I will say if you want is that there is a beautiful boulevard that is open. This boulevard is to learn enough math to use it afterwards in the right way in other areas, but first you have to start to learn enough math, that's the price to pay, it's absolutely necessary okay.

- My question was precisely in relation to the question of truth : by hearing you, one has the impression that really mathematics, it allowed to reach this truth with quantum physics, and I wanted to know, I believe it was Einstein who said that "the world is a bit like a closed box", you can just see what's going on, but you can never be sure that what we find is true. And what do you think of that, of the idea that maybe all we are explaining are theories that are ultimately false, for example, Einstein, who put everything back in cause in physics... ?

- There is always indeed the possibility of a theory above, which will simplify what's on the front floor etc. But we can still see that progress, from this point of view. What I was trying to get across, it's the extraordinary subtlety, the richness of nature, where you are. The fact that each time we will have surprises and we will have extraordinary surprises. Since at the end of the XIX$^{\text{th}}$ century, there were physicists who said that we had understood everything. And precisely, we were before the quantic era, before all that, before general relativity. It's true, perfectly true, what you say. But, I will insist more on the wonderful imagination of the nature, what, I mean, we are surely very, very far away, there may be civilizations, there are surely civilizations, in other inhabited planets, in which people have gone much further than us. That's really possible and they would take us for primitives. It is possible, it is absolutely possible.

- You said that mathematics were not in the domain of known, and globally, the sciences. But don't you think that this is because mathematicians, physicists and others do not participate not enough to debate, for example, when defining scolar programs, we hear philosophers, historians...

- That's right, there is truth in there, there is truth. But on another side, that's not really the problem. I will not say that the problem comes from fact that it is not popularized enough. I think the problem comes more from the slow absorption by the whole of society of elaborate notions. For example, I take a typical example, which for me is important. You see, when the printing press was discovered, the notion of number has been transmissible. That is to say, there have been books, etc., etc. Now, we are at the point where it is no longer the number that is transmissible, but it is the notion of function, graph, etc. And there is a vocabulary that has passed in the general public, for example, when we say that we are going to reverse the unemployment curve *(laughs)*. So that involves the cancellation of the second derivative. If we were told "we're going to cancel the second derivative...", there would be something to laugh about *(laughs)*. Well, well, you kind of see... So it's sure that, if you will, there are now some mathematical notions, including the notion of growth function, decrease, primary derivative, cancellation of first derivative, etc. that have gone out to the general public, okay, but there are much more subtle notions, like the notion of topos, like the notions that come from quantum etc. who have more trouble to pass in the general public. How to get them across to the general public ? No doubt through education, but at that point, we should be very braver, in the school system, it's obvious, it's absolutely obvious. We should not be in the current renunciation which is dismal. I knew that in my time, well I mean, we didn't stop, I did not stop and my friends did not stop, making problems of geometry. We came home, and we made problems, and they were not easy problems at all. And now it's over ! Now we learn recipes, we learn to apply recipes ; of course it's much easier for a teacher. One day when you have children, you will notice one thing that is absolutely fundamental is that when you have a small child, you have a choice. The little child, he tries to do something, you have two possibilities : the first possibility, is to do this thing for him, you believe you are helping him, in fact, you don't help him at all, you harm him by doing that ; the second thing is to be patient, and to wait for it to happen by itself. And there you do something really useful, okay. So good, in what we do in the current school system, which is to learn ready-made recipes for solving ready-made problems is doing things instead of the child. That's exactly what we do, it's exactly that thing. So that the real discovery of work, the real discovery of school, it must be between 11 and 12 years old. And it must be done by drying in front of problems which we are not given the solution for, but which we ask you to find, okay, where you are asked to dry. And from the moment where at that age, we understood what real work is, it's okay, it's okay, it's OK. And that is not the case in the current school system. Of course, far from there, terribly. Of course. There is Laurent Lafforgue who tried, and of all his strength, to go in that direction, well, he spent a lot of energy ; to say that he got there, that would not be true, I will say, in any case, there are people who made a colossal effort to go the right way, now afterwards, there is a terrible resistance from the system.

A topo on topos
Alain Connes

Abstract : Alain Connes presents the intellectual approach that led Alexandre Grothendieck, from an "annoying" writing that he had to do for Bourbaki on homological algebra, to discover and perfect the concept of topos and he tries to explain in what sense this notion has a considerable scope, thanks in particular to the nuances it introduces between the true and false (Seminar organizer : Frédéric Jaëck (ENS), Transcription : Denise Vella-Chemla)

So I hope that I will stay in the spirit of the seminar and I think that the spirit of the seminar is Grothendieck, above all. So what I'm going to do is trying to get as far back as possible Grothendieck's thought and trying to explain precisely, as I said in my abstract, the journey that brought him to the topos, and above all, I will try to give you an illuminating metaphor for what is a topos, and explain to you what is extraordinary about this discovery, in the sense, especially for philosophers, in the sense that it introduces considerable nuances into the notion of truth. I will try to explain this by an example, because nothing like a good example to explain. There will be one of my slides called "at two steps from the truth" and I'm really going to give you an example of a topos which allows us to say that we are for example 10 steps from the truth, or that we are 15 steps from the truth, etc. So listen carefully. I'll make you hear Grothendieck, Grothendieck's voice, because Grothendieck did 100 hours of lectures in Buffalo in 1973, and in those 100 hours of lectures, there are things that interest us. Of course I will not make you listen to him for a long time but I will make you listen to a point in time, where he explains what a sheaf is to people who don't know this at all. And he explains how he will do his course on topos. We will see, there will also be an even more funny interlude at some point, we will not hear Grothendieck's voice but we will hear Yves Montand's voice. You will see, finally, you will hear all that.

So the first image I show you is an image that I owe to Charles Alunni who sent me an email one day telling me that he would have liked to have Grothendieck's second thesis. But at the time, when we were doing a thesis, when I did my thesis for example, there was always a second thesis. This second thesis was not written. It was a second thesis that we had to defend in front of the jury. And we had a subject that was given to us. What is quite extraordinary is that in Grothendieck's case, he did his thesis on nuclear spaces, on topological vector spaces and on nuclear spaces, and he made a fundamental contribution to functional analysis and what is extraordinary, it is that one can think that what made Grothendieck branch off, and which eventually led him to the idea of the topos, to this wonderful idea, this is his second thesis. Why ? Because the Grothendieck's second thesis is written in this text, it is on the sheaves theory.

And then on this page, if you are perceptive, you will find that there is an error, which shows that you are never safe from mistakes. Because there is one of the examiners, if you look closely, who is called *Georges* Choquet *(laughs)*. So I looked, to say maybe I was wrong, there are 3 examiners, there is Henri Cartan, there is Laurent Schwartz and then there is Georges Choquet. So I have searched on wikipedia to see if there was not a mathematician named Georges Choquet. In fact, no, there is a clergyman called Georges Choquet who died during the second world War. So there is no problem, it is a mistake, and it is Gustave Choquet who was Grothendieck's examiner. So he sustained his thesis in 53.

And already in 55, he was of course interested in sheaves, which was a wonderful discovery of Leray. And so there, I will start with some exchanges of letters between Serre and Grothendieck because finally, it is in these exchanges of letters that we see appear what is a so famous article that we call it The Tohoku. This article appeared in a newspaper called Tohoku Maths Newspaper but the article is so famous that in fact it is called Tohoku.

So that's what Grothendieck says ; he says :

"My Dear Serre,
Thank you for the various papers that you generously sent me, as well as for your letter. Nothing new for my part."

Lecture by Alain Connes, November 7, 2017, as part of the ENS seminar "Grothendieckian Lectures" that can be followed here http://savoirs.ens.fr/expose.php?id=3257.

So that justifies what I wrote when I announced my presentation :

"I finished my annoying writing of homological algebra."

So we will see very gradually what is the philosophy that Grothendieck uses all the time when he works, that is to say that he never hesitates before a task than any normal mathematician would consider it to be uninteresting, off-putting, not going to bring him anything. So he continues :

"... that I sent to Delsarthe who lacked the editorial staff for the typist."
He says :

"I proposed it to Tannaka for Tohoku ."

Tohoku is Tohoku Maths Journal.

"It seems that the river articles do not put them off."

So it is true that Grothendieck, in general, when he writes, the minimum is at least 100 pages. Then he talks about Weil. I'll spare you what he says because he says :

"I have read at least the statements in Weil's books on abelian varieties in the hope that we have arranged since the really discouraging demonstrations at Weil, his language disgusts me."...

More ! *(Laughs)* I pass... He says :

"I spend my time either learning or writing the varieties. It's fun but long sure, but there is no question of research until you have swallowed a mountain of new things."

So then, very important, this is probably the most important thing in this excha nge, he says to a given time :

"I realized that by formulating the theory of derived functors for more general categories than modules..."

(it must be said that at the time, there was the book by Cartan-Eilenberg which was in the making, Serre calls it the Cartan-Sammy - because it's Sammy Eilenberg - and in this book, there were of course the derived functors but it was still applied to module categories. That is to say, we took the category of modules on a not necessarily commutative ring and all the homological algebra was developed like that. But obviously, it was very analogous in its formulation with what was happening for cohomology with coefficients in a sheaf. So what Grothendieck says is :

"I realized that by formulating the theory of derived functors for more general categories as modules, we get the cohomology of spaces with coefficients in a sheaf at low cost."

You should know that at the time, when we took cohomology with coefficients in a sheaf, it was always Čech cohomology. That is to say that we took covers of the topological space, and then we made a complex or a bi-complex with these covers and we defined cohomology like that. Here.

So that's what he says, and that point in his correspondence is absolutely essential. So then he continues, and so this is a letter from June 4, 1955, so we are 2 years after his thesis, so :

"Attached is the result of my first cogitations in form, on the foundations of homology."

So afterwards, I won't detail the rest since it's on spectral sequences, etc. But let's say that there, Grothendieck explains that he had planted himself on the existence of enough projective sheaves but at that time, he demonstrated that there were enough injection sheaves, and that allowed him to define, if you want, cohomology with coefficients in a sheaf without assumption on topological space. If we have good hypotheses on topological space, at that time, it coincides with the cohomology of Čech. But this is not generally true. So here is the Serre's response. So there was something else in the correspondence, there was something else that was what is called Im_1, i.e. the functor for projective limits, how it commutes with the cohomology. But what is to read is the second paragraph.

"The fact that the cohomology of a sheaf is a special case of derived functors, at least in the para-compact case..."

(because in the para-compact case, it coincides with the cohomology of Čech, so the one which is defined from coverings)

"... is not in Cartan-Sammy."

(Cartan-Sammy, this is Cartan-Eilenberg)

"Cartan was aware of this."

So Cartan was aware that when they had developed the whole cohomological theory on the modules with Eilenberg, he was of course aware of the analogy with the case of the cohomology of sheaves, but hey, they didn't want to bother themselves to do it in their book, and Cartan had conscience and had told Buchsbaum to take care of it. So it's actually Buchsbaum who, independently of Grothendieck, also defined the abelian categories. He had started to develop it but he hadn't been involved in cohomology.

"But it doesn't seem to me that this one did. So the point of this would be to see what are just the properties of the fine sheaves to be used. So maybe we could realize if yes or no..."

(it's Serre speaking, of course)

"... there are enough fine sheaves in the non-separated case.".

So the non-separated case is extremely important of course for algebraic geometry and it was a time when Serre was developing algebraic geometry from Leray's sheaf theory taking Zariski topology and Zariski topology, at the start, it is not separate. I mean, so, we have exactly that problem. So this exchange is extremely important, it's an exchange dating from the year 55. And Grothendieck's article therefore, it is marked *Received 1 st March 1957*, so it is this article *"On some points of homological algebra"* which is really the ancestor, we can really place, if you want, the origin of the topos in this article.

The reason we can place the origin of topos in this article is that in this article, if you want, first, of course, he introduces what abelian categories are. So this is extremely important, with all their properties, etc. He develops homological algebra in the context of abelian categories. So that's what he does, if you want. But, what is much more important, is that he takes two types of examples, in his article, of abelian categories. The first example of abelian category that he takes, it is the abelian category of modules on a ring. It belongs to Cartan-Eilenberg. There is no problem there, but he also takes the example of the sheaves of abelian groups on a topological space, of course ; again, no surprise since it was for unify the two that he had done his generalization work. But what is absolutely crucial is that he had another example in mind, a third example in mind, and that's what he called the categories of diagrams. That is to say, what he was doing was that he had the idea that if you take diagrams of abelian groups, but whatever diagrams you look at, well, that still forms an abelian category. Well actually, if you think right, you realize that he had the two pillars of the concept of topos. That is to say, he had the notion of topological space, which gives the notion of sheaves on abelian groups, etc., and he also had the notion of categories of diagrams and we will see that these categories there, they give rise to a topos. And these topos have an absolutely fundamental role that we will use it all the time, all the time, all the time.

So there is one thing that should be noted : he defines what an abelian category is. So what Grothendieck says is that an abelian category is an additive category. So I will explain to you in two words the misconception that we have on the additive category which satisfies the two additional axioms following : then, there is the fact that a morphism must have a nucleus and a co-nucleus, that, it's an abstract notion, if you don't know it, well, it would take a while, it's a bit of gymnastics, and the second condition is being an exact morphism. That is to say, the fact that if you divide by the nucleus and if you look at what is called the image by defining it with respect to the co-nucleus, well then you have an isomorphism between the quotient by the kernel and image. And that is extremely important, it is not true for applications in topological spaces for example, if you take an Hilbert space and if you take the continuous linear applications in Hilbert space, that does not satisfy these two conditions. It is not an abelian category and the reason is that you can have a morphism which has an image, but it is dense in its image and its image is not closed and at this time, the second condition AB2 will not take place.

So there is a misconception that Grothendieck reproduces when he defines the additive categories in his article and that is the following : that in general, people define an additive category by saying "an additive category, it is a category where we add an additional structure which is the additive group structure on morphisms." Well, this is an heresy. I will explain to you why : because in fact it is not at all an additional structure. And there is an exercise in MacLane that I noticed for you there which shows that it is an heresy. What is this heresy ? The heresy is that when you take a category like an abelian category, it has products and co-products. This is a very simple thing. And it has an object which is both initial and final, it is object 0 ; in the abelian groups, the group is reduced to 0. Okay ? It is an initial object because you have a single arrow going from 0 to any abelian group, and it's a final object because you have a single arrow going from an abelian group to 0. Everything is sent to 0. So there is an object that is both initial and final. Well, if you have that, you can tell, when the category is abelian. How ?

Well, because what happens is that you have a completely canonical arrow, completely natural, which goes from the co-product of two objects to the product of two objects. Because using the 0, you can send half on 0 and the other half on 0 and at that time you have an arrow which is completely natural. If you asked if this arrow is an isomorphism, you did half the work. Because as explained in this MacLane text, at that time, you have an addition for morphisms. Simply by using the fact that this natural arrow which goes from the co-product towards the product is an isomorphism. You have a natural addition for morphisms and once you have this natural addition, you can take the additional axiom that there is a minus sign, if you want, for this addition, of course in general, you will not have a minus sign, that's what we call semi-additive categories, they are very interesting, but if you want a category to be additive, you just ask that there be a reverse. So then, it's not an additional structure. It is an heresy to believe that an additive category is given by a category + an additional structure. It's not true. So it's a misconception.

So, now I'm going to read Grothendieck, since the principle of the seminar is to fade away in front of Grothendieck. And even at some point, we will listen to him. And then when we have read and listened enough Grothendieck, there I will take a metaphor, then we will see an example. Okay so be patient, I'm not just going to read or listen to Grothendieck, you have to be patient, but let's listen to him anyway. Here is what he says :

"The point of view and the language of the sheaves introduced by Leray led us to look at these "spaces" and "varieties" of all kinds in a new light. They did not touch, however, the same notion of space..."

So what Grothendieck says is that Leray had envisioned a space in the form of sheaves on this space, but in fact, moreover, Leray only considered the sheaves of abelian groups. And we will see the change that Grothendieck has already made even there.

"They did not touch, however, on the very notion of space, only making us appreciate and more finely, with new eyes, these traditional "spaces", already familiar to all. But it turned out that this notion of space is inadequate to account for "topological invariants" essentials which express the "form" of "abstract" algebraic varieties (like those to which Weil's conjectures apply), even that of general "schemes"..."

So here is the moment when I will make you hear Grothendieck. Why, because we are going to hear Grothendieck who defines what a sheaf is. So, I think it's important that you listen to this, okay, because you don't necessarily know what a sheaf is, we're going to listen Grothendieck talking about this, okay, and once we've heard Grothendieck talking about it, we'll come back to our sheep. This is the start of his conferences in Buffalo.

"Topoi were objects for years essentially on general topology. I mean a topos could be considered as the main object of study of topology. And so topoi is generalization of classical general topology. It's what I really like to consider... Its study require some familiarity with handling topological spaces and continuous maps, and homomorphisms, and such things, and on the other hand familiarity with the language of categories... Later we... exponentiations... and give examples. But in order to understand... what a topos means..., one would require some familiarity with the language of sheaves on a topological space. I guess

that this notions are not very familiar for everybody, therefore I think I would give rather some introduction to sheaves on a topological space. I want to assume anything known on this matters. I would give a review of standard "sheafsury"... I hope that if some explanations are not clear, or if some comments come up that you'll freely interrupt me to ask questions, or to point out errors or to make any kind of comments. It would be nice if, as time goes on, there would be some kind of participation of the audience, I am sure that a number of you know something about topoi.... You will be able to make suggestions and comments.

So I start with a kind of formal survey of sheaves on a topological space. Provided an X to be a topological space, and consider the set $O(X)$, it depends on X, of all open subsets of X. The topology of X is defined in terms of the family of open subsets of X which is a subset of $PartiesdeX$. I recall the axioms of the topology is that O should be stable under arbitrary unions and under finite intersections. All right. So $O(X)$ in particular is an ordered set by inclusion and that, that all defines a category by abuse of language.

... any all upset does. If U and V are elements of $O(X)$, are open sets on X, the set of homomorphisms of U into V will be either empty if U is not contained in V and may be reduced to just the inclusion map of U into V, if U is contained in V ; this is the definition of empty and of the decomposition of arrows... There must be at most one arrow from an object to another. So this construction of a category in terms of open sets makes sense for any all upset whatever. So the category which behave as... arrows of the graph of all the relations... Now let's first define pre-sheaves : a pre-sheaf on X, say f, is by definition a functor which goes from the category $O(X)$ to the category of sets, when I say a pre-sheaf, I mean a pre-sheaf of sets, later we will see other types of pre-sheaves. But it should a contravariant functor... It's a functor which goes from the opposite category to $O(x)$ to the category of sets, so let's recall what this means for a functor : first of all, it means for the objects of that means for every open set, $U(X)$, we associate an object fU of the category of sets..., Remarque d'Alain Connes : ce qui est très important, c'est qu'il parle de faisceaux d'ensembles, et Leray parlait de faisceaux de groupes abéliens. *It is an arrow of categories, that means that for every inclusion map U to V two open sets, we associate a map from fV into fU..., this map will be called the restriction map, corresponding to the pref-sheaf, and the axioms are the evident axioms of transitivity, namely if we have V is contained in another open set W, then we have a restriction map from fV to fW but also from fW to fV and we want the restriction from fW to fU would be the composition here and moreover we want that in the case than we take the identity arrow, the identiry arrow from U into U, you want that the corresponding map fU goes to fU would be identity. ... as maps of functors... So a pre-sheaf on f is just a contravariant functor from the category $O(X)$ to sets is just O to sets. And the category of pre-sheaves on X, let us recall it $PreSheaf(X)$, is defined as being the category of all functors from $\dot{O}$ to sets. So the pre-sheaves can be viewed as being objects of a category, the category of pre-sheaves, in the category of functors.*

So what is an homomorphism of a pre-sheaf F into another G, by definition of homomorphisms of functors, let's say f be such an homomorphism by definition, f consists in a connection of maps from fU into gU, we call this map $f(U)$ for every U object of the category that of all open subsets of X, and these maps of sets will be compatible with restriction maps, that means that whenever U is contained in an open set V, then we have also fV that goes into gV by f of V and we have the restriction maps here, from fV to fU and from gV, to gU and so that the square should commute. So that's an homomophism of pre-sheaves, that is just an homomorphism of functors, and they compose in an evident way, their composing is out and ? ? ?. There is an homomorphism from the pre-sheaf... that means for every U, an homomorphism from gU into hU, the composant of g and f is defined as associating for every U the composition of homomorphisms here.

All right, so that's just general nonsense[1] *on functors, on categories. Now, so far, we have not used the fact that $O(X)$ was the category of open subsets of X, we have just used the fact that it's an ordered set, but we are going to use it now in order to define the notion of pre-sheaf on X which recall sheaves on X, we have to introduce another axiom on pre-sheaves which will turn on to sheaves. Now, traditionnally the axioms on pre-sheaves is separate in two : you say first that the pre-sheaves are separators if the following is true : for every open set U on X and for every covering of U, open subsets U_i a covering is a union of the U_is, can be defined in terms of ordered sets ; it's just the supremum of the U_is, in terms of ... Be f a mapping of fU into each one of the fU_is, a restriction map, and therefore, we get a mapping of f into the product of the fU_is and have that two sets separated by every such choice here, this mapping is injective.*

Now, let's state this in another way : if f is a pre-sheaf on X, f of U, the elements of the sets accord

1. ?

to sections of f over U. When we'll give examples, we will see where this accord of sections come from. All right, and, the map here I already said contained in V, the given map of fV into fU will be called the restriction map... that mean the sections of ... is called the restriction of ... to U and the axiom for a sheaf to be separate means that whenever there is a union of open subsets $U_i(X)$, from which the union is U, there's a section of the pre-sheaves over U is known as we know these restrictions on the U_is. The first condition would be the restricted in ... but in geometric terms, it just means that ... is an injective arrow, that means that a section of f over U can be identified for the system of sections of f over these U_is, and then the second question that arise is to see whether we can identify the subsets here that we obtain as image of fU which are the systems of sections of f over U_is which come from global sections f over U. Now let's take a system of sections say Φ_i in fU_i for every i, for every index, here the necessary condition for this system of Phi_is to come from global sections which is the following : when you have two indices i and j, the restriction of Phi_i to $U_i \cap U_j$ could be equal to Phi_j..."

Here, I stop here, our patience is probably exhausted. But, I will say that one of the reasons for which I made you hear Grothendieck is quite complicated : we have to get used to this incredible patience he has to explain all the details, to come in, to go through all the details. And that, we will see, I mean, it is an absolutely essential quality in his approach. So I continue to read what he said about the point of view and the language of sheaves introduced by Leray. Grothendieck continues, he says :

"*For the expected "marriage of the number and the size", it was like a bed definitely narrow, where only one of the future spouses (that is, the bride) could at least find a place to nest somehow, but never both !"*

So there, if you will, he actually had in mind all kinds of developments that were of combinatorial nature, which were related to number theory as opposed to what was going on in topology but hey, of course, there was Serre's work on the use of Zariski's topology.

"It is the point of view of the sheaves which has been the silent and sure guide, the effective key (and by no means secret), leading me without procrastination or detours towards the nuptial chamber with the vast marital bed. A bed so vast indeed (like a vast and peaceful very deep river...), that
"All the king's horses
could drink together...""

We'll come back to that.

"As an old tune tells us"

(which I will make you hear later)

"As an old tune tells us that you must have been singing too, or at least hearing sung. And whoever was the first to sing it felt better the secret beauty and the peaceful force of the topos, that none of my learned students and friends of yesteryear..." (laughs)

Well, there is of course in this sentence, the fact that we know, without doubt, many of us, which is that the first to reveal a certain mathematical landscape has an apprehension of it which is incomparable (this is what happened to Galois for example) compared to the other mathematicians who come after him and understand him. This is a very striking thing, he says something more nasty, of much nastier.

"The key was the same, both in the initial and provisional approach (via the very convenient concept, but not intrinsic of the "site")" (which I will tell you about, of course) *"... than in that of the topos. This is the idea of the topos that I would now like to try to describe."*

So let's let Grothendieck speak for sure.

"Let us consider the set formed by all the sheaves on a (topological) space"

So, for the mathematician, frankly, these are sheaves of sets, that is absolutely fundamental, it is an enormous step, that it replaced the sheaves of abelian groups, which were interesting, we thought they were the only interesting ones, since they are the only ones who will give a cohomology, etc. No ! He had the idea, which seems completely naive, to replace the sheaves of abelian groups by sheaves of sets, and we will see the range it gives.

"I believe elsewhere" (and that's what he says) *"to be the first to have worked systematically with sheaves from 1955.*

What is the advantage of working with sheaves of sets, as we will see, is that when you work worth with the sheaves of sets, you can define what a group is in this stuff, you can define what an algebra is in this thing, because what you do is that you work as if you were working in sets but there is variability. That is to say, there is something that moves, but otherwise you do exactly as if you were working in sets, as usual. So you can define what a group is. So if you're looking for what an abelian group is in this category of sheaves of sets, well, you find the sheaves of abelian groups. So we fall back on our feet.

Then what Grothendieck says :

"We consider this "set" or "arsenal" as having its most obvious structure, the one that appears there, if one can say, "at first sight"; namely, a so-called "category" structure."

So of course, we were educated, at least in my time, with set theory. In fact, it was probably a mistake. The real way of thinking is category theory. So he thinks of this kind of theory that he has before his eyes as a category.

"(Let the non-mathematician reader not be confused, not to know the technical meaning of this term. He won't need it for the future.)"

That is to say, what the non-mathematician reader simply has to think is that he has an analogous, now, of set theory, in this new category, in this category of sheaves of sets. There is something that resembles set theory, and we will see that this metaphor goes very far.

"It's this kind of "surveying superstructure", called "sheaf category" (in space envisaged), which will henceforth be considered as "embodying" what is most essential to space."

So we will see a metaphor, which I will develop further, but I can already reveal a part of it. If you want, usually, when we talk about a space, I'll show it to you right away because I don't want to wait, for this metaphor. Here, the metaphor is as follows : if you want, before Grothendieck, we used to, when we were studying a space, I don't know, a curve, or wearing what, we put space... on the stage. And then we looked at it, we studied it, as a whole with structure, etc. Well, what Grothendieck does is... no! Space is not on the stage, space, it's behind the scenes. On the scene, there are the usual actors of set theory : the abelian groups, algebras, etc. But, the space in question, it is behind the scenes, like a species of Deus ex machina which introduces variability in the characters who are on the scene. That is to say that now the characters who are on the scene, they will depend on a hazard. This hazard is governed by the topos. And when we have a Grothendieck topos, there are also the constants, that is to say that there are also the sets which do not depend on the hazard. And cohomology, it is defined by comparing the two.

So it's absolutely fundamental that you gradually try to acquire a mental image, even if you are not a mathematician, to understand that the space which is given by the topos, it will appear behind, it is behind the scenes, it is not at the front of the stage, it is not it that we studied; we study set theory, but there is this good god of topos, which is hidden and which makes everything vary, which introduces variability in there, okay. So then, that's what Grothendieck says in another way : what he considers as *"this set or "arsenal" as having its structure the more obvious"*, he thinks of it as a category, this category of sheaves of sets. So then what's he saying? *"That it is a lawful thing to forget the space and to consider only the category of sheaves of sets."*

Why is it a legal thing? It's a legal thing because we haven't lost space along the way. We find the points of space. So how do we find the space points in the metaphor I gave you? Because he says to him *"it's a simple exercise to check : once the question is asked."* . Well. You can actually have fun, thinking in classical terms, if you have the sheaves of sets on an ordinary topological space, how are you going to find the space itself, that is to say the points of space? Can you ask this question. So in fact, in the metaphor I gave you, the points of space are when you take a given moment, a fixed time. When you take a time that is frozen, well then there is no more variability, and you have ordinary set theory. That's what we abstractly call in the theory of topos a point of a topos, that is to say that this is how we call a geometric morphism, which goes from ordinary set theory to the topos considered. But that is exactly like freezing things at a given time.

Grothendieck tells him *"to check is a simple exercise."*. In fact, this is not always true ; as he says *"(at intention of the mathematician) Strictly speaking this is only true for so-called "sober" spaces"*. There still must be a minimum of separation in space. And there is an extremely interesting, which I invite you to do, because you always have to... we don't do math by listening, we do math by doing exercises. So there was already the exercise on the abelian categories of everything on time. And that's another exercise now. Is that you take a curve with its Zariski topology. Or you rather take a surface with its Zariski topology. And you calculate the points of the corresponding topos, i.e. of the topology of the sheaves for the topology of Zariski. Well, you will notice that there are more points, because the space in question, it is not sober, and you will get exactly the points of the corresponding diagram. So I'm going to say, already, in this example, we see the absolutely incredible potential of this way of thinking. He says :

"... we can now "forget" the initial space, to no longer retain and use only the associated "category", which will be considered the most adequate embodiment of the logical "topos structure" (or "spatial") that needs to be expressed."

So what he explains next is :

"As so often in mathematics, we have succeeded here (thanks to the crucial idea of "sheaf", or "cohomological meter") to express a certain notion, (that of "space"), in terms of another (that of "category")."

So we replaced, always following the metaphor, space by this category, which is a category type of sets, these are sets.

"Each time, the discovery of such a translation of a notion (expressing a certain type of situations) in terms of another (corresponding to another type of situation), enriches our understanding."

Of course.

"And both, by the unexpected confluence of specific intuitions that relate either one or the other. Thus, a situation of a "topological" nature (embodied by a given space) is here translated by a situation of an "algebraic" nature (embodied by a "category") ; or, if you like, the "continuous" embodied by space, is "translated" or "expressed" by the category structure, of an "algebraic" nature (and hitherto perceived as being essentially "discontinuous" or "discrete")."

We deny, a priori, to a category the right to represent something continuous. This is the case here. This is the case because we find the points and we find the topology of the points, simply from the category, which *looks like* the category of sets.

"But here there is more. The first of these notions, that of space, appeared to us as a notion in a "maximum" way - a notion so general already, that one can hardly imagine how to find for it one more extension that remains "reasonable". On the other hand, it turns out that on the other side of the mirror, these "categories" (thus the categories which one obtains as categories of sheaves of sets) *"that which one finds, starting from topological spaces, are of a very particular nature..*

The "mirror" in question here, as in Alice in Wonderland, is the one that gives as an "image" of a space, placed in front of it, the associated "category"."

So this category if you want, it is the category of the scene behind which is the topos.

"They indeed have a set of strongly typed properties, (we will not talk about them all immediately) *that make them look like sort of "pastiches" of the simplest imaginable of them."* What is the simplest of them ? It is set theory. So the categories you get like that, from a topos, are pastiches of set theory. That's what Grothendieck says.

"That said, a "new style space" (or topos), generalizing traditional topological spaces, will be described simply as a "category" which, without necessarily coming from an ordinary space, has nevertheless all these good properties." So they're called topos, so he's going to say it by the way, wait, I have to find him...

"The name "topos" was chosen (in combination with that of "topology", or "topological") to suggest that it is the "object par excellence" to which topological intuition applies. Through the rich cloud of images mentally aroused by this name, it must be considered to be more or less the equivalent of the term (topological) "space", with simply a greater emphasis on the "topological" specificity of the concept. So..." Well, he's talking about vector spaces, etc. So I go back.

"So here's the new idea. Its appearance can be seen as a consequence of this observation, almost childish to tell the truth, that what really matters in a topological space is nully its "points" or its subsets of points, and proximity relationships, etc. between these, but that these are the sheaves on this space, and the category they form. All I did was to lead towards its ultimate consequence the Leray's initial idea - and this done, ***take the plunge****. Like the very idea of sheaves (due to Leray), or that of the diagrams, like any "big idea" which shakes up an inveterate vision of things, that of topos has what to disconcert by its character of naturalness, of "evidence", by its simplicity."*

In fact, we know when we do math, when we are on the right track, when someone tells you *"Oh, that's it !". (laughs)*

This is what Grothendieck says, so :

"... by this particular quality that makes us shout so often : "Oh, that's it !", in an half-disappointed, half-envious tone ; with in addition, perhaps, this implied "wacky", "not serious", that we reserve often to all that confuses by an excess of unforeseen simplicity. To what comes to remind us, perhaps, the long buried and denied days of our childhood..."

So there, he comes back to the concept of schema :

"It constitutes a vast extension of the concept of "algebraic variety", and as such, it has renewed from top to bottom, the algebraic geometry bequeathed by my predecessors. That concept of topos constitutes an unsuspected extension, to put it better, a metamorphosis of the concept of space.".

If you will, which is absolutely extraordinary, even at the outset, in the concept of topos, it is the way space is understood. As I said, it is no longer apprehended by points, it is apprehended by the hazard it introduces : it introduces a hazard into set theory ; he introduces a variability in set theory. And that is extraordinary.

"By this, it carries the promise of a similar renewal of the topology, and beyond this, of geometry. From now on, moreover, it has played a crucial role in the development of new geometry."

This is for l-adic cohomology, or for crystalline cohomology.

"Like her older sister (and quasi-twin)[2]*, she has the two essential complementary characters for any fertile generalization, here it is."*

First, this notion should not be too broad, I pass quickly enough on it ; he talks about topos, we talked about it ; *"the most essential geometric constructions must of course apply, that they can be transposed in a more or less obvious way."*. It shouldn't be too general, the notion you take should not be, for example, the general notion of category. If it was the general notion of category, we would not go far. So it must have this property.

He explains : *"Among these "constructions" , there is in particular that of all familiar "topological invariants"."*

He explains very well : *"For the latter, I had done everything necessary in the article already quoted from "Tohoku" (1955)."*

2. schema theory

So the origin, you see, it comes from there. As I said earlier, in Tohoku's article, there were both the sheaves on topological space and there was also, and it was extremely important - so much important, the categories of diagrams, he was talking about the categories of diagrams and we will see that they play an absolutely essential role, in the same way. So he talks about mental associations, and less technical notions, of course.

"Second, the new concept is at the same time broad enough to encompass a host of situations which, hitherto, were not considered to give rise to intuitions of a "topologico-geometric" nature - to the intuitions, precisely, that we had reserved in the past only for ordinary topological spaces."

As we will see in an example that I will give you in a relatively short time, as we will see, what happens in the metaphor I was talking about, what will happen is that as it there is this hazard, as there is this variability in set theory, we can no longer apply the excluded middle principle. On the other hand, intuitionism works. And so, what it will generate, this nuance, that's why I want to take you there step by step, we're slow, but we have to be slow, so I will show you an example, as I said at the beginning, in which the notion of truth associated with topos will be much more subtle than the notion of ordinary truth, and I will have a slide on which there will be marked "a step from the truth" and I will give you a topos in which we will be "10 steps from the truth", "15 steps from the truth", "20 steps from the truth", etc. We will take an example because until we have taken an example, as long as we talk abstractly, we don't really know what we're doing. So we're headed there.

"The crucial thing here, from the perspective of Weil's conjectures, is that the new notion is pretty vast indeed, to allow us to associate with any "diagram" such a "generalized space" or "topos" (called "étale topos" to the proposed scheme). Certain "cohomological invariants" of this topos (all that there are "dumb kids"!) seemed to have a good chance of providing "what we needed"".

He continues and we relax a bit before coming to the examples and the really crucial things.

"It is in these pages that I am writing that, for the first time in my mathematician life, I take the leisure to evoke (if only to myself) all the master-themes and great guiding ideas in my mathematical work. This brings me to better appreciate the place and the scope of each of these themes, and of the "points of view" they embody, in the great geometric vision that unites them and from which they come. It is through this work that the two innovative nerve center ideas appeared in full light, in the first and powerful development of new geometry : the idea of diagrams, and that of the topos."

And there he insists :

"It is the second of these ideas, that of topos, which now seems to me to be the deepest of them. If by chance, towards the end of the fifties..."

So Grothendieck introduced the topos in a slightly depressed period he had after the death of his mother in 1957, he introduced the topos in 58. So we will be, in the coming year, in the 60th anniversary of the birth of the topos.

"If by adventure, towards the end of the fifties, I had not rolled up my sleeves, to develop stubbornly day after day,"

This is Grothendieck : "Obstinately, day after day..."

"Throughout twelve long years, a "schematic tool" of delicacy and power perfect - it would seem almost unthinkable to me that in the ten or twenty years already that have followed, others that I in the long run could have prevented from introducing at the end of the ends (were it to their bodies defending) the notion which obviously imposed itself, and to draw up somehow at least a few dilapidated "prefab" barracks, failing the spacious and comfortable[3] residences that I had at heart to assemble stone by stone and go up with my hands."

3. address of the speaker to the audience : "there, he is talking about the diagrams"

There he talks about schemas.

"On the other hand, I don't see anyone else on the mathematical scene, over the past three decades, who could have had this naivety, or this innocence, to take (in my place) this other crucial step between all of them, introducing the so childish idea of topos (or even that of "sites"). And even assuming that idea already graciously provided, and with it the timid promise it seemed to harbor..."

You know, someone would tell you, "I'm going to do this...". "Good luck!", you would say! Okay... *(laughs[4])*

"... I don't see anyone else, either among my friends of yesteryear or among my students, who would have the breath, and above all faith, to bring this humble idea to fruition (apparently ridiculous...) "What is it to strive on the sheaves of sets on a topological space?"... (seemingly derisory when the goal seemed infinitely distant...) : since its earliest early beginnings, until the full maturity of the "mastery of eternal cohomology", in which it ended by incarnating in my hands, in the years that followed."

Good, after, he talks about details, finally, of things which are important for the mathematician, but he speaks of etale cohomology and it is on this subject, as he says :

"It was inspired by this statement that I discovered the concept of a site in 1958." So he discovered the notion of site in 1958, and it is this notion, of course, and cohomological formalism, which were developed later.

He says :

"When I speak of "breath" and "faith" ," (that's always for the mathematician), *"these are qualities of "non-technical" nature."*

Grothendieck wrote somewhere in *Crops and Sowing* that he was not fast, that he was surrounded by people much faster than him, but hey, this is an example that shows how not to be discouraged, when you are not fast, good when talking with people who you realize they understand ten times faster than you do, don't be discouraged. However, which is absolutely crucial is to be persistent, and to have faith in an idea.

That is to say, if you have an idea, you must first make it your own, make it your own. And once it's yours, you have to protect it; initially, you have to protect it like a very small child who has just been born. Do not show it too much, not too much, etc. *(little laughs)*. And then after... Not so much because someone can take it from you but because you have to test it, we will talk more about it late, you have to test it, you have to get used to it, in private.

"On another level, I could also add to it what I would call "cohomological flair", that is to say the kind of flair that had developed in me for building cohomological theories."

Afterwards, he groans a little against his pupils, but that, we are used to with Grothendieck.

We're going to take a short break : we're not going to stop but I'm going to make you listen to Yves Montand *(laughs)*.

"Yes, the river is deep, and vast and peaceful are the waters of my childhood, in a kingdom that I thought I quit a long time ago. All the king's horses could drink there together at ease and all their drunk, without exhausting them! They come from glaciers, fiery like these distant snows, and they have the softness of the plains clay. I just talked about one of these horses, that a child had brought to drink and who drank his content, at length. And I saw another one coming to drink a moment, in the footsteps of the same kid if it is - but there it did not drag. Someone must have chased it away. And that's all, as much to say."

We hear to the song Aux marches du Palais who speaks about The girl who has so many lovers that she doesn't

4. to underline the irony of the euphemism of Good Luck compared to the magnitude of the task that this represents.

know which one to take and who has chosen her little shoemaker.

Here we go back to serious things.

So he goes on, he says :

"However, I see countless herds of thirsty horses roaming the plains - and not later that this morning their whinnies pulled me out of bed at an undue hour, I am going to my sixties and I love tranquility. There was nothing to do, I had to get up. It makes me pain to see them, in the state of lanky rosses, while good water however is not lacking, nor green pastures. But it looks like a malicious spell has been cast on this land that I had known welcoming, and condemned access to these generous waters. Or maybe it is a coup mounted by the country horse traders, to bring prices down, who knows ? Or maybe it's a country where there are no more children to lead the horses to drink, and where the horses are thirsty, for lack of a kid who finds the path that leads at the river..."

So, with Pierre Cartier and Olivia Caramello, we organized a conference two years ago, in the IHES, precisely to revive the idea of topos, but really of Grothendieck's topos, and the colloquium was remarkable, it went very very well.

So what is a Grothendieck topos ? So now we come back to mathematics. So there are three ways of defining them : it is undoubtedly the first way which is the simplest. So, if you will, as Grothendieck explains when he lectures, the important thing, at the start, they were pre-sheaves, that is to say they were contravariant functors which went from the open category, but it's an extremely simple category. I mean it's a category for which between two objects, there is at most a morphism so it's really something extremely simple. So we looked at the contravariant functors that went from this category to the sets. So now, we remove all the conditions on this category, except that it is a small category. What does a small category mean ? It means that the objects form a whole. And then morphisms as well of course. So we look at a small category and we look at all the contravariant functors from this small category to sets. We forget the fact that we had them open and that we had an extremely special category by looking at the open. Okay, we take no wear which. And so now, what we're asking, of course, we're not going to take all the contravariant functors, since we know well, and Grothendieck explained it in what we listened to, that we are not going to take all the pre-sheaves. Among these pre-sheaves, we will select some which we will call sheaves. What is the important property of this selection ? There are two things that are very important in this selection : the first is that we are not going to change the morphisms ; the first thing that is fundamental is that when you take a morphism from one sheaf to another, you can forget that these are sheaves. It's a pre-sheaf morphism, okay. So in fact, when we are going to select the subcategory of the pre-sheaf category, we will take a full subcategory. Full subcategory, that means that we are not going to change the notion of morphism. Okay, that's crucial, that, if you listen to Grothendieck further, he talks about it and he says it is crucial. So first thing. Second thing, which is extremely important, is that there is a way, when you have a pre-sheaf, to harness it, to turn it into a sheaf. So that means that sheaves are special pre-sheaves, but there is a kind of projection that allows you to replace a pre-sheaf by a sheaf. So what is the correct way to say it is that the functor which includes the category of sheaves in the category of pre-sheaves, first it is full, it is faithful, because we forget nothing, but above all, it has an adjoint on the left, who is the harnessing, and miraculously, this adjoint on the left, it is exact on the left, which is normally never the case for an adjoint on the left. Normally, an adjoint on the left, we know that in all cases, it will preserve those which are called colimits, but it is very rare that it preserves the limits. Well there, it preserves the limits. So that's the condition. So if you want a short definition of what a topos is, that's it.

So now what we will see, and then we will see what a site is, there is another way to say it which is more precise : it is that in fact, we know that any harnessing, like the one of which I was speaking, in fact, it comes from what is called a Grothendieck topology on the small category which we left. So let's see what it is. And then in fact, there is a third definition of what is exactly a topos. But then, that is really a definition, how to say, very abstract but it is a definition that states properties that are true for set theory, okay. And we ask the topos to check them too. So that spawned another theory of topos, called the elementary topos theory, which are not Grothendieck toposes in general. But then, what is it missing to an elementary topos, therefore a topos which verifies naive properties of set theory, to be a Grothendieck topos ? What it lacks are the constants. That is to say that when, in the metaphor, I told you that we have variable sets, which depend on a hazard, well, when we have a Grothendieck topos, there is what is called a geometric morphism, which goes from the topos to the topos of the sets, and that allows us to talk about the constants. Now to talk about the constants, when doing cohomology for example, it's absolutely essential. Because these are the constants which allow for example to define the global sections of a beam, etc., etc. So it's not at all innocent, and there is a very big difference between a Grothendieck topos and what is called an elementary topos which would bring together elementary properties of set theory.

So the examples. Well then among the examples, there is of course the example of sheaves of sets on a topological space. This is the first example. The second example, I already talked about it, are sheaves for Zariski topology, so this is a special case of sheaves of sets on a topological space. But as I said, the point is that when we look for the points, for this topos, we find the good points of the diagram, okay. And finally, there is a third example, which is the one that Grothendieck introduced in 58 to have etale cohomology, that is to say that we start from a diagram, and there is a topos which is associated with the diagram, but it is no longer a topos which comes from a topology on the diagram. So this is something that is above and which, well, of course, there is already a topos in the sense original if you want, which is outside the topological spaces.

So I come back to Grothendieck, he says :

"The theme of the topos comes from that of the diagrams, the same year in which the diagrams appeared - but in extent it goes far beyond the mother theme. It is the theme of the topos, and not that of the diagrams, which is that "bed", or this "deep river", where geometry and algebra, topology and arithmetic come together, mathematical logic and category theory, the world of the continuous and that of structures "discontinuous" or "discrete". If the theme of diagrams is like the heart of new geometry, the theme of the topos is the envelope, or the abode. It's what I have designed more broadly, to grasp with finesse, through the same language rich in geometric resonances, a common "essence" to situations of the most distant from each other, coming from such and such a region of the vast universe of mathematical things. This theme of the topos is very far from having known the fortune of that of schemas.".

There is a kind of curse on topos. There is a curse that reigns, we may come back to it if we have time. So here is the metaphor. So the metaphor I was talking about earlier. It is absolutely necessary that you have a mental picture of what a topos is. So we used to, like I said, to put the space to be studied on the front of the stage. Grothendieck makes it play this role of Deus ex machina, who is not present, who stays behind the scenes. But what is important is to know that when you have a topos, you can do all the manipulations, you can talk about abelian groups, you can talk about algebras, etc., and if you work with a topos coming from a topological space, that would give you the sheaves of abelian groups, or the sheaves of algebras, etc., it's great, it's great to have this freedom of maneuver. So then, when one works in a topos, everything happens as if we were handling ordinary sets. So that's what one needs to know. In fact, as soon as we have bundles on a space, we get into the habit of thinking about a bundle as in a variable vector space. But there, variability is the **right** notion of variability, because it sets up the sets. Except that we can no longer apply the rule of the excluded middle. So what appears if you want is that we can no longer have, for a proposition p, (p is true) or (not p is true), we no longer have the excluded middle principle. So we will quickly see a concrete example of a topos for which this notion of truth becomes more subtle than the simple true or false than we use colloquially. For example, if you want, if you watch TV, and you watch a political discussion on television ; well, we used to say "this one is right and this one wrongly". Well, I pretend that we don't have the conceptual tool we need to judge. And I will give you examples. I'm going to show you how much more subtle the notion of truth is and how the idea of the topos allows to formalize it. So we're going to make this work on an example. To make that work, we are going to introduce topos which are other than the topos which come from a topological space and which have an extremely simple nature : these are the topos which consist in taking a small category and simply take the category of all contravariant functors to the sets. So here we does not distinguish between sheaves and pre-sheaves. We take all the pre-sheaves. They are all sheaves. So to a small category, we are going to associate a topos which is its kind of dual, if you want, which is all the contravariant functors from this little category to the sets, and we will have fun with that.

So in 91, Grothendieck was still in perfect contact with certain mathematicians, and lo and behold what he wrote in a letter to Thomasson, he said :

"On the other hand for me, the original paradise for topological algebra is by no means the simplicial category..." . So, I don't know if we will have time to talk about it, but he is talking about topos.

Indeed, topos having categories sheaves on sets $\hat{C}$, with C a small category, are by far the simplest of the known topos." And it is for having felt, that he insists so much on these categorical toposes in SGA4. So if you look at SGA4, you will see that there are two basic topos examples ; of course, there is the etale topos, and then there are the topos which are dual of a category. Okay. So we're going to have fun with that.

So what is the notion of truth in a topos ? *(laughs)* How is the notion of truth different in a topos ? So, in what sense, first of all, are we able, in the sets, to define the true and false ? So how are we going to define true and false in set theory ? We are going to be interested in trying to classify the subsets of a set, okay. You see, if you work with ordinary sets, and if I tell you "I have a functor which, if you give me a set, it combines all of its subsets." . It is a functor because if you have an application that goes from X in Y, you can recall the subsets of Y backwards, so it's a functor. So now, the question is "is this functor representable ?". It's a mathematical notion, okay, and then in the sets, it is representable because of a notion that we know very very well : it is that with a subset, we associate what is called its characteristic function. That is to say that when we have a subset of a set, we define a function : this function, it is 1 if we are in the subset, and it is 0 if we are not in the subset. So this function, this makes that it has a fairly miraculous property : it classifies, that is to say, it represents this functor. In the case of sets, there is a privileged object Ω which is the object which is formed of the set $\{0, 1\}$, the two-point set, and when you look at all the subsets of a set, it comes back to look at all the applications of this set towards the set $\{0, 1\}$. Because when you have a application which goes to the set $\{0, 1\}$, the subset, it is defined by the subset on which it takes the value 1. But where it does not take the value 1, well, it necessarily takes the value 0. Well, if we think enough, in logic, thinking in the language of topos, we realize that it is this simple fact that there were only 0 and 1 in the sets which makes it possible to have the principle of the excluded middle.

So now we're going to have fun with a topos which is a little bit more complicated. We will take... this is what I call "at two steps from the truth". So, what are we going to take as a topos ? We will take the category C which has only one object, and which has for morphism the powers of one morphism. That is to say that I choose a single morphism which goes from this object in itself and I raise it to powers, okay, T^n. So what does that mean, an object from the category of contravariant functors from this category to sets ? It simply means a set with an application from X in X. That's all. Why do you only have one set ? Because the category had only one object. So you only have one set. And in fact, the category, it had only one morphism, well, we raise it to its powers, but, I mean... you just have to know it, you just have to know its image. What is its image ? It is a transformation T from X into X. What is the topos ? Well, these are the sets provided with a transformation. Well, the sets with a transformation, that makes a topos. It's a category, okay. Why is it a category ? It's a category because if you have two sets with a transformation, you have the applications from X to Y which respect transformation. That is, they verify that the image $f(TX)$ of TX is $Tf(x)$. So you have a category, and this category is a topos. Why is it a topos ? Because it is the dual of the small category that I gave you.

Okay so now we're going to look for Ω, for that, so we will try to classify the sub-objects of an object. So why is it annoying to try to classify the sub-objects of an object ? Well, let's try with $\Omega = \{0, 1\}$. We will try with the characteristic function, as we did earlier. After all, if I take a set with an application, if I take a sub-object, it's a subset that is stable by application, okay. So if I take my X, I will take a subset Y which was invariant, which was invariant by applying T. Well. Very good. It is invariant by applying T. So I'm going to associate the value 1 on this subset, okay. On the subset, I will give 1. Why can't I give the value 0, on the complementary ? Well because there can have complementary points that will end up landing in the set in question. I'm not at all assured that the complementary will be invariant under T. It may very well happen that a point in the complementary, after a while, tac !, it will type in the subset in question. It's not because the subset in question is invariant that its complementary is invariant. Of course no ! The transformation I took is an action on N, not on Z. So how do we go do ? It's annoying ! It means that the application which went towards 0 and 1 does not work. Well ! Well, you have to think about it a little bit. What do we have to do ? Well, when I take an x which is in X, there will exist a smaller integer, a first integer, such as when I apply T n times, it hits the subset. Okay. So I'm going to associate this integer with it. This integer, it will be infinite, of course, if we never get into the subset.

So we see that we have to replace the set $\{0, 1\}$ by the set $\{0, 1, 2, 3, \ldots, \infty\}$. And how are making this set a set with a transformation ? Well, we realize that if I look the h, i.e. the smallest integer for TX, well the smallest integer for TX, it's going to be the smallest integer for X minus 1 unless it becomes negative ; if it becomes negative, it doesn't work ; so I take the *sup* with 0. Okay.

So you see that for this topos, then the notion of truth which before was simply 0 or 1, main-holding, it is given by the set $\{0, 1, 2, 3, \ldots, \infty\}$, with the transformation which replaces by $N - 1$. So what does that mean ? Well, that means we have an incredibly simple example of a topos which allows to say "Yeah, what you do, yeah, I would say it's 10 steps from the truth...". Me, I've always said that people who do string theory are infinitely close to the truth. *(laughs)*

So you see that this notion, innocent as it is, stupid as it looks, in fact, it has an absolutely rich potential. And what I claim is that our mind, our training logic, is extremely primitive because we are used to, when we listen to a discussion policy of decreeing "yes or no", "such a person is right, such a person is wrong" and we are wrong by doing that and if there were philosophers, well, I dream, if there were philosophers knowing math, and who understand the topos from the inside, and there are very few, who understand the topos from inside, they would be able to give models, which would be useful, to appreciate much better these kinds of discussions, these kinds of situations, which are actually much more subtle compared to the notion of truth, that this notion of absolute ineffectiveness, that we use all the time, and that is "Such is right or so is wrong.". So I absolutely wanted to give you this example, so that you keep it in mind, and try to build other similar examples ; there are examples of course, don't be afraid that there are n that goes from 0 to infinity, in fact, you can very well imagine finite constructions, okay. The finite constructions, there is a wealth combinatorial in the topos which makes finite constructions have extraordinary potential.

So what is a sieve ? This example will allow us to define what a sieve is. What is a sieve ? I gave you an example of a sieve. The Ω in general, when you take the topos, which is given by all the contravariant functors from a small category to sets, well we build the Ω and how is the Ω built ? The Ω, it is constructed from a sieve. So what is a sieve ? Well a sieve on an object of a category, the Ω will be constructed from the objects of this category, let's remember that the category which I mentioned earlier, it had only one object. So for the moment, we have nothing. We have a single object. So, a sieve on an object X, on our object X, is the data of a family of morphisms $C(X)$ which is contained in all the morphisms whose image is X... finally, it goes from a set Z to X, whose codomain is X, and which is stable by right-hand composition.

What are cribles, in the earlier example ? We had only one object ; the morphisms which went into this object, it was just the integers, since it was the powers of T, there was T^0, T^1, T^2, ... What is a sieve ? Well, a sieve is a kind of ideal if you will, that is to say that it is a family of morphisms which is stable by right-hand composition by any what morphism. So in the case of earlier, what is the composition on the right ? It add to an integer, well, that adds any integer to it. It's like looking at all the intervals infinite on one side. So, among the infinite intervals, you have what, you have 0, up to infinity, that is what is called... finally, it is a sieve which must always be present ; it is the sieve that is formed by all morphisms. And then, we had all the morphisms that were from a certain integer n. That was a sieve, okay, and it was when we were at a distance n. And then, there is the sieve where there is none to none, it's the empty set, and that corresponded to infinity earlier. Here.

So it turns out that in general, we can define the Ω, the truth values if you want, for the dual of a small category, and we define it exactly from the sieves. When we calculate Ω therefore, we construct this object, simply as always as a contravariant functor of a set etc., but we build it from the sieves on each of the objects in the category. In our case, there was a single object so it was very simple. It was very very simple.

So I was fascinated for a long time by the idea that Grothendieck had called sieve and that he was not unaware that this name had already been used by mathematicians, and that there is for example a sieve which is well known and which is the sieve of Eratosthenes. So I finally found the answer, I finally found why the sieve of Eratosthenes is a sieve, in the sense of Grothendieck and that, it comes from a common work we did with Katia Consani and in which the category we take is very similar to that from earlier, where there was only one transformation, but this time, it is a little more complicated anyway, because instead of having (we always have a single object, as before), but instead of having the powers of a single morphism, we have an action of the multiplicative integers. That is, for each integer, we have a morphism, and when we make the product of two integers, the morphisms are composed. So it is an exercise to demonstrate that the sieve of Eratosthenes is a sieve in the following way : it is very funny. Because... what is the sieve of Eratosthenes ? The sieve of Eratosthenes, that is to take the first non-trivial number. We're going to fuck 1, we don't care 1, okay. So we take the first non trivial number which is 2. And what does the sieve do ? The sieve considers all multiples of 2, all even numbers except 2. And then there are things, good. There are 3 left for example, then it takes all the multiples of 3 except 3. And then there are things, 4 we have already taken since... So it takes all the multiples of 5 except 5. Well, I pretend that if you look at the integers like the morphisms, multiplicative integers, such as the morphisms of a category that has only one object, and if you look at everything I just told you, i.e. if you look at all even integers except 2, all the multiples of 3 except 3, etc., that makes a sieve in the sense that I gave you earlier. And it shows you how subtle the notion of truth is for this category, because I only gave one example of a sieve. To check that it's a sieve is trivial, it's not the question, it's not the difficulty.

So now, once we have the notion of sieve, we will see the notion of Grothendieck topology. I couldn't give a talk on topos without giving the definition of a Grothendieck topology. So, I will tell you the moment which for me was crucial in the appreciation of the concept of topos. The crucial moment was this : before, when I was presented with a topos, they always presented me with a topos saying to myself "I take a category, a small category, and I suppose that it is stable by fiber product." At that time, my ear closed and I was thinking of something else, okay *(laughs)*. And the reason is this : it is that, when we say that, and after that we write what is a base, etc., we obviously have topological intuition in mind ; that is to say that when we say that the category has fiber products, we think of two open ones that have an intersection. And from there, good, we can develop things. And so what was crucial for me was when I understood in fact that, already in SGA4, Grothendieck had defined the sites, and the fiber products on the sites, *without any hypothesis* on the small category, *without any hypothesis* on the small category, we have absolutely no need to assume *anything* about the small category, and the huge advantage is that when we do that, we understand better what we are talking about. You know, in math, there's one thing you need to understand is that the main difficulty when you are faced with a problem is to manage **to think right**. And thinking right, it sounds silly, it looks like... trying to think right... but once we get there to think right, things fall like ripe fruit, but you have to know how to think right. And it's not just to think of asking the small category to have fiber products. Thinking right is to think what there is there, that is to take the maximum sieve, the fact that when you have a sieve... So, what is a Grothendieck topology, it is a collection of sieves, we give for each object a collection of sieves, and we have compatibility conditions. But what is the intuition that you need to have behind ? No matter the detail of the axioms. What is the... When you do topology, you have the intuition of open collections. This is a very delicate intuition, I will explain to you why it is very delicate. Take for example the interval $[0, 1]$. And then take in $[0, 1]$ only rational numbers. They are dense, so you will recognize the open ones, with the rational numbers, since the open ones are interval meetings. An interval, I know it by its intersection with the rational. Okay ? What will change ? Why is it that if I take the topos that is given by the rationals with these open ones, I get something different than the topos which is given by the interval $[0, 1]$ with its ordinary open sets ? They look alike, they seem to be the same. Well, if you search, you will find that there are actually a lot more openings for rational than there is for the real. For rationals, there are open covers that are there, while they are not there for the reals. Here. Typically, what you want is that if you take a continuation of larger and larger openings but whose limit is an irrational number, well, that's fine appear as an cover at the rational level but it will not be a cover at the real level. Okay ? That is to say that at the real level, if you take the complement of that, the combination of the two, it will not be an open overlay. So in fact, there are a lot less open collections for the real that there are for the rational. When we think topologically, we think like that. When we think at the topos level, we think differently : how do we think at the topos level ? We think that sieves mean small things, they mean small objects. Sieving will give objects that are small. And at that point, the axioms, they become almost absolute - clearly obvious. And what does it mean that an object is small compared to an open cover ? What does it mean that an overlay is small compared to an open overlay ? It means that it passes through, it means that it is contained in one of the opens of the covering : it passes through a hole. So this is the intuition that you need to have : the intuition of the sieve is that these are things that are small, and pass through the holes. Okay.

So having said that, now we have the intuition of a Grothendieck topology when there is a basis, etc. ; I'm not going to bother you with that. So there is an essential notion in topos but it is similar, I will not talk about it too long : it is the concept of point. And especially the notion of geometric morphism. So if you want, the topos... It happens that once you just think about topos, the same properties that are true for topological spaces continue to make sense, but obviously, they are much more subtle. Typically, what happens, and that, I copied a page of SGA4, this is what a morphism from one topos to another, what is called a geometric morphism.

So to understand what a geometric morphism is, that is to say a morphism from a topos in another, you have to have some familiarity with the sheaves on a space. Why ? Because when we have a continuous application from space X to space Y, well, there are two ways to connect sheaves on X with sheaves on Y. There are two ways to do this. And these two ways, there is one that is tautological, almost trivial, which consists in taking a sheaf on X and pushing it forward towards a sheaf on Y. And that, in what sense is it trivial ? It's trivial because it's enough when you take an open on Y, take its inverse image and look at the sections of the sheaf on X on this open, on the reverse image. So that makes a beam, there is no problem. So this definition, it will self. But there is another way to connect the sheaves of X and the sheaves of Y which goes in the other sense, that is to say it sends a sheaf on Y towards a sheaf on X, and that one is much more interesting, it is much less trivial. It's visually obvious if you think of a beam as a space spread over the base space, and this is particularly the case for sheaves of sets, but, where it is extremely interesting, is that this application that goes the other way, it has a wonderful property, it has a totally unexpected property. First, it is deputy on the left the other. That is true, it is not a big thing, we could have defined it like that. So it is the adjoint to the left of the other, of the one that went forward, very good. But it has a wonderful property, and this marvelous property is the property that it is exact on the left, that is to say that it commutes with limits. So this is an extremely powerful, extremely amazing property, and I think that the example that is due to Pierre, the most striking example of that, you have to be struck by an example, until you are struck by an example, you will not understand. The most striking example of that is what we call simplicial sets, simplicial complexes. So what you do is that there is a small category, therefore a little more complicated than that of earlier, *(intervention by Pierre Cartier : which Grothendieck does not want)*, taken over by Alain Connes, which Grothendieck does not want, precisely. I'm going back to Grothendieck's page because he doesn't want it. It's fun, by the way. Here.

It's the one Grothendieck doesn't want. This small category is called Δ^{op}, it is the category that is semi-simplicial ? These are the finite sets, completely ordered, with the not decreasing applications. This category is very important for the following reason : in topology, in years 40-50, a concept developed, at the beginning, it was formulated in a little too simple, which was the concept of simplicial complex. We took a space and we triangulated it. When we take ordinary space, we can triangulate it, or in a larger dimension, when we triangulate it, we can give a combinatorial data which encodes the triangulation. This combinatorial data, we can formulate it by looking at what is called the simplicial complex but in a fully combinatorial way, taking simplexes, etc. So it happens that if we do things like that, it doesn't work very well at all for the product. That is, since the product of two simplexes is not a simplex, by example, the product of two intervals, it's a square, it's not a simplex, but it doesn't work at all for the product. But it's because we didn't think right. And it's because we don't have done something, that seems trivial when you do it, but it is fundamental. And this thing which is trivial when you do it, but which is fundamental, is that you have to understand a lot better geometric realization of this combinatorial object, and this geometric realization of the combinatorial object, in fact, it is a point of a topos. It happens that this category is associated with a topos, the topos bébête, the topos of contravariant functors that go from this category to the category of sets, and that, it is a theorem that we can easily demonstrate, the points of this topos, in a direction on which we do not go to drag on, the points of this topos are exactly the intervals. That is to say, these are exactly the totally ordered sets which have a smaller element and a larger element. So the points of this topos are given exactly like that. And when we have a point in the topos, well, the inverse image functor, which goes to the sets, well, this functor is the geometric realization functor if we takes for totally ordered space with a smaller element and a larger element, if we take the interval [0 , 1], that gives exactly the geometric realization of the simplex, of the simplicial complex.

So now, wonder of wonders : this functor preserves finite limits and therefore, it preserves products. And so, when we take the stupid product of two simplicial sets, that is to say of two contravariant functors from this little category to the sets, well, when we take the geometric realization, it will give the product of geometric realizations. It's an immediate exercise to check that it is compatible with the topology. It does not present any difficulty, the difficulty, it is purely set designer. And, Pierre, it was you who demonstrated this theorem for the first time, right ? *(Pierre Cartier answers "Milnor")*. Yes, Milnor or you. But, what you have to see is that the concept of topos includes that thing. It understands this thing and generalizes it to an absolutely incredible point, that is to say that a point of a topos now, will precisely preserve not only the arbitrary colimits, but will preserve the finite limits, therefore will preserve the products, etc.

And that's why when we take a point from a topos, it takes us to set theory but respecting everything we know. That is to say that it will transform an abelian group in the topos into a real abelian group ; it will transform all the elementary notions that we can have into a real notion in set theory. So, there is one step that I will not dwell on at all, but which is extremely important and in which precisely, there are very very interesting works which is done now, which concerns the classifying topos. That is to say that exactly as there is a classifying space for bundles, or vector, etc., there is a classifying topos for logical notions. And one of wonders of that, which answers a little bit to Grothendieck's question when he says *"the age-old category Δ^{op}"* is that the topos which is associated... not directly this category, but the topos which is associated with this category, it is exactly the topos which classifies the intervals. That is to say that if we define abstractly what I explained earlier, that is to say an interval, a completely ordered set, but we should not speaking of the whole, in an arbitrary theory, well we see that this notion has a classifying topos and that this classifying topos is exactly the dual of the category Δ^{op}. Well.

So we're not going to go into details. Now we're going to do something else : I don't want to go back in the technical details, I don't want to. We're going to come back to Grothendieck, we're going to read again Grothendieck and then we will finish by reading the end of the exchange between Grothendieck and Serre in their correspondence. So that's what Grothendieck says. Well, it's very important to have talked about topos, but it's still more important to try to perceive Grothendieck's way of working, because that's what we need. Okay, sure, maybe we will use topos to do all kinds of things, but we also need, terribly, in our civilization : when we are now witnessing a speech that is done in public, we realize that there is a third of people who have their computers open in front of them and who do their emails, *(laughs)*, or who do something else, or cell phones. But it's a disaster, because when you read Grothendieck and when you immerse yourself in his way of thinking, you notice a thing, the most striking thing is the time he had. One has the impression that he had an infinite time, infinite time, that he was not constantly disturbed. You know, now, we're talking of generation Y, that is to say people who do 3 things at once. We believe we save time, but it's not true. We now have a basic need in our civilization to isolate ourselves, and to be able to think slowly, and to take the time to check everything, to be sure, to do it twice, to do it three times, etc. That's why I made it last, when Grothendieck was talking about sheaves, it lasted, *(funny)*, it lasted, but it was on purpose that I did it, I did it on purpose, because I wanted you to realize this fundamental slowness. It is a slowness which, when you feel it in the first degree, is irritating. It's the slowness of the turtle and the hare, if you will *(laughs)*. And it is she who wins. So here's what Grothendieck says :

"When I'm curious about something, mathematical or otherwise, I question it. I question it, without worrying if my question is perhaps stupid or if it will seem such, without it being at all costs maturely weighing. Often the question takes the form of an affirmation - an affirmation which, in truth, is a blow probe. I believe it more or less, to my assertion, it depends of course on the point where I am in the understanding of the things I'm looking at. Often, especially at the start of a search, the affirmation is downright false - it still had to be done to be able to be convinced. Often it was enough to write it down." .

One basic thing that Grothendieck often does is that he is able to write an idea that is not yet ripe. He is able to start writing, that's a fantastic quality.

"Often, it was enough to write it so that it was obvious that it was wrong, whereas before writing it there was a blur, like a malaise, instead of this obviousness. Now you can go back to the burden with this ignorance less, with a question-affirmation perhaps a little less "next to the plate". More often still, the statement taken literally turns out to be false, but the intuition which, awkwardly yet, tried to express itself through it is just, while remaining blurred."

I stop for a second : when he talks about writing, the computer is still a disaster, because we write better, in this kind of situation when writing on paper with a pencil, because when you write on the computer, it has to look perfect. We are going to ask ourselves questions of LaTex, we are going to ask ourselves questions like that, but it's completely ridiculous, we're not there yet, we're at a point where we want to leave the pencil that does what it wants on the sheet of paper. This is very very important. So this is what he says :

"This intuition will gradually settle down from a gangue just as shapeless at first of false ideas or inadequate, it will gradually come out of the limbo of the misunderstood that is just waiting to be understood, the unknown who only asks to let himself be known, to take a form that is only hers, refine and sharpen its contours, as the questions I ask of these things in front of me arise more precise or more relevant, to define them more and more closely. But it also happens that by this gait, the repeated soundings of the probe converge towards a certain image of the situation,..."

That means that we are in the process of forming a mental image.

"... coming out of the mists with features marked enough to lead to a beginning of conviction that this

image there expresses well reality - when it is not however, when this image is tainted with an error of size, likely to distort it deeply. The work, sometimes laborious, which leads to sieve of such a misconception from the first "takeoffs" noted between the image obtained and certain patent facts, or between this image and others who also had our confidence".

It must be said there, that it is very good, in these cases that he describes, to take a step back, to do other thing, and Grothendieck often had, Cartier often told me that he had 100 irons in the fire. When we see that things tend to mess around a little bit, it's better to take the field, because in fact, we are viscerally attached to the ideas we had, and we don't want to accept that they are false.

"This work is often marked by increasing tension, as we approach the knot of contradiction, which at first becomes more and more glaring - until finally it breaks out, with the discovery of error and the collapse of a certain vision of things, arising as immense relief, like liberation. The discovery of the error is one of the crucial moments, a creative moment among all, in any work of discovery, whether it is a mathematical work, or a work of self-discovery. It's a time when our knowledge of the thing suddenly probed renews itself."

And now here is one of the most magnificent paragraphs I know :

"To fear error and to fear the truth is one and the same thing. Whoever is afraid of making a mistake is powerless to discover. It is when we fear to be mistaken that the error that is in us is made immutable like a rock. Because in our fear, we hold on to what we have decreed "True" one day, or what has always been presented to us as such. When we are ripe, not by the fear of seeing an illusory security vanish, but by a thirst for knowledge, then error, like suffering or sadness, crosses us without ever freezing, and the trace of its passage is a renewed knowledge.".

If one day, you are not in the mood or all that, read that sentence again. It is a kind of talisman.

So I'm going to finish in... I started with the discussion between Serre and Grothendieck, at the very beginning, on Grothendieck's Tohoku article, and I'm going to close with a rather different note, in a very different tone, which is precisely the reaction of Serre when he received *Crops and Sowing*. So, good, I don't know if you know Serre, but I mean, he's not used to mince words, and he don't really like moods in general and so, I mean, it's extremely interesting that in the correspondence between Serre and Grothendieck, they continued their exchanges, at the time when Grothendieck had deliberately isolated himself from the mathematical world. I mean, it's not the mathematical world that had chased him, it was Grothendieck who chased himself, who isolated himself from the mathematical world, he wrote this text ; all the passages that I read to you from Grothendieck are in *Crops and Sowing*, so it's an admirable text, and you have to read it with a certain perspective, of course, because there are times when, if you like, he says things that are not ideal, but in any case, he expresses himself. So here's what Serre says after receiving it :

"Dear Grothendieck,
I have received the Crops and sowing *leaflet that you sent me. Thank you so much. The penultimate booklet is still missing to me, booklet of which I only have a few isolated pages." (laughs)*

Well obviously, there are so many pages. I must tell you, moreover, that it is a text that you must read... you must not read more than 5 pages at a time. I remember spending an amazing summer by reading in parallel *Crops and Sowing* and Proust's *In search of lost time*. And I mean, from the same way, that is to say, of course, people who are looking for crisp anecdotes, they will read by skipping pages, if you do that, you lose everything. It's exactly the same with Proust. Proust, you can't read it by reading more than 5 pages at a time, you have to meditate on it, you have to rethink to it, etc. It is necessary to let yourself be penetrated by an atmosphere that is absolutely extraordinary. So this is what Serre says, he says : *"One thing strikes me. In the texts that I could see, you are surprised and you are indignant at what your alumni did not continue the work that you had undertaken and largely carried out. But you don't ask yourself the most obvious question, one that every reader expects you to answer : "Why did you abandon the work in question ?" " (laughs).*

It is still quite a question. And then what is great is that Serre has an answer, and it's not an obvious answer at all. No, no, but it's a letter from Serre but he continues his letter and he has a proposition as to why Grothendieck left. So this is what he says ; he says :

"I have the impression that despite your well known energy, you were just tired of the enormous work that you had undertaken."

Anyway, everyone will understand. I mean when I talked about the time, it means he had little time to do something else. So I mean, it's huge. At the beginning, I read you passages in which he talked about everything he had to absorb, etc., well I mean, it's monstrous such an amount of work.

"Especially since there were also SGAs who were falling behind, year after year. I remember, in particular, the rather disastrous state of SGA5 where the editors got lost in masses of diagrams, whose commutativity they were reduced to assert without proof, to the nearest sign by being optimists." (laughs)

"And these commutativities were essential for the future. It's in this disastrous and not idyllic state, as you would think to read Crops and Sowing *that my sentence from the Bourbaki seminar refers to, the version definitive of SGA5 which should be more convincing than the existing presentations and handouts."*

It's Serre spit.

"We would like to have your impressions on all this, even modified by 15 years of burial for borrowing your terms, we're hungry."

So now he's going to go to a much deeper explanation :

"We can wonder for example if there is not a deeper explanation than simple tiredness to have to carry so many thousands of pages at arm's length. You describe your approach to math somewhere, where you don't attack a problem head-on, but where you wrap it up and dissolve it in a rising tide of general theory."

This is what I was talking about earlier when I was talking about thinking right. And for example, there is an anecdote, Cartier will not contradict me, which is that once, going up from the cafeteria to the IHES, there was... I believe it is Demazure who is asking Grothendieck a question on $SL(\mathbb{Z})$ or on... So what Grothendieck says that this is not the right way to formulate this question and the result was SGA3, i.e. Grothendieck's theory of algebraic groups *(laughs)*. So that's Grothendieck, it's... He can have a specific question, we can ask him a specific question, but he will say "this question is not in the right context". And he's going to develop a general theory to let the question become natural. And from the moment when the question is natural, and when we took the pain and time to think right, it will fall like a ripe fruit. So that's what Serre says when he says :

"...but where we envelop and dissolve it in a rising tide of general theory."

So the question dissolves. And in *Crops and Sowing* elsewhere, Grothendieck has very beautiful pictures, he talks about a nut, and he says that there are two ways to deal with the nut : the first way, is to take a hammer and break it, and the second way is to let it soften in water, etc., so that eventually it opens on its own.

"Very good. It's the way you work and what you've done shows that it works, less for EVT[5] *and algebraic geometry."*

So that's what Serre says, and that's serious. He says :

"It is much less clear for number theory, where the structures at play are far from obvious, or rather, where all possible structures are at stake."

And I prefer to finish on this. That is to say, if you will, it's extremely striking to see these two ways of thinking about mathematics. Grothendieck's way, okay, which is a way which is precisely to try to think right, and to try to formulate, if a problem is given, to formulate it so that it falls on its own, and if you want, explore every corner, every single one nooks. In his house, the house of which he speaks, there is no corner which is dirty, which is not explored, etc. He wants everything to be flawless. And he can only think when it is like that. And the price to pay, it's a colossal job. But it's a job that is not really difficult, in the sense that, we develop things, etc., etc. At no given time if you want, we are on a steep

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cliff, and we risk fall, at none of these times. It's kind of like you know Israel, like how the Romans wanted to attack Masada, I don't know if you know. Okay, this is something very striking because they backfilled with earth, earth, earth, so that it finally happens to the level... and it took them, I don't know, I think it's ten years or something like it *(The public gives their opinion, 3 or 4 years).*

That's the Grothendieck method. And what you see in hindsight is all you can learn from this method. All we can learn... As opposed to another method, that I like a lot, which consists of, in the corridors of the École Normale, in time, when I was at the school, there was a friend who had posed a problem to me : he was on the third floor and I was on the ground floor, and then I went for a weekend, and then I spent my whole weekend trying to solve... good. This is the problem solver, if you want, we give you a problem, and you are looking for to solve it, well, you're looking to solve it in the most efficient possible way. These are two completely orthogonal ways of acting and in fact Grothendieck has a whole discussion in *Crops and sowing* on these two ways of acting and he distinguishes them, well, he formulates them with yin and yang. Well, but it's very very important : he says that the method he has is more feminine, if you will, the other, which is a male method. It's hard to say exactly. But it's very important, when we do math, to imbibe this idea, actually, that there is this need, and that often, we don't believe it. For example, recently, I had a colleague who had posed a problem for me at the Academy that I ended up solving, but I was flabbergasted that I solved it when I started to think right. It amazed me ! Because I would be told "but you want to solve a problem, but why do you worry about that ?". No, that's not true, it's something fundamental, getting to think right, it is something absolutely fundamental. It will never be useless to try to think right. Never it won't be useless, okay.

So I hope I made you want to read *Crops and Sowing* and then, above all, to manipulate the simplest topos, and try to use them, compared to our logic, which is very poor, even in circumstances completely outside of mathematics. Obviously, it requires work, it requires work that is very slow, etc. which is that of appropriating the concept. And it's a notion which is cursed, it is cursed : with Pierre, and then especially with Laurent Lafforgue, for example, we tried for several years to support a very very brilliant mathematician, who is Olivia Caramello, and we encountered hostility, not to say contempt, from the mathematical world in general. And we were able to experience on this occasion how there is a kind of, I don't know, fatality, on the notion of topos, there is something that irritates people, because probably, they feel, this is what Grothendieck says, he says it so well, he says it explicitly, he had already felt it in his time, no doubt, they feel that there is something, but they do not really understand it. And to really understand, you have to do it, of course, but there will come a time when the concept will belong to you and you will come to appropriate it. And the best way is this metaphor is that space is not front of the stage, it is behind, it is a kind of Deus ex machina, and it's him which makes the sets rotate, it's him who introduces a hazard, a hazard in the sets, in set theory. Just as there is a hazard in prime numbers, which we all know, and even that there is a quantum random, so here, we must keep all this in mind, and I will stop there.

The journey of a mathematician
Alain Connes

So, so I apologize in advance for the narcissistic side of my presentation. But I mean, it's part of the game and hey, I was asked to talk about my journey. And so, I'm going to explain a few things that you won't find in other conferences and that you will not find anywhere. The first scene, if you want, it performs at Lycée Thiers, so in Marseille in the year 1966, and it's in May and I'm taking the exam of École Normale Supérieure. And this is the first test. A test which lasts six hours. I am sitting on a bench and I have an immediate neighbor. We are given the math problem and my neighbor, he begins to write, to write, to scrape. I read the problem statement. And then, an hour passes and my neighbor continues to write. I do not understand the statement of the problem. It's not coming... I mean. After two hours, I look by the window and I think about nothing. My neighbor writes. Three hours, it lasts six hours, three hours, nothing. Four hours. 5. Nothing. 6 hours. I leave the room in practically making white copy and leaving the room, I find the solution of the problem. Well, then in such a case, normally, the conclusion is clear. But I had a group of great friends, I had a group of extraordinary friends. They told me "You can't do that, you can't stop, we're going to go and swim in Cassis !", so they take me to swim instead of going home and moping. They take me to bathe in Cassis. We bathe and all that, I start to relax, all that. Then, the next day, I say "I'm going back". Go on hop ! Then I was received at the École Normale. So if you want, what I thought about before I made this conference, it's trying to give you tips that can really serve you. So when we talk about tenacity, what does that mean ? Tenacity doesn't that we are stubborn, etc. No, it means that when the circumstances are the worst they could be (and there were, I mean the main test, 6 hours, nothing) so, when the circumstances are the worst they can be, eh well, don't give up. When we can continue, we must. We must continue. Here. That's what happened at the beginning, at the very beginning.

And so, I went back to École Normale Supérieure and there, I found an absolutely extraordinary atmosphere, that is to say that at the time, we were not forced to nothing, we were not forced to do anything and in particular, we were not forced to pass the aggreg. And what mattered was simply to ask each other problems and to try to solve them, etc. That was the atmosphere of the School and at the end, three or four years later, I had a teacher who was Gustave Choquet and I had been duit by a theory called standard analysis, which still exists. But I hadn't realized when I was seduced by this theory that in fact it was something that was a bit chimerical. And my teacher Gustave Choquet had sent me to a summer school of physics which still exists, the summer school of Les Houches, which was created by Cécile de Witt

Lecture given as part of the Maths for All cycle on 18.12.2017 at the École Normale Supérieure, in Paris, viewable here https://www.youtube.com/watch?v=QfZLKxKTS2c

and which is an extremely interesting place for the communication, precisely, of more experienced people with students. And so, well, I'm going to listen to lessons at Les Houches Summer School and I was spotted at this, then, and I was sent the following year... I was invited to Seattle, to United States, at the Battelle Institute, and I was invited there. I was young married and I had decided to accept the invitation. It was mainly to visit the United States. And then, with my wife, we went to see my brother Bernard, who was at that time in Princeton. It was July and in Princeton, the weather was terrible. But you know, the humid heat, and there was only one place on campus that was nice. It was the Book Store. So with Danye, my wife, we spent the afternoon inside, at the Book Store. Well, at the time, I didn't have much internet. There was really very interesting books and I was looking for a book because I knew that we had decided to cross Canada by train. Instead of flying to go from Princeton, from New York to Seattle, we decided to go to Canada, and to cross all of Canada by train. But it took five days. I said to myself five days with nothing. It will do a lot. I will try to find a book from math and then read it during the trip, I hesitated a lot. I hadn't looked a lot of books, I hesitated between a lot of books. I ended up taking this one, and I had looked at it, I had tried to understand it, I did not understand everything. I had tried to understand it during the Canadian train ride. There were great plains that we crossed for days and days and days. And then, we got to Vancouver and Victoria, Victoria Island and finally we got to Seattle after a week.

And in Seattle, when we arrive, I went to the Battelle Institute to watch the conference program. And to my amazement, I found that the author of the book, which I bought completely by chance when I went to Princeton was lecturing there. At that time, I thought... Ah really, if you want, it's something completely extraordinary happening, so in fact, I will not listen to any other conference. I'll only go to this one. They were conferences on von Neumann algebras, my first subject and so here, von Neumann, it is known above all for something other than von Neumann algebras. But it's him, if you will, who developed something called operator algebras. And the theory's conceptor that was behind the book I bought, it's a Japanese mathematician called Tomita. And in history, it is also a completely extraordinary, so I continue to tell you stories. I hope it's OK.

Tomita, if you will, is a Japanese mathematician who miraculously escaped the war between Japan and the United States. He was deaf since the age of two and as he was deaf, in the regiment in which he was, he was exempted from going to the expedition that had to be made, because he was deaf, he couldn't hear well orders. They left him alone. The whole regiment was wiped out in the former that they made. He found an absolutely brilliant theory, but he had a hard time communicating and it's another Japanese mathematician called Takesaki, who wrote a book in those years on the theory of Tomita, the theory that Tomita had invented. And so, if you will, what happened is that I listened to Takesaki's lectures on the theory of Tomita and so that was a fairly new theory, etc. And when I returned to France. I decided to go to Jacques Dixmier's seminar. So, if you like, Jacques Dixmier had a seminar on operator algebras, operator algebras that had been invented by von Neumann. They were invented to understand quantum mechanics, to understand what are called subsystems.

In quantum mechanics, therefore, there was a formalism of quantum mechanics which had been well developed and von Neumann wanted to understand the subsystems and there are subsystems which correspond to factorizations of the Hilbert space. But in fact, von Neumann, with Murray, had discovered other factorizations and he had discovered three types of algebras called von Neumann algebras. There are what are called type I algebras which are very simple. There are type II, which are much more incomprehensible, and type III were the others, those that were left.

So when I got home, there was the Dixmier seminar. So I went for the first time at the Dixmier seminar and in the Dixmier seminar, Dixmier proposed a subject which was the classification of algebras and he distributed articles that then, we had to exhibit in his seminar to explain to others. So there too, I have raised my hand to get an item, I went to get the paper and when I got home in the suburban train, something completely became obvious to me, that was that the article he gave me to expose it in the Dixmier seminar, which was a priori on a totally different subject, was in fact perfectly connected to the theory of Tomita. And that was the start of my thesis. So, in fact, what I did, I wrote a small letter to Jacques Dixmier. He said "Okay, your letter is only half a page, it is not detailed enough, etc. Send me a more detailed letter.". I sent him a more detailed letter. I went to see him in his office and the only thing that he said to me when I went to see him, he said to me "Go for it !". And from there, the things went naturally but there was a kind of competition circumstances that made it happen, at least in the beginning, absolutely wonderful. Then, afterwards, there was a period, of course, when I had to do extremely complicated calculations, etc. And there was a moment when there have been... if you will, in a lifetime, there are very few moments like that. There may be two or three at most. There was a day when I brought Danye at his high school and I was driving home and at one point there was a red fire. So there, I realized, my brain realized that there had one thing that was completely obvious, that was before me, that I had not seen before, and which made it possible to completely unblock the situation. What was it ? It was the following. It was that Tomita and Takesaki, therefore, had demonstrated, if you will, that if we have an algebra like that, which we call a von Neumann algebra. If we give a state of this algebra, it's a rather complicated thing, we automatically have a group with one parameter of automorphisms of this algebra. But what was absolutely unclear was that it depended on a state of the algebra. What does it happen when we change the state of algebra ? What I understood when I was at the red light, it was because when we change the state, the group with one parameter practically does not change.

It does not change. If you want. If we neglect what are called inner automorphisms which are the automorphisms which appear naturally when we take a non-commutative space, when we take a non-commutative algebra, automatically, the automorphisms that come from non-commutativity, when they are cleared up, the ambiguity disappears. So what was, if you want, the message, what was the philosophical consequence of this thing. It's something that followed me all my mathematical existence. It is the fact that non-commutativity implies, generates time, implies a temporal evolution. So, I did not make a transparency on that, but to explain what non-commutativity is, you have to remember that. I'm telling you again a story that explains how it was discovered in physics. It was discovered in physics by Heisenberg. So Heisenberg was a physicist and at one point he was, I believe, in Göttingen and he was victim of a hay fever that was terrible and at that time, they could not treat him effectively. The only way to cure hay fever was to send people to an island where there was no pollen. They sent him to an island called Heligoland and hey, so he was on this island, he was housed by an old lady and there, he got down to do calculations. He was doing calculations and towards 4 am, he had an illumination. He understood. In fact, he saw a species of landscape that was revealed before his eyes. He understood that the physical quantities, for example, if you write $e = mc^2$ or put c^2 times m, it does the same.

Heisenberg understood that when we look at a microscopic system, it is not the same. That is, if you multiply the position of a particle by the moment or the moment by the position, you don't get the same result. And that was something huge he found. And what he says in his memoirs, he says that instead of going to bed when he made this discovery, he could not. He went to climb a rocky outcrop which was on the edge of the island and he waited for sunrise of the sun at the top of a rocky peak. So if you want, the extraordinary thing, is that the physical quantities at the microscopic level do not switch. That's it which completely unlocked quantum mechanics and that's what resulted in von Neumann to develop von Neumann algebras because von Neumann algebras are precisely, if you like, the algebras that are going to fit into quantum mechanics. So what ? So the contribution that I made in my thesis, if you want, it was to understand, precisely, that in fact this non-commutativity, it will generate time in a completely canonical, completely natural way. So that, of course, gave a bunch of invariants for these algebras. It allowed to classify them, but of course, it took a long time to classify them, i.e. that it also solved type III. It helped reduce it. That's what I did in the second part of my thesis, it is to reduce type III to type II with automorphisms. But of course, afterwards, we had to classify types II at least in the hyperfinite case and classifying automorphisms, it took, it took me an absolutely considerable duration and there was a period of my mathematical existence which was very, very conducive for that. It was the period of my military service. So of course, you will laugh because if I had really done my military service, that would have been very difficult to do math. But hey, I was lucky. I did my military service in cooperation with underdeveloped countries, with English Canada...

So it was still pretty cool. So I was in a small university which is the University of Kingston, Ontario. And there, again, I want to say, we have met extremely extraordinary people, humanly, around us, in a small group. And the fact that this university was not a central university, it was not Princeton, etc. It gave freedom of thought which is incredible, that is to say if you want, it helps in a small university, or if you are in a place like that, to be a bit offbeat. It allows not to have the weight of science, of all knowledge, etc., on the shoulders, and that allows to have some freedom. So I felt this freedom maybe more than ever in that place. So I was very happy at the time since I had practically finished what I wanted to do which was to understand the factors, to understand type III, etc.

And then, I returned to France and when I returned to France, I was invited at the IHES, at the Institute for Advanced Scientific Studies. And there, when I got there, I had a hell of a shock, that is to say that I went to lunch where people go to lunch usually, there is a small cafeteria and there were people talking about math. I had absolutely no idea about what they were saying. That is to say, if you want, I was a specialist in a sharp subject, of course very difficult, but I was not... I did not know, for example, what it is like a De Rham complex, etc. There was all this stuff going on over my head and I got lucky again. I was really lucky. I met a mathematician named Dennis Sullivan who, at the time, was one of the pillars of the IHES. That was in 1976. And this mathematician, he had the following property, very, very unique. He had the following property, which was that when he saw someone new at IHES, he would come and sit next to him, and he would start asking questions, with "What are you working on ?". So, we were chatting with him and at the start, we thought he was completely silly, because he was asking naive questions, if you want. Then he went on, like this, and he kept asking questions, of a naive nature. And then, after a dozen of questions, you realized that you didn't understand what you were talking about.

He had an absolutely extraordinary Socratic power. So, I started to discuss, discuss in detail with him, etc. And by talking to him, he taught me a lots of things, he taught me a lot of things I didn't know. He taught me the geometry and what bothered me a lot was that the work I had done on von Neumann algebras, if you will, it seemed to be in a place quite out of the center of mathematics. There is a kind of mathematical landscape and in this landscape, there are places that are really quite in the center. And then there are places that are much more peripheral and I had the impression that it was related to physics, of course, but I had the impression that the von Neumann algebras were something that remained quite peripheral. And I realized, talking to Dennis Sullivan, that in fact you could associate with a geometric datum which is perfectly known, which is what we call shovel a foliation, you could associate it with a von Neumann algebra. And what did it work ? It allowed to illustrate the classification I had made from geometric objects that were perfectly understandable geometric objects. The simplest foliation, you know, foliation, you have thought of a reverse lamination. It's basically. It is a space like here, the torus. But the leaves of the foliation, these are the lines that wrap around here. But what is striking in a flipping is the fact that while the total space is compact and finite, the leaves there in the winding, are infinite, that is to say that the leaves, when looked locally like on the right, we see something that is very simple, it's a product. But when you look at the leaves overall, they don't come back at the same location. They wind up endlessly, okay. So it was a point absolutely crucial because the foliation, in geometry, people know very well what it is. And they have lots of examples. And it turns out that the classification that I had done with the factors of type III_2, type III, etc., which seemed something rather odd and quite remote, fairly off-center in the mathematical landscape, in fact, it was perfectly illustrated by the most natural object that you can imagine. For example, this page is a sheet-floor of Type II. And if we take for example a geodesic and cyclic foliation, it is a type III foliation, etc. So if you want, that, that gave sense to the algebraic objects that I had found, it gave them a geometric meaning. Another meeting that was absolutely crucial to me at that time was in 78, I was invited to give an hour-long lecture at the International Congress of mathematicians. I was very struck by the following thing, that when I did my presentation, I had prepared and prepared, prepared. When I made my presentation, of course, my talk was about my results, about the classification of factors. But at the end, I had added results on the foliations, an appendix, and then so very, very odd, for me, the part on the leaves of a foliation, it's a complete trivial part and on the other hand, the really really hard, very, very hard part was the part on operator algebras. After my presentation, Armand Borel came to see me and he was all excited about the part on the foliations.

And suddenly, he invited me to Princeton and I was invited to Princeton during the year 78-79 and there, at Princeton, I had a meeting that was going to play an essential role in my mathematical life. It is the meeting of my collaborator who is called Henri Moscovici, who is a professor at Colombus, in the United States, and with whom, if you want, I really collaborated, most of my articles were written collaboratively with him. I also need to tell you another story about Princeton. I had a colleague at the École Normale who had stayed at Princeton before, for a very long time. I think he had stayed while we were still students at the École Normale. And he was quite fat. And while he was a little round, he had passed a year in Bristol and when he came back, he was absolutely skinny. We had asked him "But what happened ?". He told us that he had spent a year in Princeton, that he hadn't spoken to anyone, that he hadn't spoken to anyone, that is to say that it was a place that was so, how to say, prioritized, etc., that in fact, he hadn't managed to speak to anyone. So there was that side, there was that side, really, in Princeton. And I had an incredible chance which was to meet Henri and with Henri, of course, we immediately started working and we collaborated together.

So, if you will, what key point has emerged so far ? What was the essential point ? In fact, I understood at that time, because of the leafing through, I understood the scope of what was going on, because in fact what is going on, if you will, when you look at this type of space, what happens is that we actually have a space which, if we try to find its cardinality at the set level... If we look the leaves space of a foliation, okay, if we look at the space of the orbits here, if we look at the space of the leaves of a foliation, we will see that if we take the theory of ordinary sets, this set has the cardinality of the continuous. But we cannot biject it with the continuous in a constructive way.

In fact, we realize that it is impossible, we can demonstrate it, it is impossible to build an injection from this set into real numbers. So, in fact, I saw very, very gradually that these spaces in fact, if we tried to understand them in the usual way with the theory of functions, etc., we would not get them, absolutely not, and that the only way to understand them was to associate to them a non-commutative algebra. And this is the beginning of non-commutative geometry. Why is this the start, the very beginning ? Because the algebra of functions associated with such a space sees this space only at the level of the theory of measurement.

Now, measurement theory is an extremely fuzzy theory that allows you to tear the space into parts, etc. But who doesn't give the topology, who doesn't give everything else. And gradually, non-commutative geometry, it's a theory which allowed, if you will, to redefine, to reconstruct all the concepts which range from the theory of measurement, of course, to topology, geometry, differential geometry and even real geometry, Riemannian geometry. If you want, in the commutative case, spaces like that, in fact, what came to light at that moment was that there was a whole new universe, of completely new spaces, just waiting to be understood. But at that time, it was first necessary to know that it would necessarily be interesting.

Why were we sure it would be interesting ? We were sure it would be interesting because such a space was automatically a dynamic object. That is to say, such a space, automatically, has its own time, it generates its own time, while an ordinary space, like a variety, etc., it is static, it doesn't move. So these non-commutative spaces rotate over time. And this is something absolutely extraordinary. So we knew it would be quite extraordinary. But hey, well sure, afterwards, it was necessary to develop the theory and therefore it was necessary to develop the geometry. If you want, spaces in the fields, they do not commute. The first example, of course, this was the example that Heisenberg had found, that is, the example of the Quantum mechanics. So, we had to completely find, redefine the geometry for these spaces.

So when we talk about geometry, of course, well, in mathematics, there are all kinds of geometries that people have invented and that are more or less elaborated.

But hey, the most, the most relevant, the most important geometry is the geometry of the space in which we live. So, in fact, that's what will interest me, it has interested me for years. It is the geometry of the space in which we live. And what is quite amazing is that in fact this work is based on the quantum mechanics, it is based on quantum formalism, etc. In fact, I answer to a question that was asked by Riemann in his inaugural lesson. In his lesson, Riemann was fully aware that the notion of geometry that he had formulated, from Gauss, etc., from what we knew at the time, the notion that Riemann had formulated was not necessarily a notion that would continue to make sense in the infinitely small. What does it mean ? It was clear in his day that it covered great distances, but Riemann was extremely cautious. And what he explains is that the reasons why he doesn't not believe that it continues to have a meaning in the infinitely small, it is that the concept of solid body or the concept of light ray has no meaning in the infinitely small. In fact, we are immediately in the quantum domain when we look at this. So he explicitly wrote in his writings that in the infinitely small, it is all quite the same as if the concept of geometry will not conform to what it is, at the one he gave in his inaugural lesson. So, in fact, what is happening, so, of course he continues and continues. And he also explains that in fact, the founder metric relationships must be sought in the bonding forces that act in the space. I don't know how he got this intuition absolutely extraordinary. So what Riemann says, if you like, is that to understand really the geometry of the space in which we live, we must in fact understand the forces that hold things together. So, of course, since Riemann, there have been absolutely extraordinary progress compared to what Riemann said, in that, of course, there must be non-commutativity.

This leads directly to the domain which is physics, which is not part of math. But he explains how crucial mathematical thinking is for that.

And what is very important, above all, is that Riemann refers to Newton. And Newton had also sensed that when we go into much smaller distances that cannot be seen with the eye, there will surely be new forces that will appear. So what Newton says is that the pull of gravity or magnetism, and electricity are visible from a great distance. So we can observe in the usual way. But of course, we know everything that will happen at those distances.

The much shorter distances escape observation. And there is a book that I recommend to you on the history of particle physics whose author is Abraham Pais and in which he explains precisely how, in 1895, it was after Riemann, since Riemann was in the 1860s. Between 1895 and now, we succeeded at the level of vision in the infinitely small, to increase vision by a factor 10. It's a colossal thing and by doing that, in fact, the real microscope, the real microscope which allowed to see in this very small distance, it is the LHC.

Okay ? It was at the LHC, in fact, that we managed to pierce the structure to a level much smaller. But when we talk about much smaller distances, it is like saying much larger energies. So now, effectively, we arrive at 10 TeV, that is to say at 10 at the 13 electronvolts.

So what happened was that the formalism that I had to develop for purely mathematical reasons, to make the geometry of non-commutative spaces, proved, this formalism, in the 85s, from the moment when I returned to the Collège de France, it proved to be incredibly suitable to take the geometric structure of the space in which we live. But from experimental results, that is to say to come to understand that space in what we live, it is not simply the continuous at all scales, that it has a fine structure, but this fine structure, in fact, it is exactly, in the words of Riemann, dictated by the forces acting in the infinitely small.

So how did the paradigm change ? I can perfectly explain it. The paradigm has changed in the following way. So the purpose of the trip ?

If you will, what we have achieved in the very recent years is to take this kind of huge mechanism called the standard model coupled with gravitation. But understand it as just gravitation, lie on a space which is more subtle, which is more complicated and which has a structure finer than that of ordinary space. But then, what happens at the level of concepts ? Conceptually, what's going on is something very simple to understand. At the time of Riemann, at the time when Gauss, etc., defined their metric, the distance measurements were made while trying to take the shortest path from a point A to a point B.

This is what is shown here in this picture and in fact, there was a whole expedition which was made at the end of the Revolution and then until the 1799s, by two French men here. I don't know if you've heard of this, but they are the ones who measured the Meridian. They are the ones who tried to define the unit of length by measuring the distance between Dunkirk and Barcelona and from their measurements, well, there were all kinds of episodes, but from the measurement, we defined a unit of length which we called the meter.

And when I went to school, we learned that the unit of length was the meter-stallion which was deposited at the Pavillon de Breteuil, near Paris. It was a bar of platinum. But things have changed. It is the old geometry which consists in measuring lengths like that.

But what happened, something extraordinary happened which is that one day there was a meeting of the weights and measures conference. And there is someone in the room who said “your length unit, well, its length changes.”. It’s annoying anyway if the unit of length changes in length. And what had happened ? What had happened was that the man in question had measured the standard meter by comparing it to the wavelength of krypton.

And he realized that the length was changing. Little by little, physicists have thought about it a lot and they came to define the unit of length either as being the standard meter which is deposited somewhere at the Pavillon de Breteuil, etc. It is necessary to think well that when you say the unit of length, it is the standard meter deposited at the Pavillon de Breteuil, if you want to unify the metric system in the galaxy, if you explain to people on another planet in our solar system that only for measuring their bed, they must come to the Pavillon de Breteuil, it will be a bit complicated, so they found a much better solution.

They found a much better solution which was to define the unit of length. First, they took it from wavelength, krypton, etc. Then they defined from the wavelength of what is called the hyperfine transition of the cesium. Cesium has a certain hyperfine transition in the wavelength. This is a microwave type wavelength which is of the order of 3.5 cm and this allows to measure very, very effectively.

So what ? Obviously, that changes everything. Because if, for example, we defined the unit of length from the spectrum of hydrogen, for example, hydrogen is present throughout the universe. So there, it would be perfectly valid. Okay. So, it turns out that the transition between the old definition of the meter located in Breteuil and the definition from the wavelength of the cesium spectrum, this is exactly the transition between old geometry and non-commutative geometry.

It’s exactly that.

It is a spectral type geometry, spectral in nature. And so, that’s it, the change is the change in the unit of length. So, it’s a spectral geometry in nature and in addition, if you want, there is non-commutativity of algebra, it is practically imposed by what are called the gauge theories. Physicists have discovered strong interactions, for example, they discovered that there is not only electrodynamics, but that there are also strong interactions that hold quarks together in an atom.

And for that, they needed non-Abelian gauge theories. Well it is this that is really at the root of the fact that space has a very small structure, a fine texture which is non-commutative.

So there is a saga, a very, very long saga that I do not want to tell you. But

I'll just tell you the end, there have been ups and downs, that is to say I had collaborators, like Chamseddine, with whom we therefore made a model, if you will, a model of space-time that was based on these ideas, on the hyperfine structure, on the structure which comes from non-commutative geometry. It has had its ups and downs.

There was, in 98, the discovery of the neutrino, the mixture of neutrinos, the models of Calabi-Yau. So there, we gave up, we gave up for a number of years. Then we came back in 2005 with a new idea which was to change a dimension. Everything worked. Great. Except that in 2008, there was an exclusion of the mass of the Higgs who contradicted our work.

For example, I wrote a blog, I wrote quoting Lucrèce who talks about people who rejoice themselves in the misfortune of others. So indeed, theren we were very unhappy. And then, there was a period, so that, that was from 2008. And then, there was a kind of resurrection again, simply, it's the reason I'm telling you that you should never get discouraged.

Never be discouraged.

There was a very long period of discouragement from 2008 to 2011 and in 2012, my collaborator sent me an email and he said the following thing, he told me, "Look, there are 3 different teams of physicists who have managed to stabilize the standard model until making it compatible with the mass of the Higgs.". Good then okay, I keep reading his email and he says "They did that by adding a scalar field which verifies certain coupling properties with vacuum.". Then, I continue to read his message, he said to me : "This scalar field, we had it in our article of 2010, but we neglected it.". So in fact, we had it. In fact, we were discouraged. What ? That is to say that we had said : "The scalar field, it changes nothing.". If we had been courageous, truly courageous, we would have taken it into account and we would have seen that everything worked wonderfully. So that's it, it's for that side.

And in fact, therefore, what happened if you will, in terms of my mathematical development is that after having developed the theory called non-commutative geometry, so it's a theory that is still... I shouldn't want to make you believe that this is a theory that is simple, it is a theory that is very elaborated, it has a lot of relationships with a lot of different concepts, etc. And basically, each of the concepts involved in the geometry we are used to has to be changed and we look at it in a completely different way.

Even the integral, even the notion of integral is changed, it is replaced by this which we call the Dixmier trace and which is a concept which was invented by Dixmier and which plays an absolutely central role in this theory. Okay, it's connected to a bunch of other theories, but I didn't mean to bother you with that. Now, I'm going to explain that to you...

So in fact, there was another rather bizarre phenomenon, which is that in the year 1996, I was invited back to Seattle. I couldn't refuse if I was invited back to Seattle, and it was for the birthday of Atle Selberg, who was a great number theorist, and I was invited because in my collaboration with Jean-Benoît Bost, we wrote an article in which we found that we had what's called a phase transition on a statistical mechanical system and we found that the partition function of the system was Riemann's zeta function.

So, as this lecture was a lecture on the Riemann zeta function, they invited me, so I did well, I followed the same route a bit than before. I stopped in Victoria, then after I went to Seattle and in Seattle, I gave my lecture and after the lecture, I saw Selberg who said to me : "It is not clear that what you are doing will be related to...". If you want of course, we know the famous conjecture. He told me that, he told me that. And well, of course, lying, I mean, we also like to be provoked, we like people to tell you critics. There's no shortage of that in mathematics, no problem on that.

And besides, I have to add one thing, it is that not only do people yell at you, but you have also good friends who pass the critics on to you. So it doesn't not lack, but it has a positive side. It has a positive side because not only you have to be catchy, but you have to be able to be, to transform frustration that you have when people criticize you in positive energy. This is absolutely crucial. It is an essential quality, that is to say if someone criticizes you on one way or another, you have to take it and consider it as an energy potential, not as something negative. You have to be able to distance yourself from yourself and see it as positive energy.

Okay, so when I went back from Seattle instead of... I didn't mind jet lag. That is to say ? I stayed on Seattle time, I stayed on the Seattle area and what I did was that for a week I didn't work.

I was reading the book called *The Staff*, I don't know if you know this book, it's a book about astronauts and Apollo 13, etc. On this whole story, it's a wonderful book. Well, I was reading this book, I read it, I tasted it on the spot, I could have read it in a few hours.

But in fact, I tasted it little by little and after a week, I understood that in fact, what people were looking for when they were looking for a spectral realization of zeta zeros, they were all looking for it in the form of what is called an emission spectrum, that is to say a spectrum in which you will have a black background and you will have emission lines. For example, if you are taking sodium and you heat it, the sodium will give you a spectrum. If you pass the light through a prism, it will give you a number of very bright lines but well isolated like that, on a black background. Okay, but actually it's not like that we saw the spectra for the first time. The spectra, it is Fraunhofer who really discovered them and he discovered them by taking the light that came from the sun, we had already looked at this light through a prism, the prism decomposes light in different colors, the colors of the rainbow.

But what Fraunhofer did, he had a great idea. He had the brilliant idea of looking at it under a microscope. And when he looked under the microscope, he realized that in fact, there were lots of black lines. First, he cleaned up his thing, okay. And then, in fact, he realized that whatever instrument he took, there were the same black stripes. So what ? The wonderful story that was behind is that afterwards, there are Bunsen and Kirchhoff who have succeeded in heating bodies like sodium, etc. produce the same spectrum, but to produce them as emission spectra. Not like black lines, the reverse, except that he still failed to try with the light of the sun. There were black lines that we couldn't produce by broadcast. So obviously physicists are always clever, they said these black lines correspond to a new body that they called Helium, like the sun, of course. When they have it called Helium, of course, it's a bit like dark matter.

You will tell me what is this story ? Except there was an eruption of Vesuvius. And when they did the emission spectrum of Vesuvius lava, they found helium in it. Okay, so what I understood when I got home from Seattle was that in fact, instead of looking for an emission spectrum, which was what people were looking for, but there was a "-" sign that didn't work, there was always a "-" sign in the formula that didn't work.

In fact, the spectrum had to be sought as an absorption spectrum. So what ? I already had the non-commutative space I needed for that. I already had the non-commutative space that was necessary. And I knew how to demonstrate that this non-commutative space gives good terms in what is called the Riemann formula. But I had seen so after, of course, as I knew we had to look for an absorption spectrum. I looked to see if that, indeed, gave the spectral realization. And then, the miracle is that it gave the spectral realization.

That's what I understood when I got back from Seattle. After a period of total boredom, that I cannot recommend highly enough. There were no emails at the time. Me, I was not connected. Okay. It was a good time to leave the brain function because I was reading something else. I was not doing math. Of course. I was reading something else and after a while it came as something which has become completely natural. So what did that mean ?

But that meant that this very abstract geometry that had come from afar, you see, who came from von Neumann, who came from Heisenberg, etc., it looked like it was working, it seemed to work as well for the space in which we live and for probably the most difficult space that exists, which is the one which understands the nature of prime numbers, of the set of prime numbers, because what is behind the spectral realization, the state function, etc., is exactly the nature of the set of prime numbers. So that was the starting point. I wrote a little note to the Accounts. I was invited to Princeton.

And then there, what happened, me, I call it a gag, I found it very funny. This means that I gave my conference at Princeton. Good, etc. It is true that what you have to know, is that the subject in question, which is Riemann's hypothesis as soon as we approache, it's a subject that is mined. So you should know that it is protected by mountains of skepticism. Okay. But there is always that I had found something anyway.

I went there. I made my presentation, I surely understood something, if you will, it was the understanding of the Riemann-Weil formula in the form of trace formula. And then there are two people that I knew and that I know very well. There is one person I had collaborated with, Paula Cohen and Bombieri who made a hoax. In fact, a hoax for April 1, that is, April 1, you know, we usually do hoaxes, we see stuff. So they sent a hoax by email saying that after my conference, there was a Russian who was there and who had succeeded in demonstrating Riemann's hypothesis. Okay, so it was very funny.

I laughed, etc. Except that I hadn't realized that, it was in 98, in March 98, and I didn't realize that this hoax was going to be sent everywhere. So it was sent pretty much everywhere. Inevitably, there were people who took it seriously. So what ? What is absolutely incredible is that it was taken seriously. It was in 1998. It's a year when the International Congress of Mathematicians was held in Berlin. Well, I have nothing against the Germans, but hey, if you want, there was, basically, there was Germans who took it seriously. So what did they decide to do ?

This is something that is absolutely incredible. I realized only recently because I mean, back then I had ignored this game. I shrugged. What did they decide to do ? They decided to invite for talk for an hour at convention, the person who was best suited to be my competitor. Okay. So instead of inviting me, for example, they invited a person who was working on the same subject and they gave him a boulevard.

And well, I remember that Selberg, when I saw him again the same year, it was after the congress. He told me that he had never heard such an empty presentation. And me, I didn't know at the time, I hadn't looked, I looked recently and I was flabbergasted because I realized that this presentation, in fact, is a presentation who used my ideas without really quoting them. Or rather by quoting them with what Grothendieck calls the technique thumb. The technique thumb, it's the following if that can serve to you, it's... you know, you're borrowing someone's idea. But you don't really want to quote him. You put his article in bibliography, but you quote him for something else. It works very, very well. Okay, so. Okay, so this is the prototype of what happened to me at that time. It's the prototype of the experiment which happens when we approach this subject, when we are interested in this subject, etc.

But on the other hand, you can't be afraid. You do not have to be afraid. That is an absolutely essential thing. You have to be able to endure this kind of slander without drawing any consequences. And precisely, trying to transform them into positive energy. Okay, so what has happened since then ? I will, I will not delay... I do not know. What has happened since ? If you want, what happened since that time, this is the next thing. Is that I collaborated with Katia Consani. And at the time, at the time of Grothendieck, I had only one idea, it was to flee the subjects that Grothendieck was dealing with. Why ? Because there was a kind of snobbery around. There was a kind of courtyard surrounding it, etc. It's for that I had done operator algebras. So there I started to learn the algebraic geometry with Katia Consani.

And in 2014, it was not long ago. It means that you should never discourage. In 2014, we made an absolutely incredible discovery. We have discovered that this non-commutative space that I had used to make the spectral realization, etc., but that people could think of as a completely weird because non-commutative, etc., in fact, it was the points space of a Grothendieck's topos of an incredible simplicity, which one calls the Topos of the frequencies, simply the half-line produced semi-direct by action, by integers, multiplication. So it's something that is wonderfully simple. But when we calculate the points of this topos, because the notion of Topos, it is sufficiently subtle so that when you calculate the points it's something. It is in general very, very difficult when you calculate the points of this topos. You find a non-commutative space and this non-commutative space is exactly the space that I built to have spectral realization.

And what did it give us with Katia Consani ? It gave us on this space the structural bundle, that is to say before, we would never have imagined that and we saw, we understood that this structural beam, it was in fact a beam which is called tropical, that is to say which is connected to what is called tropical geometry. And so, that allows us to move forward. It allows us to move forward. I did my last two lessons of the Collège largely on it.

So that allowed us, if you want, we are not far from the goal. Of course, you can't say it until you've got there, you can't say anything. One can say absolutely nothing.

What if we said something ? People would hit us with a hammer on our heads. Above all you have to say nothing, but if you want, what we discovered, it doesn't matter if you arrive or not. Because in this guess, what is wonderful, is that if we really know the underside of mathematics, we realize that this conjecture is at the root of most concepts that were developed during the XX$^{\text{th}}$ century.

I will not give you a description, but in fact, in almost every mathematical concepts that were developed, they were developed with that in mind, behind, so here, in fact, we came to a space. Now, it's also progressing : we're arrived at a space which is much more, how to say, geometric, which is beautiful and suddenly more understandable, but which is, how to say ? which is understandable because Grothendieck had this wonderful idea for topos. And this idea of Grothendieck's topos, in fact, this idea has the same relationship with non-commutative geometry as the relationship that exists in the Langlands program between the Galois theory and automorphic functions. So it's exactly the same, the same relationship that still appears.

So to finish, just one thing I wanted to say is that there is another collaboration that was crucial and that is what we managed to do with Chamseddine and Mukhanov who is a researcher who does cosmology, so what we managed to do : we managed to understand what was the deep root of the standard model coupled with gravitation. Because before, we put the fine structure I was talking about, we put it from experiences, we started from experimental results and everything that, it was said that it needed such algebra for it to stick with experience. And we had no conceptual reason to say why it was necessary to put this algebra and not an other one. And that, that reason, we found it and we found it by a contest of circumsctances. We were looking to solve a geometric problem, purely geometric, and by solving this geometric problem, we came across the good Clifford algebra that we had put at hand before.

So that's it. And there is a very deep theorem that shows that we made all the varieties like that. It is a theorem which, geometrically, is based on this kind of images. So that's it, I think I'm going to stop. I forgot to tell you something. Okay, but I forgot to tell you something, it is that in fact, what underlies my presentation is something that had already been understood by Shakespeare.

I'll tell you what Shakespeare writes, there's the translation too, but I tried to translate. Shakespeare is always much better than his translation. So, what Shakespeare says is this :

There is a tide in the affairs of men,
Which taken at the flood, leads on to fortune.
Omitted, all the voyage of their life

Is bound in shallows and in miseries.
On such a full sea are we now afloat.
And we must take the current when it serves
Or lose our ventures.

Il est une marée dans les affaires des hommes,
Qui, prise à son apogée, conduit à la fortune.
Ignorée, tout le voyage de leur vie
Est confiné aux bas-fonds et aux écueils.
Sur une telle mer, nous sommes maintenant à flots
Et devons suivre le courant quand il forcit
Ou réduire à néant nos projets.

It's the only thing I want you to remember : when you see the tide, you have to follow it. But you have to feel it, of course, you have to feel it is there, okay. It's intuition. This is called intuition, it is something that is impossible to define. It's not something you can rationalize, but it's something which is fundamental in the job we do.

((Applause).

One of the students of the ENS organizing the Maths for all Cycle : Thank you very much for this presentation. If you have any questions do not hesitate. We have a little time, I think.

Question : I believe Shakespeare said "The world is a theater.", is it the same for Mathematics ?

Yes, there is a lot of truth and it's great that you said that because that allows me to explain to you what a topos is. It's great, it's absolutely great, I'll explain what a topos is. Th common theater in mathematics, it is set theory. We all know the sets theory, we know the groups, we know the algebras, we know... the usual theater of mathematics. What is a topos ? It's something extraordinary because... the theater is the same, the actors are the same... But behind the scene, there is a kind of Deus Ex Machina, which makes things to vary, which introduces variability.

That means that in set theory, there will be variability and that means something extraordinary, which is that a geometric space, it is not perceived by what it is, it is not at the center of the stage at all. It's the Deus ex machina backstage who makes things happen. And it is by understanding how things vary that we understand the geometric space that is hidden behind all that. It is purely theater, it is purely theater.

Ordinary set theory is a static theater, but the topos is much more interesting. Okay, that's a fundamental idea of Grothendieck, but you, you will never see it explained like that in the books.

Question : This is a question that may have been asked a bit often, but can you discover mathematics, it's something that exists without rationality of man ?

Of course. So, of course, we discover... The reason... I had this very long discussion engaged with Changeux, with Jean-Pierre Changeux, I will say... Jean-Pierre Changeux wanted to demonstrate that in fact mathematics were a construction of the brain.

But no. But in my opinion, it does not hold. And the reason why it doesn't hold, it's the next thing, is that thanks to mathematics, we explain the table from Mendeleïev, we explain, if you like, why there are chemical bodies, etc., etc. Okay, so the comparison I always take, I take two comparisons. The comparison I take is, take Watson and Crick. When Watson and Crick discover the double helix structure of DNA, they discover it, they don't have invented it. They were not the ones who invented it. Math is exactly that, it's exactly the same, that is to say... And especially now when at the computer. If you want, it's terrible now, and that, I haven't talked about it. But having computers and computers that are so powerful, it makes it possible to collide to mathematical reality. But all the time, all the time. That is to say, whatever problem you have, you can always test it with the computer, always.

And if it is a problem of symbolic calculation, you can solve it with the computer. So, I'm going to say, the computer, it doesn't invent. I'll say shit, we tell it a problem, okay, I mean, it tells you if what you found is correct or not, etc. So no, it's a real reality, it's a real reality. It's not a reality which is concretely realized in the world as it is, but it's a reality that is just as resilient, just as impossible to change as the external reality.

That's for sure, for sure. We invent tools, because Watson and Crick observe the double helix. They use the electronic microscope. We invent tools, but there is a reality that is there, a reality that is there, that is impossible to change.

Question : Still in the same vein, do you think that we could also discover intuitions that were previously completely hidden from us, and that we end up doing math with things that come out of human power, recently discovered and that a new branch that we are exploring...

What kind of power recently discovered ?

Continuation of the student's question : That is, for example, when you are dealing with subjects like topos for example, these are not things that directly, we could grasp *by intuition which is not mathematically educated. Is, according to you, after a mastery of the field, more or less relative mastery, that we could approach an intuition...*

But then, this is a very good question. This is a very good question because the human mind is not trained in quantum. The human mind is trained to the classic and therefore the human spirit in particular used to always give a classic image of quantum things. What is certain is that there is more instruments now that are based on quantum. For example, there is an instrument that makes random numbers, which is made by a Swiss man and with a mobile phone, we make random numbers, we make them with quantum and quantum is more and more widespread now. So what ? If we actually managed to train in quantum, to train, well... it's obvious that quantum hasn't been used much for natural selection so far, so we are not trained for that. But if we did manage to get to know each other a lot more with quantum optics, with quantum, etc., it's absolutely obvious that we would make progress. It's obvious. This is obvious.

I'm not talking about machine learning and artificial intelligence because for me, it's exactly the opposite of what we do in math, that is to say that we seek to understand and we seek to invent, to invent tools, therefore to find the concepts behind what we discover. And that, well, machine learning solves problems. But if you solve a problem without knowing how, it's not really interesting.

Question : I have a question on this paper that you wrote with Mukhanov. I have got the impression that you are working with a riemannian metric but our world is not riemannian. So, does it work too ? or...

Of course it works. But the real answer is this, the real answer is that when we do that, physicists know this very well, is that when we do the field theory and when we do the functional integrals, etc., there is something that we call Feynmann's I_ε trick, that is to say that we add to the propagator a term I_ε, and if we think about what it means, it means that we are working in euclidian. So, in fact, we want to make the functional integrals in Euclidean and the true functional integral that we want to do, in this case, it is that we take two three-dimensional riemannian spaces and we look at the cobordisms between the two and we do the Euclidean integral on it, okay ? But it's perfectly true what you say.

Question : Do your mathematical theories allow you to understand what is quantum entanglement ?

Ah, it's... So there, I left again for an hour... So there, wonderful question, of course, but I have already made presentations which are on the Internet on this. Yes, if you want, that's a wonderful question. Why ? Because what I got you explained is that a non-commutative object generates its own time. So it is obvious that we want to understand in what sense, it is related to the time with which we are familiar, etc. We thought about it a lot and I had an episode that I tell, that we tell in our book *Le théâtre quantique* with a physicist named Carlo Rovelli.

And if you want what happens, it's the following. I thought a lot more, much more, afterwards, on this time that appears, etc. And what we said in the book on quantum theater, we have a sentence that sums up the idea. The sentence is "The quantum hazard is the ticking of the divine clock.", which means this is the quantum randomness, the fact that when we do an experiment twice, we pass an electron through a very small slit and we look at the place where it comes.

It will never arrive in the same place. We only know the probability, so that's the quantum hazard, and the theory that's based on it, if you will, is that precisely this quantum hazard generates time. But in fact, when we think about it further and because of the entanglement, we realize that we commit everytime a mistake. All the physics we know, it is written by equations as a function of time.

To go back to my year in Math Spé, I had a special math teacher and once, he had put me a curve on the board and he said "Mister Connes ? What is the parameter ?" *(AC draws a curve in the space in front of him.)* I thought, thought... Then, after a while, I said "It's time !". He was very happy. So you see, all the equations of physics, are written as dt, dt^2. This is that I actually think, to answer that, and I will answer your question.

I think that time is only an emerging variable and that true variability since we attribute all variability to the passage of time. But I think that true variability is more primitive than time and that this true variability, it is the quantum hazard. So what is the meaning of entanglement, the meaning entanglement is that the quantum hazard is synchronized in two events that are correlated. It is synchronized, that is to say instead of being completely random and purely independent, it is synchronized. So there should be a very deep reflection, which I am not able to have, and which would consist in saying that variability in physics, it comes from the quantum hazard and that time is just an emerging phenomenon and we should understand entanglement that way. Because the entanglement is incomprehensible, otherwise. Why is it misunderstood ? Because what says the entanglement of a phenomenon of quantum mechanics which is entangled, is that you're going to have, as soon as you make an observation on A, it's going to be reflected immediately on B. But it will not transmit information, but however, these are two events that are causally independent. So you cannot say that there is one that is before the other. So, it's completely incomprehensible. And what I'm saying is that precisely, the error, the terrible heresy, is to try to write everything over time, and that we need a deeper reflection and which should be based on these things.

This is a second question that one must ask you often too, but for you, is Riemann's hypothesis true ?

Well there, I can't resist the temptation. I can't resist the temptation, but I have no right to speak about it. It is that we are finishing a book whose we gave the preprints today to Odile Jacob and in which we tell a story which is the story of a mathematician, much like me, who has enough worked on this hypothesis and who is ready to sell his soul to the Devil. So he is ready to sell his soul to the Devil, not to recover youth or whatever else, because hey, he's sick of it. He is discouraged, he wants to know, he wants to sell his soul to the Devil. And then, if you like, the problem is how to meet the Devil. And in fact, one day he goes to a conference on machine learning. Then, he did a bunch of calculations, if you will, a bunch of stuff, and he recognizes in calculations that the guy does, because you know, we say in mathematics "the Devil is in the details", he recognizes in the talk of the guy, the machine learning specialist, he recognizes the Devil. So, I'm not going to tell you the rest of the story because you will know it in a book, I am not telling it to you in advance.

You don't want me to tell, Danye ?

So I'm not telling you, but you'll see that : it's a very, very elaborate story, and at the end of the book, well, there is a stuff like that, that's it. Now, there you go. So I'm going to say, like I said from the start, you cannot know anything until you are at the end and no doubt that we shouldn't get to the end, because there is another aspect of mathematics which I did not mention, because it is a more, how to say, more difficult aspect.

It's always the fear of being wrong and I imagine that if we got to the end, we wouldn't would live more because we would be constantly afraid of having made a mistake somewhere. And that would be an absolutely unbearable situation. Okay, this is not desirable, really. Okay, there you go. In any case, you will have all the details soon from the history, this history of the Devil.

One last question : Yes, it may be a bit prosaic, but just now, could you talk about computers ? You said you could solve symbolic calculations or that we could face reality. But yourself, you use computers or... ?

Terribly, terribly.

Right now, I'm plugged into the big machines at Polytechnique to do a calculation. I use it terribly.

Terribly, of course, of course. It's amazing. It's great because as soon as we have a little practice, we manage to put any problem. For example, the weirdest problem you can think of, you will think "no we can't have it resolved by a computer, yes, yes.". And we can understand a lot of things, a lot of things. Even for geometry problems, etc. So, because above all, the visualization, the ability to visualize, the *Manipulate* and all that.

It's amazing. It's amazing. It's a wonderful tool, wonderful. I cannot say enough that it is a wonderful tool.

Last question, yes.

Last question : I have a question. You seem to demonize the machine learning, there is something wrong with machine learning. So I would like to understand, precisely. Machine learning is yet another area, what have you tried to do with machine learning ? In fact ? and what dit you get upset that you were not able to do ?

Imagine that machine learning tells you "Riemann's hypothesis is true." but doesn't give you the reason, don't give you concepts that were invented for the occasion, etc. It would be sad, sad to die. So, what I blame, I don't blame, but I mean, I understood at one point, talking to Alain Prochiantz, the analogy that there was between natural selection and the machine learning. It is true that we arrive at a result, but if we arrive for example at a result, and we do not understand why and we do not develop a concept. I'm frustrated. Personally, I am very, very frustrated. If it's not renewable, if it can't be repeated.

You have to draw positive energy from it ;-)

What ? Ah yes, draw positive energy from it, there, I mean, there is something to do. Of course, of course. No, but I'm not saying, but for the moment, it doesn't work so great, because when you have a machine learning on the phone, it's really not terrible. "Repeat... I don't understand what you are saying...". Okay, it will get better, that's for sure.

The organizer : Thank you very much, again, for your presentation.

(Applause)

Alain Connes : So at the start, we could approach mathematics by considering it as a part of physics, as a language that is developed to better understand the physical world around us ; and effectively, we realize that mathematics has this remarkable efficiency in this domain there. And very often, we have an understanding from the outside of these things that is in a way too superficial, and one example I wanted to take is the version which we now come to, with physical reality.

So physical reality is nothing other than the superimposition of possible imaginaries. And I will not give the formula but in this sentence, there is something extraordinary, it is that imaginary numbers, complex numbers are implicated, and these numbers, by the same name, as imaginary numbers, at the beginning, were mathematical fictions.

So there is this incredible efficiency of mathematics in the physical world which even upsets our philosophical conception of what reality is and which questions, of course, materialism as a somewhat naive idea because the materialism is a theory based on a partial understanding of things and which identifies the real with the material.

Now from what I said, precisely, the fact that the real is this superposition of imaginary possibilities shows how much more subtle reality is. But in fact, after a while, we realize that there is a clean journey, inside from the mathematical world, which becomes disjoint from the physical world, and the main tool that makes it possible to start this journey and begin, is the analogy ; it's the fact that the human mind is able to see between very, very different areas, certain reflections, certain correspondences, and from these reflections, these correspondences, to transpose, to transplant ideas that were valid in a domain, to transplant them into another domain ; this is how mathematicians have discovered whole swathes of mathematics that have nothing to do with the physical world, and that it would be illusory to want to find in the physical world.

This is called, for example, the p-adic world. The p-adic world is the world in which for example the integer two is very small, and when, by analogy, we transplant the concepts we have for real numbers, we notice, a little as Alice in Wonderland, we discover 36 things we wouldn't have never suspected.

So this is the first thing, the first thing is that there is a world of mathematics, which is absolutely not subject to the world of physics and when we explore this world much further, we realize in fact that in this mathematical world, there are for

Transcription of a video that can be played here https://www.youtube.com/watch?v=i08kF-NLVS4 example things that are true but not demonstrable.

So this is something that is quite difficult to explain, I tried to explain it in a book we wrote with Lichnerowicz and SchÃ¼tzenberger, by the fable of hare and turtle. So it would be technically complicated to explain but itâs a typical example of a mathematical statement, which we know is true, we know also that it is unprovable in what is called Peano arithmetic, it is said in simple ways, and the reason we know it to be true is what is called ordinals theory, set theory, Zermelo-Fraenkel' theory, etc. So in fact, we have since managed to see that if we take simple statements, arithmetic statements that can be formulated in a simple way, and in fact, we know that most of these statements are unprovable, that is to say that the proportion of statements that are true but not demonstrable tends to one, that is to say that most of the statements that are true are actually unprovable. And the image that must be kept in mind for the relationship between the mathematician and this mathematical reality that I call archaic mathematical reality, because rightly, there are things that are true but that we cannot perceive in a direct way, this is the same relationship as the one between external reality and a court : in the court, you have data, so in mathematics, these are called axioms, and from this data, you can make a number of deductions, this is what the mathematician does while working. But, it would be wrong to identify the deductions that are made inside the court with external reality and this for obvious reasons.

Alain Connes

I wrote this book *Le Spectre d'Atacama* with Danye Chéreau and Jacques Dixmier, so one is literary, the other is a great scientist, a mathematician and the general theme, the general idea, it's a scientific novel, it's a new genre and we want, in fact, to make the reader discover deep ideas both about mathematics and physics, but in such a way that they are accessible, that gets to get a feel for it by really reading the book like a novel. The story, the novel, which is an adventure novel, features three characters, who are three scientists ; on the one hand, there is a mathematician who well knows physics ; there is a computer scientist who will play an essential role ; and there is a physicist who is a survivor of a stay in the quantum. In fact, the central character, we can say at the scientific level, it is the spectrum. This notion of spectrum is a notion that is common to physics and mathematics. It was first identified by Newton when he decomposed light with a prism. And that's what we see in rainbows, but it has also become an essential concept in mathematics at the beginning of XX^{th} century. And we realized at one point that the two notions coincided, that is to say that we could explain the spectra of physics, the spectra that we receive for example from very distant stars, galaxies, that we could explain them by mathematics. And what we tried to do, I hope that we have succeeded in doing it, is to make this theme perfectly understandable, to make it sensitive, to pass it without being dogmatic, that is to say without quantities of explanations, etc., but by reading the book, by reading it like an adventure novel.

It can really be read as an adventure novel, there are scenes, really, that are happening and that are quite extraordinary. Through this reading, there is an osmosis that occurs and we come to understand what are spectra, what is the music of the form, and the message must get across. There is another message which is important in the book, and basically I can summarize it in the following form : is that we are living in a transition period that everyone knows, I want to say that there are more and more machines and in particular there is what is called the machine learning which will replace, soon, when we call a company, instead of having "press the 1 button, etc.", and then finally, we have somebody, we will never get somebody, we will only have machine learning. So in fact, this novel is a plea for concepts ; it's a plea to try to resist to this drift of the world that makes us abandon our individuality and we do it consciously, that is to say that people who are on facebook, etc. they consciously give up, it is them who do it, who give their individuality, and in fact, there is a message very important at the end of the book, which is that we have to save our souls, we have to save our souls by escaping this temptation, this temptation that would make everyone of us a kind of cell in a much larger body, whose soul we would totally escape. And the role in the book of machine learning is held by the Devil and fortunately, there is another central character who intervenes, who is spirituality, and that finishes the book in an optimistic way, that is to say that there is a message

Transcription of a video that can be played here https://www.youtube.com/watch?v=qPA5DywLFeI that we have to listen to, and that basically sums up this idea that concepts are so much more important, and human-made concepts like the topos, the concepts are so important compared to this current drift in which we let ourselves slide ...

Great interview with mathematician Alain Connes

Nicolas Martin : We often find in great scientists this common point, this line of escape or this beautiful escape to the world of the arts. Poet, painter, musician or novelist, in this case for Alain Connes, Fields medallist and Gold medal from CNRS, mathematician at the origin of noncommutative geometry, a branch of mathematics that aims to embrace GUT, the Grand Unified Theory, the theory of everything that would reconcile general relativity and quantum mechanics. Novelist, therefore, musician too, but above all and forever, obsessive researcher. Alain Connes is our great guest for the hour to come. Welcome to *La méthode scientifique.*

Thank you and hello Alain Connes.

Alain Connes : Hello.

Nicolas Martin : A thousand thanks for accepting our invitation, so I'm going to a quick summary presentation that I will leave you to complete for our listeners who don't know you yet. You are therefore a mathematician in this paradise for researchers that is the IHÉS, the Institute of Advanced Scientific Studies in Bures sur Yvette.

Alain Connes : I'm first at the Collège de France, let's not forget, the Collège of France.

Nicolas Martin : I'm coming, also Professor emeritus at the Collège de France, holder of the Chair in Analysis and Geometry, member of the Académie des sciences française. But other Academies of Sciences, including the Academy of Sciences in the United States, but also in Denmark and Norway. You got the Fields medal which is, I repeat, the greatest distinction mathematical in 1982 for your work on operator algebras.

You could say that you have somehow revolutionized algebra by founding noncommutative geometry, you will talk about it again and the CNRS has awarded you his Gold medal in 2004 for solving the mathematical problems raised by quantum physics and relativity. And you just published your second novel after *"Le théâtre quantique"*, *"Le Spectre d'Atacama"*, co-written with your wife Danye Chéreau and your former thesis director Jacques Dixmier. It's up to Odile Jacob editions. What

interviewed by Nicolas Martin on 17.5.2018 as part of the radio program *La Méthode scientifique* on France Culture
https://www.franceculture.fr/emissions/la-methode-scientifique/la-methode-scientifique-du-jeudi-17-mai-2018

should one add to this description, Alain Connes ?

Alain Connes : Let's say that if you want, indeed, it's a scientific journey, we can now look back on it. And I will start by saying, if you want, that each mathematician is a special case. So I mean there are no generalities to make and in fact, the course that I followed, I put a lot of time to find my way, that is to say, initially, if you want, I had started by doing logic with non-standard analysis, with Gustave Choquet. I had done some number theory too, and it's finally with Jacques Dixmier that I found my way. And so, in fact, the journey begins with it, with the operator algebras and in fact, with, if you will, what von Neumann had understood from the discoveries of quantum mechanics, it was von Neumann, who had formalized the quantum mechanics and therefore the formalism that he had developed, if you will, hasn't changed since, you could say that this framework he created, the framework of Hilbert space, vectors space in Hilbert space, states, etc., is something that has never been questioned since the 1930s. But it has been shot further, with a collaborator called Murray, and basically, if you will, von Neumann asked himself the question of when could we define a subsystem of a quantum system. That is to say that when we take a system quantum, normally, it involves all operators in Hilbert space. No, that's a bit technical, but von Neumann had asked himself the question of knowing when do we have a subsystem ? And at the start, you would think that simply when you have a subsystem, Hilbert Space is factorized into the product of two subsystems, but von Neumann had thought much deeper, trying to understand at the algebraic level, so, we come back to algebra, at the algebraic level, how this factorization manifested itself. And with Murray, they had an extraordinary surprise, that is to say that they found that beyond the very simple factorizations of the Hilbert space in tensor product, as we call this in mathematics, there were algebraic factorizations, which gave the notion of factor, that is to say that in the language of operator algebras, there is an essential notion that we must come to understand from the start as coming from an essential problem of quantum mechanics, which is to know when we can characterize a subsystem. So what ? The wonder that happened is this, the creation of factors by von Neumann. Dieudonné called them von Neumann algebra, since they were due to von Neumann, Dixmier worked a lot on it. And when I arrived, I was lucky to arrive at a good time. It was at the time when a Japanese mathematician named Tomita has, maybe 5 or 6 years ago, find a very, very interesting theory. And I was lucky to discover that fact, the evolution over time that was associated with each state, normally, in an algebra, after doing very, very, very complicated calculations during months and months, I finally discovered that this evolution over time, it was unique, it was actually independent of the state, modulo the so-called interior automorphisms, it is something invisible. So, in fact, I understood at that point that, if you will, these von Neumann factors when they were of a fairly exotic type called type III, they spawned their own time. And because they generate their own time, it has created a lot of invariants that allowed to completely unlock the classification of these factors. These factors appeared to be intractable before, and in my thesis, under Jacques Dixmier, in fact, I showed how we could, if you will, reduce these factors to much simpler things thanks to this evolution in the weather and how they had, if you will, all kinds of invariants, like the periods, etc., etc.

Nicolas Martin : It is noncommutativity, that is what must be remembered.

Alain Connes : What you have to remember, at an abstract level, at the conceptual level what must be remembered is that noncommutativity was discovered by a physicist, by Heisenberg. So, that was a discovery, almost, how to say after... almost from experience, that is to say that Heisenberg has based himself on the laws of spectroscopy. That is to say, we observe spectra. These spectra have very specific properties and Heisenberg understood, from what we call the Ritz-Rydberg principle of composition, that in fact, if you want the algebra which was underlying quantum mechanics, he understood that in 1925 and it's a fundamental discovery, was a noncommutative algebra. I don't recall the anecdote, of course, which was that he was on the Helligoland island, that he was alone. He could finally work quietly because he had no more lessons to give, because he was sent there, because he had a cold hay. He lived with an old lady, he could work as long as he wanted. He was doing calculations, very complicated calculations. And then one night at 4 a.m. he understood.

Nicolas Martin : Eureka ! It exists, therefore !

Alain Connes : It exists. And he had before his eyes, he says, an absolutely wonderful landscape, which was almost scary of novelty. It was the landscape of quantum mechanics and it was the landscape of the noncommutative. What Heisenberg understood was that when we work with a microscopic system, a very small system, we no longer have the right to swap letters when doing physics calculations, you know, when we write $e = mc^2$, we could write $e = c^2$ times m, it would be kif-kif, it would be the same. Well.

Nicolas Martin : That is commutative.

Alain Connes : That is commutative. But what Heisenberg understood is that when we work precisely, for example, with position and momemt, well, it is the speed multiplied by the mass of a microscopic particle, at that time, we have to be careful, just as we are careful when we write, you see. When we write, obviously, we don't have the right to swap letters, since if we swap them, that makes an anagram, we can get anything starting from something. The first book we wrote with Danye Chéreau and Jacques Dixmier,...

Nicolas Martin : *Le théâtre quantique...*

Alain Connes : Yes, *Le théâtre quantique*, we cite anagrams, so, I mean, for example, *L'horloge des anges ici-bas* and *Le boson scalaire de Higgs*. We see that by swapping the letters, we can change the meaning completely.

Nicolas Martin : It is the happiness of our colleague Etienne Klein.

Alain Connes : Absolutely. Etienne Klein is a great specialist in anagrams.

Nicolas Martin : Alain Connes, that makes Nicolas Anne, you see, it's about near my math level.

Alain Connes : No, but I received an email from someone. I did absolutely not understand what he meant. I thought he had gone mad, but there was the anagram of my name five times. Okay, which is easy to find, I mean... So going back to Heisenberg, if you will, he made this extraordinary discovery, which is that when we work with a microscopic system, what we calls the observables, the natural variables of the system no longer switch between themselves. What does it mean ? It means that when we take what we call in physics the space of the system phases, it is a space which no longer corresponds to the description that Descartes made and which was at the source of all the algebraic geometry, what is called algebraic geometry, i.e. on the one hand, there is geometry, and on the other side, there are the coordinates of space like Descartes defined them, but these coordinates usually switch. Heisenberg's discovery is that when we take the space of the phases of the physical, we can no longer assume that the coordinates commute. And what is the root of noncommutative geometry is exactly that. That is to say, there are spaces which are in fact natural spaces, they are not pathological spaces or whatever. There are natural spaces in which, precisely, the coordinates do not more commute. So actually, if you will, what made the theory interesting, what made the theory really interesting, because generalizing algebraic geometry in cases where the coordinates no longer commute, it seems a tedious task, and which does not hold big surprises. But what motivated me, if you will, to develop noncommutative geometry, this is precisely the work I had done in my thesis under the supervision of Jacques Dixmier, and which had shown that a noncommutative space, i.e. a noncommutative algebra, generates its own time. And then, if you want, it's so new, compared to the ordinary geometry ... What does that mean ? It means that ordinary geometry is commutative, it is static, it does not move, while noncommutative geometry automatically generates its own time. And this time will allow us to do things that we would have no idea of doing otherwise. In particular, it allows to do the thermodynamics of a noncommutative space. It allows for example to have a noncommutative space, for example, and to cool it. So this is completely unexpected, if you will, it is something that is completely new. And that's what, of course, motivated me for years and years, for practically all of my scientific journey, exploring these spaces, exploring geometry for these spaces which are completely new.

Nicolas Martin : And this notion of time, by the way, that you explored also with Carlo Rovelli, whom we received here very recently. We go and put the link back on Twitter feed. I would like to question, Alain Connes, something that I often do with great scientists who follow each other at this micro, that's the question of vocation. You describe yourself at the start of your career that you come telling us brilliantly and this anecdote, that you ended up to tell us, of Heisenberg, finally, despite having said you would not. One describes you as a young mathematician with exceptional talent. What is the feeling... What is your conception of this vocation, of this attraction for mathematics ? When is it born ? How does it germinate ? Is there a vocation or is there not a vocation, is it a mishap ?

Alain Connes : Well, I think it's something that comes about rather slowly, that is to say that, if you want in my studies, indeed, quickly enough, I spent much more time trying to develop my own ideas, and to create my own ground than to be at school and to take classes, etc. So it happened actually happened very early and I remember, for example *(small laugh)*. I remember that when I was a child, I think it was in course of Seconde or Première : I had a math teacher and he said in the class that there was no formula which gave the number of prime numbers smaller than n. So obviously it's not true, I mean. I think what he had in mind is that there is no simple formula ; in fact, by the way, there is a simple formula, but it is not very, very useful. I can give it to you, so we'll see.

Nicolas Martin : Give it to us.

Alain Connes : So it's not a formula for $\pi(n)$, the number of prime numbers smaller than n. It is a formula for $n - 2\pi(n) - 2$. But anyway, whatever. And then it's only true for n greater than 13. Okay ? But still, it's very simple. It is the integer part of the sum from 1 to n of cosine of $\pi\Gamma(k)$ over k. Okay, we can't say it's very complicated. Okay. So, the next day, I came back, I came back and I gave my teacher a formula that was much more complicated than that. But from that moment on, I had taken a step which is an essential step for the young mathematician and this step essential is to believe in yourself, that is, not to give credit to authority. And that is extremely important. And I think math are a subject in which it is possible. It would be much more difficult in chemistry, in history, etc. Because there, knowledge plays an absolutely essential role...

Nicolas Martin : The observation ?

Alain Connes : Not only, but the accumulation of knowledge, while in mathematics, we can very well find ourselves face to face with a problem. The problem is very easy to apply and a priori, there is no reason why if we find a solution, it is not fair. So math is very special in this sense where there is not, if you want a priori, a kind of bead of knowledge, of knowledge that prevents a young person who is starting to understand something no one else understood. This is extremely important.

Nicolas Martin : Neither God nor master in mathematics.

Alain Connes : Yes, in a way. I'm going to say, and in fact, one of essential conditions in the path of a mathematician is to arrive at questioning yourself, that is to say if you want, if, from the moment one believes that one is stronger than the others, etc. There, it is the beginning of the decline. I think it is absolutely essential to never think that you have acquired enough knowledge, etc. or I think that it is essential not to believe that the path we are following is necessarily the right one. I think that this is one of the essential subjects of our book with Danye Chéreau and Jacques Dixmier. So, in *Le Spectre d'Atacama*, we describe the journey of a mathematician who is called Armand, Armand Lafforêt. And we just highlight this quality, that essential property of doubt. Why ? Because nothing says the way in which we are engaged when we want to solve a problem be the right one. It is necessary to constantly question yourself. We must constantly ask ourselves the question of know if, of course, if we reach the end, so much the better. But when the problem is very difficult and the problem we are talking about in the book is an extremely difficult problem, in these cases, actually, there is no other way out than constantly doubting and being able to question yourself.

Nicolas Martin : On precisely this relationship to your work, on this relentlessness to solve the problems, you tackled Riemann's conjecture, the Riemann's hypothesis which is the eighth of 23 mathematical problems for the 20^{th} century of Hilbert, that has until now not been proven, and I speak under your control. You talk, Alain Connes, about that and on your work in general, of the mathematician's obsession, there is something obsessive in this work ?

Alain Connes : Yes, in fact, I was interested in this problem completely by chance. You have to know that. What does this want to say ? That is to say that as it is one of the big problems, my starting principle, it was the opposite, that is to say it is always to remain marginal, to remain a little ambush and never take an interest in a problem like this one. Except that what happened was in 1996, and I was invited to a conference that was for the 70th anniversary of Atle Selberg. So Atle Selberg is a very, very great Norwegian mathematician who worked, for him, tremendously valiant on Riemann's hypothesis and found great things. And on the occasion of this meeting which took place in Seattle, in 96, there was a day... Well, I did my conference because I found with a collaborator, with Jean-Benoît Bost, we had found a system of quantum mechanics which was related to Riemann's zeta function, but it appeared to be related to it in a very peripheral manner, that is to say the zeta function appeared as what is called the partition function but it was peripheral. Now, what happened is that I gave my talk. And then, at the end of my conference, Atle Selberg came to see me and he said to me "It is not so clear that what you do is related to the Riemann hypothesis."

Nicolas Martin translates this sentence in french.

Alain Connes : Exactly. Then afterwards, there was, we had a meeting, etc. Then when I got home, I was really thoughtful, for a week. At the time, there was no email. I was not looking at my emails. I could be completely disconnected. I was jet lagged for about 8 days. And then, after 8 days, I realized that in fact, the system that we had defined with Jean-Benoît Bost gave exactly the space that people had sought to about that. So hey, then I said "gave *the* space". Nothing says it is still the right one. Nevertheless what it showed immediately, it showed that a formula which is essential in this theory called the explicit Riemann-Weil formula appeared completely naturally from the geometry that we had defined. So, if you want, I wrote a note to the Accounts. And then, thread by needle, I was caught in this kind of situation in which one does not control more, because it's true that if you want, as soon as someone is interested in this problem, basically, I'm kidding of course, but basically, if you want, the other mathematicians wish you that you fall over and above all, that you does not resolve it, and for a good reason.

Nicolas Martin : The mathematical world is even more anarchic than we have ever imagined it before starting this show.

Alain Connes : In fact, it's more complicated than that because in fact, how to say, it is very interesting, the sociology of the mathematical environment. But this assumption, Riemann's hypothesis, you have to understand in fact that without this being obvious, it is behind countless very fruitful developments in mathematics of the 20$^{\text{th}}$ century. It started with the theory of almost-periodical Bohr's functions. It continued with everything André Weil did and then, well sure, Hasse, Artin, etc. on geometry with finite characteristic, what Deligne did, what Grothendieck did. So if you want, there is a huge influence from this conjecture on the development of mathematics. And in any case, what would sadden me terribly is if it were resolved anecdotally. And in fact, I recently, for example I made money from a number theory journal which quite often receives articles which claim to demonstrate this assumption. In fact, they send it to me and they pay me when I find the error. Why ? Because therefore, I mean. It's very, very complicated. It's an extremely complicated, extremely interesting, extremely mentally interesting problem, because in fact, what is likely is that it will be demonstrated only when the surrounding landscape will be fully revealed. It's a little like a mountain peak. But before we really understand what's behind, apparently, there is no way to cut, if you will, there is no shortcut and if there was one, it would be a bit catastrophic because it would mean that the magnificent landscape that we have to discover about this guess, well, it wouldn't have been released.

Nicolas Martin : And at 4 :22 p.m., we continue our interview with Alain Connes, who has just published a novel, his second, *Le Spectre d'Atacama*, co-written with Danye Chéreau and Jacques Dixmier, since we were talking about the vocation in your mathematical career, Alain Connes, where does this will come from, this desire for romantic writing, for fiction ?

Alain Connes : Ah then that, it's a desire for freedom. In fact, that means that we have discussed a lot together, the three authors. But mathematical work is work in which, of course, imagination plays a significant role, it is obvious. But this imagination is terribly corseted. That is to say, there is a mathematical reality. I use the computer a lot, a lot. And this mathematical reality, it is undeniable. That is to say that if we have an idea of a formula, etc., we can try to check it out, and if it works, it works and if it doesn't work, it doesn't work. So imagination plays a big role. Of course, it has the role especially, I would say, of the creation of mental images, that is to say that when I say imagination, it's a lot more, the fact that by looking for a problem, even if we cannot solve it, by done, when we can't solve it, it's better because it means it's a problem that allows us to improve ourselves, at that time, we create mental images. When you see someone on the subway reading a music score, if you're not a musician, that doesn't mean anything to you. If you see a mathematician who reads a sheet of math, it doesn't mean anything to you neither. And on the other hand, a mathematician is fine with this speak right away. It will speak to him right away because he has mental images. And these mental images, they wake up as soon as he sees the corresponding formulas. So this is extremely important. Unfortunately, it is very hard. This is very, very difficult, that is to say, well, we can have an idea. And then, after a moment, when we try to write a demonstration, no, there is something that doesn't stick, etc., so if you want, we come up against a reality which is extremely resistant. On the other hand, in the romantic writing, which is this pleasure that we all had the three, these two times, but especially the second, because we spent a long time to write this second book, in this romantic writing right there, imagination can unfold. And in fact, what strikes me when I look at this book, it's especially with current developments, what strikes me is the infinite freedom enjoyed by the hero, who is Armand.

Nicolas Martin : In which it is difficult not to recognize you.

Alain Connes : Yes, but in fact, it's not true, there are a few ingredients.

Nicolas Martin : He is a mathematician, he works at IHÉS...

Alain Connes : Yes, of course, but in fact, no, no, no, no. In fact, he's a character from a novel and a character from a novel who enjoys a freedom which, unfortunately, will be more and more difficult to have. For example, well, he's going to Chile. Afterwards, he decides to take a boat, he goes to the Staten Island, etc. So what ? We say to ourselves that right now, he would have had a Facebook account, that people would have glimpses that he is no longer responding, that he is no longer there. They would have gone and look for him. So this notion of fundamental freedom, this magnificent freedom is present in the book. It shows, if you will, how essential it is to maturation of an idea, etc., precisely when the mathematician is obsessed with an idea. But what scares me is how much it risks disappearing. You know, I always tell this anecdote which had struck me so much, which was at the time of an unexpected visit by the President of the United States to the Princeton Institute and the Director of the Institute was showing him around the offices since hey, he wanted to show... At one point, they knocked on the door of a mathematician and they entered the office. They had found the mathematician lying on his table, sleeping. So here, when now, they would have entered, what would they have seen ? They would have seen the mathematician, in front of his computer, answering 36 solicitations, in general completely without interest. He should have written a report or do etc. But if you want, this notion of doing nothing, this notion of leaving your mind completely freewheeling and being able, precisely, at some point, to wake up and tell you there it is, there is something, etc., well, this notion it is, unfortunately, very, very threatened by technology, by the fact that we are more and more regimented, more and more corseted, more and more labeled. So, I'm going to say, reading this book, I think, gives pleasure in that sense, that is to say we see this pure state being threatened, disappearing, unfortunately.

Nicolas Martin : We will come back... because you are speaking, in fact, in this book *Le Spectre d'Atacama* of Artificial Intelligence on which you don't have, will we say, a very mild look. But before that, maybe, a word, there are a lot of things that come to mind, but maybe, for the rest of our interview, for listeners, a word on what *Le Spectre d'Atacama* says, three people, you said it, Armand, a mathematician, Charlotte, a physicist, and Ali, a computer scientist, these are the same characters as those of your first novel, moreover.

Alain Connes : Of course, they are the same characters. In fact, the novel is the story of a spectrum, which is received by the Alma Observatory...

Nicolas Martin : So it's not a ghost.

Alain Connes : It's not a ghost ; a spectrum, you know, that's something that has both physical and mathematical meaning. The first way in which the spectra appeared, they were called absorption spectra. And it was Fraunhofer, a physicist who was optician rather, a German, who had this idea awesome which was to look at the spectrum of the sun which had been for example disclosed by Newton. That is to say, we pass the rays of the sun through a prism and we of course gets the colors of the rainbow. But he had this extraordinary idea of looking at this spectrum, for example, with a microscope, and he noticed that there were black lines. So of course, at first, we can think that those black lines are caused by the lens that's dirty or something like that. In fact, it was not. These black stripes, he had listed about 500 and this is the first spectrum from which we got the physical trace of a chemical element. And this is called an absorption spectrum. What does that mean ? It means if you want that these black lines, in fact, come from the signature of certain chemical bodies which are contained in the solar corona, it means that the light which comes from the sun is absorbed by these chemical bodies. And since they have a chemical signature, it allows us to know what is the composition of the solar corona. So after, we realized that they were not only absorption spectra, but they were also emission spectra.

For example, when you take sodium, heat it, and pass the light matter coming out of sodium through a prism, this time, we get brilliant lines on a black background and these bright lines on a black background matched exactly to some of the black lines, on the light background, in the spectrum of the sun. But there was also something great that happened. Because among the spectra that we could recognize, there was one which was completely mysterious, which did not correspond to a chemical body on Earth. And the physicians and chemists had this great idea to say "Oh ! It's a chemical body we don't know". And they called it helium like the Sun, well sure. So the wonder of wonders is that there was an eruption of Vesuvius, I don't know more in which year and that we could observe in the lava of Vesuvius exactly the helium spectrum. So the circle was complete. These are the spectra in physics. At the beginning of the 20$^{\text{th}}$ century, mathematicians and physicists understood how to calculate these spectra of physics, from mathematics, and from a notion of spectrum which comes from mathematics and which is central in the quantum mechanics von Neumann's formalism. So an operator's spectrum exists, it's its variability, it's its vital space, if you will. And in fact, then the starting point of the book is Armand, who is a mathematician, who is a bit obsessed with a problem, etc. And then one day he receives a message from Rodrigo, who is a friend of him, who is an astronomer at the Alma observatory and who said to him "Come and see me, come and see me, there is something extremely mysterious, in Chile, in the Atacama Desert". And finally Armand, well, his friend is no longer available because he had a cerebral stroke, eventually he recovers the spectrum, so he gets a very, very bizarre, very bizarre spectrum. And then after, he will embark on this spectrum in all kinds of adventures which are a species of escape from the how to say, the hubbub of the modern world. He tries to escape the hubbub of the modern world to try to focus, to understand what this spectrum is. So he will go from adventure to adventure. Like that, he'll...

If you want, his physical journey is a metaphor for his intellectual journey, of course. Okay ? And then after, what is absolutely incredible is that he will find along his adventures, another of the characters in the first book, which is Charlotte and Charlotte, had also had an experience... Charlotte who is a physicist at CERN, had really lived in her flesh, an experience that looked crazy in the first book. I think a lot of people who read the first book considered it a completely crazy experience...

Nicolas Martin : in *Le théâtre quantique*, published by Odile Jacob.

Alain Connes : And so, in fact, for this experiment, we understood what was behind, in the second book, and we have, how to say, we explained what had happened to her because in reality, it's actually very funny because Armand had fled, and when he learns the experience of which Charlotte is a survivor...

Nicolas Martin : A quantum life experience

Alain Connes : A quantum life experience, when he learns this new, in fact, it is juxtaposed with an article from *Le Monde* on a representation of *The Sleeping Beauty.* And what's behind it, because there are a lot of things hidden in the book, you have to read it several times, there are a lot of things that you have to understand. What you have to understand is that Charlotte's experience, it was exactly the experience of *The Sleeping Beauty.* That is to say that she had been pierced by a needle and she was woken up by a CharmingPrince, in this case, Charming Prince is Florimont. It's a computer.

And we learn in the book that while in the first book, we said "she is ressurected" : well, she was dead. She died since she was pierced in CERN, in one of CERN accelerators. His brain was completely taken over by computers, etc. And then, she is resurrected by the computer. So we didn't understand. In fact, in the second book, we understand that she never died because her death was in fact only a brain death. And when her brain was recovered by the computer, then we might ask "but why did the computer want to resuscitate it ?". In fact, she rises herself. And she has risen herself by adding something, she added a little more thing in her brain. And in the second book, precisely, what is absolutely amazing is that because she had added something more to her brain, when she is near Florimont, it allows her to work a lot better, it is not by chance. That's the name of the resuscitating computer, in the ballet, it is Florimont who plays this role. And so, in fact, what happens is that as she is, in fact, she is a little trans-human, that is to say that she knows how to add something to her brain, which makes her work better when she's close to the computer, so what is very amazing, is that it allows her to decanulate, as we say in mathematics, i.e. to understand another spectrum, which is also sent alternately by the Alma Observatory. But it is Armand who finds the meaning of the first spectrum.

Nicolas Martin : Don't tell us everything because... there is a lot of future adventures, indeed, to be read on different levels, and then, there is something also very important in this novel whose cover you will have to explain to us also, because it remains surprising, and this other element is the place of music and especially this music.

Musical extract from the Messiaen's Quartet for the end of time.

Nicolas Martin : Here it is, an extract from the Messiaen's *Quartet for the end of time*. Why, Alain Connes, in a few words, because we will hear your co-author, Jacques Dixmier, on this subject, why this importance of Olivier Messiaen ?

Alain Connes : So why Olivier Messiaen in particular ? Because in fact, if you like, music and time, as I said, played a permanent role in my evolution, in my mathematician journey, and about zeta, so the Riemann's zeta function, while most other researchers on the problem, seek zeta zeros, i.e. the spectrum, like an energy spectrum or a spectrum of frequencies, I realized that in my approach, it appears as a spectrum of time, a spectrum of lengths. And then, when we look at what corresponds to zeta, but which has already been understood through work... by André Weil, if you like, it's an analogous, but in simpler case. Well, in this case, we get times, just like in the case of zeta. And these times verify an extremely particular property as attack times in a melody and this extremely peculiar property is a property which had been highlighted by Messiaen, under the name of non-demoteable rhythms, and it comes down to palindromy. And this property is an essential property of the corresponding zeta function. So I thought it was an amazing opportunity to connect, precisely, zeros of zeta function, for the case analogous to that of André Weil, with Messiaen.

Nicolas Martin : Don't say too much since we are going to hear precisely your co-author and ex-thesis director on this subject on the link between mathematics and music, since you went to meet Jacques Dixmier, Céline Loozen.

Céline Loozen : Yes, hello Nicolas, hello Alain Connes, hello everyone. I went to see your former thesis director, Jacques Dixmier, to understand a little bit the link between maths and music because in the writing of your novel, you drew ideas from Messiaen who inspired you on the question of rhythm and time. And you discovered a direct relationship between the concepts developed by Messiaen and mathematics, that then gave you the idea of composing music yourself, from prime numbers, to create what are called non-demoteable rhythms.

Jacques Dixmier : The aspects of music in question here are relatively elementary. We were specially inspired by the Messiaen's *Treaty*, *Treaty of music and* *ornithology*, I think so. And since there was a lot of talk about spectrum in the beginning of the story, it's related, in the story, it's related to observations then it's related to earthquake waves. Finally, the waves, eigenvalues often intervene. So it wasn't so surprising that the music intervenes since it is a question of frequencies, all the same : the high of a sound means the frequency of the air's vibration.

CÉLINE LOOSEN : What is the relationship between prime numbers and the arithmetic musical composition which is highlighted through one of the pieces by Olivier Messiaen.

JACQUES DIXMIER : So that's at the end of the book, actually, where we use prime numbers to make rhythms. To find certain rhythms, we then use a mathematical theory which is extremely elaborate. It's called the theory of algebraic curves on finite fields and the action of Frobenius'automorphism *(JD laughs)*. It's related to that. The diagrams that we find towards the end, with the different rhythms, are related to this problem.

CÉLINE LOOSEN : And you find the idea of the spectrum which is omnipresent through your history.

JACQUES DIXMIER : Yes, the idea of spectrum intervenes in the book, to many aspects. This is not surprising, given the work of Alain Connes. For him, the spectrum of an operator is something fundamental. But then, what amused us is that it can intervene in a story and not in a mathematical brief...

CÉLINE LOOSEN : In particular, it is about space, musical range of a space. And what is found in the music of Messiaen that is quoted all throughout the book ?

JACQUES DIXMIER : Well, he talks about rhythm.

CÉLINE LOOSEN : But non-demoteable rhythms. What does it mean ?

JACQUES DIXMIER : That means that we can turn them back in time and that they are identical to themselves. These are rhythms, therefore non-demoteable, it is a rather particular relation to time which are obtained by the method of curves on the finite fields that are in the book. There are non-demoteable rhythms. You can find the designs. Here, there, for example, the axis represents time and small vertical bars give the attack times of the notes. The fact that it is non-demotion, you can see it very well. For example, take the rhythm that is there and if you go back and forth, look. You have two, two notes close together here too. There, it is therefore rhythms, not ups. You see that if you read backwards...

Céline Loosen : Is it like a palindrome ?

Jacques Dixmier : Yes, it is also the term he uses. But it is more clear here, look. It starts here or there. And then, you have two very close notes, two very close notes, etc. So you can read it backwards.

Céline Loosen : There is a form of symmetry.

Jacques Dixmier : Well, a mathematician would rather talk about symmetry. Moreover, when we speak of Frobenius'automorphism of the curves on finite fields, we speak of the symmetry of its spectrum. Yes, but then, so you see the numbers that are there, these are prime numbers. Finally, these are between 43 and 83. And in particular, therefore, they give rise to a rhythm. This is the point of view of the book. We explain how to each prime number we can associate a rhythm. That's it.

Céline Loosen : And what does this have to do with physical space ?

Jacques Dixmier : In a way, there is none. Except we're talking about spectrum in both cases. In physical space, there are... Well, for example, you have heard of gravitational waves, which we have just highlighted and experienced experimentally. Well, like all waves, they have wavelengths, and frequencies. So it's just starting out. So we still know almost nothing, but we will measure vibrations of the entire universe. We will be able to do that in a few years. There will surely be operator spectra that we can analyze mathematically. We will verify experimentally and this will be the analog on the scale of the universe, vibrations of a drum, vibrations, in history, of the Atacama desert, when there is an earthquake, guitar notes, it all comes out of a fairly general scheme.

Back to the initial interview.

Nicolas Martin : This is Jacques Dixmier, co-author with you, Alain Connes, of *Le Spectre d'Atacama.* A word on this analysis, this interpretation in any case of your use of music in the novel ?

Alain Connes : Yes, so let's say that indeed, there are two aspects. First, if you will, when we look at the analogous case of the zeta function, but which had been resolved, by André Weil, as Jacques Dixmier explains, what will it give ? It will give rhythms, it will give attack times, but who have this particular property of palindromy, of symmetry and which Messiaen calls non-demoteable rhythms. But what does he have in mind? He has in mind that if we downgrade them, we will get the same. So it will not give anything new. That's the idea. So, in fact, so there, these are rhythms. But I had to compose, for each prime number, pitches which would then be played by these rhythms. So that's what we're going to hear. And to compose these heights, I harnessed myself to the task, if you want to associate a melody with each prime, but in a purely mathematical way. That's what we're going to listen to.

Nicolas Martin : That's what we're going to listen to. We can recall, of course, to our listeners who really do not have the mathematical fiber, that a prime number is only divisible by itself and by 1. Here.

Short musical moment.

Nicolas Martin : What did we just hear, Alain Connes?

Alain Connes : So what we just heard is very surprising, we just heard a melody that is different for each of the prime numbers between 7 and 67 and how it was constructed, this melody, it was constructed from a purely mathematical method, that is to say that what we did was the following : we took the spectrum of the guitar. When you look at the frets on a guitar, actually, these frets, they are not equally spaced at all. And when we look at what this mathematically signifies, these are the powers of a number and this number... so they are spaced exactly like the powers of a number. It is q power n, let's say. And that number is the twelfth root of 2, but it's practically also the nineteenth root of 3. And that's what is behind the music.

Okay, so, in fact, what we did to define this melody which is associated to each of the prime numbers between 7 and 67, was to look at each prime number to do its development in what's called a continued fraction, but to take them in relation to these powers of q, that is to say in trying to write it as a power of q and at that point we get automatically a melody which is palindromic here. And the way we heard it there, we heard it so that whenever there was a prime number, there was a corresponding melody. This melody was different for each of the prime numbers since we know that their development in continued fractions are different. And now we're going to hear this melody play for each prime number which was played equally. We will hear him play by a rhythm which is a Messiaen rhythm, but which is associated with a zeta function, as explained by Jacques Dixmier, which is associated with a curve. So if you want the fundamental difference, it's going to be that the zeta function is going to give you a way of playing that is going to be different, in terms of rhythm, in terms of note attacks. But otherwise, the melody will be exactly the same. Here, so if you want, what we heard, we saw that there, there was "ta ta tata" *(accelerations, very short notes)* , so the way of playing is completely different. So what is quite extraordinary is that we did the calculations for 6 different curves, and we see that each curve, so, it has, how to say, its personality and its plays in a certain way... but in a way that is consistent. So it's a kind of interpreter. So what we're saying here is that we get to perceive by hearing, we can perceive by hearing something that is normally very difficult to understand, which is precisely, well, these proper values of the Frobenius which do this analog of the Riemann zeta function, but which are perceived this time rhythmically, which are perceived as times, okay, so that's a...

Nicolas Martin : Where does that come from in the book ?

Alain Connes : It plays a crucial role in the book because basically one of the messages in the book is the following : if there is a way to communicate with extraterrestrials, with extraterrestrial intelligence, mathematics is an extraordinary tool for that. And in fact, therefore, the Alma Observatory has received two spectra which are sent alternately. This is what we learn near the end and the fact that we receive these two spectra and that we have all understood thanks to Charlotte for the first spectrum, thanks to Armand for the second, what these two spectra mean, well, there is a revelation, it is necessarily, that they have come to us from intelligent beings. So intelligent beings, in what sense ? Intelligent, in what meaning ? There, it is the pinnacle of intelligence. This is something that was found by Bernhard Riemann, who was a mathematician of the 19^{th} century. And it's probably the pinnacle of intelligence is to have understood that what governs primes is music. What governs prime numbers are what we call precisely these zeros of the zeta function. They are the ones who govern the hazard that is in the prime numbers and the Riemann conjecture we were talking about earlier is exactly extremely significant for the following reason. What it says, basically, is the corresponding spectrum which is the Atacama Spectrum, which is the cover of the book. In this spectrum, what Riemann's conjecture says is that there will always be extremely precise lines, there will not be what are called resonances, that is to say that there will be no place where, at instead of having a precise attack time, there is a diffuse attack time. That's what the guess says. What it says about prime numbers is that fact, although primes look completely random, they are governed by a random, but a hazard which is perfectly controlled, if you will, and which is perfectly... Yes ?

Nicolas Martin : I would like, because we have just a short time left, a small minute, just a word, all the same, Alain Connes, to conclude, on this temptation that I want to call holistic temptation : you are a mathematician, novelist, a musician, the will to understand, to integrate this mathematical language as a universal language, what do you think ?

Alain Connes : If you want, what I think is the following : one of the great discoveries come from the wish of human spirit to understand, good. The human mind, in the 19^{th} century, I spoke of Bernard Riemann, but I could have talked about Galois : Galois was able, without having a computer, without having means of calculation, etc. to understand how to completely capture the rational relationships between the roots of an equation, by associating it with another equation and solving this one tautologically, practically. But he said at the time : “Let’s jump with attached feeton the calculations.” I had to make, at the Academy, a presentation on Galois for the two hundredth anniversary of his birth, I showed, since we can now do the calculations with the computer, I showed for a very simple fifth degree equation, what the calculations would give, in this case : we can see that for Galois, that was absolutely impossible. Nevertheless, he perfectly understood, conceived all of which was behind and the message of the book. A message that is very, very important is that nowadays, we tend to let go to the temptation to do without understanding as opposed to understand without doing of Galois, and as opposed to the creation of concepts which is the prerogative of mathematics.

Nicolas Martin : It will be the last word, Alain Connes, since it is 4 :52 pm, I remember the title of your book : *Le Spectre d’Atacama*, co-written with Danye Chéreau and Jacques Dixmier, and published by Odile Jacob editions.

TABLE DES MATIÈRES

Chapter 1

The principle of uncertainty

This is to retrace a scientific journey, in this case the mine, but it's sort of secondary. What matters before everything is meetings, and above all scientific fields is concerned.

After entering the École Normale Supérieure, I decided not to pass the aggregation because I didn't want to start over cram. I had started, at the School, to do research in math but I really did find a topic that interested me only after getting out, with Quantum mechanics.

I always had with me the little book published by Heisenberg in 1930 (The Physical Principles of Quantum Theory) and I had been extremely inspired by the way he explained how he had discovered the mechanics of matrices which underlies the Quantum mechanics. I start there, because this discovery of Heisenberg played, throughout my journey, an absolutely essential role.

Before Heisenberg's discovery, there was a model for the atom, which was called "Bohr's atom" and which postulated electrons in stable circular orbit around the nucleus. And there were completely ad hoc rules, which had no conceptual justification but allowed to find, for example, how the spectrum of hydrogen was made. Heisenberg was taking care to calculate spectra of atoms, i.e. to determine mathematically the set of wavelengths present in the light emitted by the atom in question. By a contest of circumstances - chance plays an important role in science - he had been sent by his university to the island of Helgoland, in the North sea, to treat a serious hay fever: at the time, the only remedy was to take refuge in a place totally sheltered from pollens. It is a very small island, where he was staying with an old lady and had plenty of time to think and make calculations. He had developed his new mechanics but his theory seemed to him contradictory: energy conservation, which plays an essential role in classical formalism, posed a problem and had to remain true in its new formalism. So he made calculations with the system that he had created and he finally realized that the energy was well preserved! He describes this moment very vividly in his autobiography. It was then 3 or 4 in the morning, and he said that at that instant, he had a landscape in front of him that almost frightened him by its immensity. Instead of going to sleep, he climbed one of the peaks that borders the island and waited there for sunrise.

This discovery of Heisenberg was my starting point. When I graduated from the École normale, I was a student of Gustave Choquet and he had the idea to make me learn physics by sending me to Les Houches summer school, in 1970. There were Oscar lectures of Lanford, who explained what von Neumann had done after Heisenberg. What Heisenberg found was that when you do physics calculations for microscopic systems, like an atom interacting with light, a phenomenon quite extraordinary happens: you can no longer have the freedom you have usually to swap the order of terms in an equation. When we write $E = mc^2$, we might as well write $E = c^2m$, the result would be the same: this is the essential algebra rule that we say "of commutativity", which means that if we swap the two terms of a product, the result is unchanged. But Heisenberg found that, when working with a microscopic system and multiplying observable quantities, for example the position of a particle by its speed, or more precisely by its moment (its speed multiplied by its mass), we can no longer swap freely the terms of the product. The corollary is very well known: this is the Heisenberg's uncertainty principle, who says there is a limit to the precision with which we can simultaneously know two properties of the same particle associated with observables which do not commute. For example, if the more we know with great precision is its position, the less we know precisely is its speed, and conversely.

This is the physical part of what is going on, we will come back to it.

Consequence: there is a kind of permanent novelty, of freedom, in mechanics which means that when we repeat certain experiments at the microscopic level, we don't get the same result. For example, if we send an electron through a slit whose size is of the order of the electron wavelength, it arrives at a target placed beyond the slit, at a specific location.

But we cannot reproduce the experiment in such a way that the electron arrives again at this same precise place. All that we know, it's the probability that it will arrive at such and such a place. There is no way, it's the

Heisenberg's uncertainty principle that says, repeat the experiment so that the electron arrives exactly in the same place. So there is a kind of fantasy of the quantum that manifests itself at all times, every time we do such an experiment at the microscopic level.

Mathematically, it's a different story, because the discovery of Heisenberg taught physicists that they needed to be careful when handling these observable quantities in the framework of what has become quantum mechanics, that is to say the mechanics of microscopic systems. This may seem confusing but this is actually a phenomenon we are accustomized to: it manifests itself every day when we write. If we swap the letters used between them, like when we make anagrams, we change the meaning of the sentences: thus gravitational waves "includes the same letters but does not have the same meaning that" the distant stormy wind.

Both have the same value when working in Commutative algebra, where you can swap letters. For that, the sentences do not lose their meaning, we understood that we must do pay attention to the order of the letters in a sentence. And Heisenberg has shown that when we work at the microscopic level, we no longer have the right to simplify as we simplify in physics calculations ordinary. It's a major discovery because it has a considerable impact, not only in physics, but also in mathematics. As far as I'm concerned, I spent most of my existence of scientist to exploit it mathematically.

Max Born and Pascual Jordan understood that the calculations that Heisenberg made were what are called, in mathematics, calculations of matrices. No need to know what a matrix is. Which is essential is that matrices have this property, compared to ordinary numbers, not to switch between them. The product of two matrices in the order "ab" will generally have a different result of the product "ba". Born and Jordan understood that Heisenberg had rediscovered the matrices, but in a natural form, from observations.

Chapter 2

Spectra

In common parlance, spectra are ghosts or in any case report strange things. In physics, the word spectrum denotes a reality, just like in mathematics. One of the miracles which happened in the XX$^{\text{th}}$ century is that the spectra of physics have could be calculated as spectra in the mathematical sense in the most important physical examples.

The physical meaning of the spectra is understood in the way next: when, following Newton, we take the light that comes from the Sun and we pass it through a prism, it gives, once decomposed by its passage through the prism, a rainbow, that is, it breaks down into simpler elements which each correspond to one of the colors of the rainbow. But in refining this experience, we realized that we were watching at a place of the rainbow a black line, which is called the black line sodium. We considered this line as an isolated element, until what the German optician Fraunhofer had in the XIX$^{\text{th}}$ century the idea extraordinary to watch the rainbow obtained after the passage of the sunlight through a prism with a microscope. They are then saw that there was not a single black stripe, but did listed about five hundred. They are what physicists call an "absorption spectrum", which looks a bit like a barcode. Years later, Robert Bunsen and Gustav Kirchhoff, among others, have noticed that by heating certain bodies, like sodium, you could get the same configuration, not with black stripes on a rainbow background, but with stripes bright on a black background. We then understood that these lines were a kind of signature of the chemical body in question, and succeeded in reproduce, with different chemical bodies, a number of lines that appeared in the spectrum of the Sun.

So these barcodes, these "absorption spectra", appear like black lines when we look at the sunlight at through a prism, with the extraordinary precision that a microscope. But we observe lines that we do not manage to relate to a known item. This is where physicists and chemists stepped in to say they might be the ones of an unknown chemical body. And, as it comes from the Sun, we have it called "helium".

Then occurs, at the beginning of the XX$^{\text{th}}$ century, the eruption of Vesuvius. With the same spectrometric processes, we analyzed the light from its lavas and helium was found there. That's wonderful. That explains what what are spectra in the sense of physics, it gives their meaning and their importance: each of them is a signature. Each body different chemical has different signature, barcode different. When the chemical body is pure, its signature is not a superimposition of different signatures, it is also pure.

Obviously, we immediately tried to understand what was the nature of this signature. We looked for the available body on simpler, hydrogen, because it was more difficult for helium. We have took a while to figure out that if we looked at a general spectrum, not in wavelengths, but in frequencies, this one had a remarkable structure. It was actually a spectrum formed by the differences A - B between any elements A, B of a simpler set of frequencies. That is to say, you had to index frequencies that appear in a general spectrum by two indices like (a, b) or (c, d). Of course, if we take the difference (A - B) between A and B and that we add it to the difference (B - C), between B and C, this gives the difference between A and C.

This gave a general rule of composition for frequencies that appear in a spectrum called the "rule of Ritz-Rydberg composition". Heisenberg's genius is having builds its mechanics from this rule. He understood the thing following: if classical mechanics had been valid for a body microscopic, we wouldn't have had this rule of composition but the rule of a group, that is to say that two frequencies u and v of spectrum add up to give a new frequency u + v of spectrum. We would have obtained, by a mathematical process that we called the Fourier transform, the algebra of observables.

Heisenberg had this wonderful idea of saying that these are the physics, the Ritz-Rydberg principle and chemistry which must prevail.

As this is what we find experimentally, we will base the observable algebra on this Ritz-Rydberg rule.

It was Born and Jordan who explained to Heisenberg that the mathematical structure he had found was well known to mathematicians, who call it matrices. A matrix is not nothing but an array, and instead of being indexed as a sequence by a single letter, it is indexed by two letters. When we multiply two matrices, we use the Ritz-Rydberg rule.

Shortly after this discovery of Heisenberg, Schrödinger made a another not extremely important. Thanks to him, we made the link between the spectra that appeared in physics and those that appeared in mathematics. Because, and it is remarkable, the word "spectrum" was already known in mathematics, by the Hilbert school for example, and known because of what are called operators and the spectrum operators. There is no question of explaining it precisely here, but it is something that has a perfectly defined mathematical meaning.

Schrödinger was the first to calculate the spectrum associated with hydrogen by mathematical calculation, while physicists apprehended him by measures. What is extraordinary is that Schrödinger's theory and Heisenberg's theory are the same, which gave rise to a formalism carried out by one of the greatest mathematicians of the time, who was not only mathematician: John von Neumann.

Von Neumann understood that there is a mathematical formalism existing, developed by the school of Hilbert, and which uses as a framework common mathematics what is called Hilbert space. This space, consider it as a kind of abstract, unique joker, which will play an essential role in everything that follows. There is only one Hilbert's only space and it's going to be the seat of mechanics quantum. This is the most suitable framework that we know up to present.

We know the Euclidean space, the plane, we also know the space of dimension 3. To go to Hilbert space, you have to do some not difficult, and even if we do not understand all the details, it is necessary know that it exists. The first step is to move from a space real to a complex space, which is not too difficult yet understand. We are very used to real numbers, but they are not not very flexible and easy to manipulate to do physics. We had need to add to the real numbers another number, baptized "Pure imaginary number", which checks that its square is equal to 1. he is very valuable for physics and in particular for electromagnetism.

The next step is much more difficult to accept: the Hilbert space has an infinity of dimensions. It is thanks to this that an incredible number of wonders will appear.

The first of these wonders is that there is a coincidence between Heisenberg's point of view (which is extremely practical, extremely concrete, because the observables he discovered become operators, that is, something that acts in this Hilbert space) and that of Schrödinger (who discovered how we could calculate the spectrum of a chemical element, which also manifested by an operator in Hilbert space). That seems very mysterious, but if we understood something about the nature, on reality, on quantum mechanics, it's good that the corresponding mathematical scene is that of Hilbert space and that the actors are the operators in this space.

Chapter 3

Operator algebras

This very beginning of history took place from 1925 to the 1930s.

Von Neumann then gave his formalism to quantum mechanics. But he didn't stop there. He asked himself the question, absolutely fundamental, concerning subsystems of a quantum system. He understood that ordinary quantum mechanics is formalized through Hilbert space. With a collaborator, Murray, he tried to understand what it meant to have a subsystem, that is, not knowing all the information about a quantum system. This is what we called algebras of operators, and with them that my mathematical existence has begun.

For recall, it was after having gone to the summer school of Les Houches, in 1970, that I was spotted by an American organization as a "promising young mathematician". Therefore, they invited me the following year to Seattle for a conference. I was so young married man and we took this opportunity to visit the United States.

We didn't really like the plane, so we decided to join Seattle by train, crossing Canada, four or five days across large, somewhat monotonous plains. I looked for first stop at Princeton to buy a math book to read during the trip and ended up spotting one from a Japanese author who seemed interesting to me. I only took this one and reading it absolutely fascinated me. Arrived in Seattle, I went to the Battelle Institute to learn about the program. What did I read ? The author of the book was there and gave a series of lectures!

Because of this, then, I applied what Brutus said in Julius Caesar of Shakespeare:

There is a tide in the affairs of men,
Which, taken at the flood, leads on to fortune;
Omitted, all the voyage of their life
Is bound in shallows and in miseries.

I decided that I would not go to any other conference than this Japanese's one and that I will work on the subject he exposed. Returned in France, I went in September to the only seminar that existed on operator algebras: the one of Jacques Dixmier. This one has explained that, that year, his seminar would be on another subject, which a priori had nothing to do with that of the Japanese. He asked who among the audience wanted to make a presentation. I carried myself voluntary, and he gave me to read an article on tensor products infinite. When I got home, by train, I realized that there was an extraordinary link between the article Dixmier had given to me and the works of the Japanese, and it was this confluence that was the point of departure of my thesis.

I wrote a small letter to Dixmier, half a page. He had me replied that what I had written was incomprehensible and that it was necessary that I give details. I gave them to him, then went to see him, and he said to me, "Go for it!" "That's how I happened engaged. Ultimately, the starting point of my career, it is this link with the work of the Japanese. Which, in fact, was two. The one who found the theory in question, which is called the theory of modular algebras, was called Tomita. But deaf since the age of two years old, he had trouble communicating, and it's Takesaki, another Japanese mathematician, who shaped and communicated his theory. It was the latter who spoke in Seattle.

I immediately linked Tomita's theory to work on type III factors made by Araki and Woods. What I found, a few months after defining general invariants using the theory of Tomita is that there is a phenomenon quite miraculous of independence which makes it possible to calculate these invariants.

The evolution over time does not depend on the choice of a state of algebra, provided that we work modulo interior automorphisms: there is automatically an evolution over time which is not completely canonical, but canonical modulo these interior automorphisms! A von Neumann algebra is precisely an algebra like the one Heisenberg had discovered, that is to say non-commutative, in the sense that we no longer have the right to swap between them the terms of a product. To summarize, when you don't know all the information about the quantum system, this partial knowledge is at the origin of an evolution which miraculously emerges from the very fact that our knowledge is imperfect. It allowed me not only to write my thesis, but to completely decanulate all these algebras which seemed extremely mysterious, and to understand their structure. Something was still missing: how did this miraculous appearance of time could be related to physics.

It was totally mysterious in my work, which was purely mathematical. I was missing this element and that would come much later.

Chapter 4

Mille-feuille cake

So I found these results, then, after my thesis, I found others results, very important, on the same algebras. Then I was invited to the Institute of Advanced Scientific Studies (IHES) in Bures-sur- Yvette. And there, I was shocked: I had worked on a subject when even quite specialized and I did not know at all the extent of the rest of math. When I got to IHES, people were talking about things that I didn't understand. I was immersed in a totally different environment from the specialized environment I was used to. My situation was a little embarrassing because I wanted to absolutely participate in this development of mathematics, which seemed so important - and it was.

Grothendieck has already left, but someone at IHES played a crucial role for me: Dennis Sullivan. He had this particularity quite extraordinary to ask newcomers about their research in math or physics with extremely naive questions. One had the impression that he hardly understood. But after a while, his interlocutor realized that it was him-self who didn't understand what he was talking about. His power was absolutely incredible, and it was he who taught me the differential geometry. I understood at that time that I had a considerable asset: there was a way to fabricate the algebras that I had classified, those of von Neumann, from objects of well-known differential geometry called foliations.

What I had done so far could be illustrated with objects that people doing differential geometry could perfectly understand.

What is a foliation pastry? A mountain can have a stratified appearance, that is, strata of smaller dimensions make it up. Another typical example of a mille-feuille foliation pastry, which results from a stack of sheets. The structure of a mille-feuille cake is very simple. It is made up of two parts: leaves themselves, and the set of all those leaves. A set of sheets, in a notebook for example, is very simple since it is simply indexed by the page number. But in math, a foliation can have a structure much more complicated, like a spool of thread in which the thread, instead to be rolled up so that after a finite number of turns it comes back on itself, is rolled up in an irrational way. That is say it never comes back on itself it will keep on wrapping indefinitely. What is extraordinary is that, whatever the flipping, the resulting algebra is always non-commutative.

There are other examples. In a conference I attended in the 1980s, Roger Penrose explained that he had discovered very explicit quasi-periodic tiling. Because if it is relatively simple to pave a space with hexagonal tiles, for example, since you can give a tiler the recipe to do it, the quasi-periodic tiling is more complicated, and in particular has the next peculiarity: they can have a pentagonal symmetry that no classic paving can have. What explained Penrose, it was that these tiling has a quantum side: when we takes two, we can superimpose parts as large as we wants, although they are not identical. This kind of almost coincidence, but never complete, has a quantum aspect that it had well felt intuitively. I realized at that time that the Penrose paving space had the same characteristics typical of the leaf space of a foliation, and that, thanks to the associated von Neumann algebra, it really corresponded to the Quantum mechanics.

Chapter 5

Non-commutative geometry

So that's kind of the starting point for non-commutative geometry. It is an almost direct consequence of Heisenberg's discovery.

Descartes explained that one can makes the geometry of plane entirely algebraic. For example, if we want to demonstrate that the three medians of a triangle intersect, we can use the axioms of geometry. But there is another way to demonstrate this theorem: algebraic calculations. It is then a question of calculating the barycenter of three points, using the coordinates of each of them in the plane, and the theorem is immediately demonstrated.

What is the advantage of transforming a geometric problem into an algebraic problem? For example, demonstrate geometrically in dimension 5 the analog of the fact that the medians intersect will be difficult, while the calculation is immediate. We calculate the barycenter, and the demonstration is made.

It was Descartes's idea, these coordinates, and that was the basis of algebraic geometry for years. These coordinates called "Cartesian" commute. But the coordinates in what we calls the phase space, which corresponds to the microscopic system, they no longer switch: it is the Heisenberg's discovery. This is what led me to develop the geometry for spaces whose coordinates no longer commute and which are therefore calls non-commutative geometry.

One would think, and it would be normal, that to generalize the geometry to a case where the coordinates no longer commute would feasible. It's actually quite tricky and finds its justification essentially by quantum mechanics. But if there had only been that, it would not have been enough for me. What motivated me is what I found in my thesis, i.e. the fact that such spaces have something extraordinary: they generate their own time. They are not static like ordinary spaces, but dynamic: they evolve over time.

After this initial discovery, I told myself that this extraordinary property of generating its own time made this geometry was necessarily extremely different from classic geometry, and all the more interesting. After the foliation finding, I had enough examples and when we try to develop a new theory, besides a good reason, you must have a large amount of examples. Indeed, if we have too much little, we risk developing a completely formal theory which will make no sense. The meaning is given by the variety of examples.

From the start, I understood that the most famous foliation gave the most exotic factors. I also understood that the associated von Neumann algebra only perceived one side relatively crude of the non-commutative space in question. The fact that these leaf spaces came from geometry gave them many other structures from differential geometry and that it was necessary to understand the non-commutative case.

It was the starting point for a whole development during the 1980s, in which one of the most important contributions was the cyclic cohomology, which I found, and which now plays an essential role in many other areas.

This made it possible to understand and develop the analog of the differential geometry in the non-commutative framework, to find the analog of de Rham's complex, cohomology etc. There have been all kinds of surprises, for example the Godbillon-Vey invariant appeared miraculously in this completely different setting. However, I still had this frustration of not knowing how to link this emergence of time with physics.

Chapter 6

Emergence of time and thermodynamics

In the meantime, I have always continued, as a hobby, as a somewhat parallel task, getting interested in physics, about which I read a lot. But not just any physics: quantum physics of course but beyond that, what's called the field theory. Around 1994, I was invited for several months at the Newton Institute in England at a session whose subject was gravitation. I went there because I wanted to complete my knowledge. On the spot, I got a little bored because there had almost no collective activity. One day I saw an ad for a conference whose title seemed to me extremely pretentious: We know what quantum space-time is. As we do not know, in fact, still not what it is, I got stuck with the speaker a bit. The conference was by Carlo Rovelli. We then discussed at length and I glimpse that he had an extraordinarily philosophical point of view.

I thought it was great, because in our community, people are overwhelmed by technique, by their specialization, and that there is ultimately very few philosophical discussions, unlike the time of Einstein and Heisenberg. I dared to explain to him what I found in my thesis: this extraordinary emergence of the time. He then left me without saying anything to return a few minutes later with two articles he had written the year previous. For purely philosophical reasons based on his thinking about what would happen if we tried to quantify gravitation, he placed himself at a level called "semi-classical", that is to say say not yet quantum. His idea was that when we write the Wheeler-De Witt equations, we find that when we try to quantify gravitation, time disappears. And it disappears because what's called the Hamiltonian, which normally generates evolution over time, is one of the constraints. We do not know more what we talk about when we talk about time.

Carlo had tackled this problem and, by mere philosophical reflection, he had an idea: the only way that time can emerge, it's from thermodynamics, because we let's bathe in a kind of thermodynamic bath, a bath of the 3 degree Kelvin radiation from the Big Bang. The idea is therefore not completely abstract, it relates to something very concrete. And it is this heat bath which would have caused the passage of the time.

His idea was very attractive because the passage of time such as we know wears us out. When time goes by, what we collide, it is wear and tear. And this wear comes from the temperature, because we're in a thermodynamic bath.

To implement his idea, he wrote an equation, and I recognized it right away when he showed it to me, the semi-classical limit of the equation used to have this magic flow intervening in the quantum. It was then that the junction was made. I had tried to understand how this emerging time could be related to physics. I tried to do it using quantum fields theory, but I didn't get there because the real where it happens, it's not in field theory, but in gravitation, when we try to quantify it.

We wrote an article in common, but it has neither Carlo Rovelli's philosophical qualities, nor my mathematical qualities. It was more to take a date to show that we had recognized these equations, but we haven't gone far enough in interpretation of the result.

This interpretation, I will try to explain it, because it has played an essential role in the development of non-commutative geometry. The paradigm I had arrived at in the years 1980 for non-commutative geometry can be explained very simply from quantum precisely and what is producting there.

Chapter 7

Variability

The idea, almost easier to explain, and more fundamental than the coincidence found with Carlo Rovelli, is as follows: the quantum has this extraordinary fantasy, this extraordinary imaginative power, which means that every time we repeat an experience microscopic, we get a response that we can neither predict nor reproduce. We are touching on a central problem which I will call “variability problem”.

Normally if you ask someone about what the fundamental variability, everyone, not just the physicists, responds that the only variability is the passage of time.

Any variability can be reduced to the fact that time is passing. If we look carefully, we realize that almost all of physics is written in terms of what is called a differential equation, i.e. that we write that the derivative of a physical quantity with respect to the time is given by a certain relation with other quantities.

All of physics is written on this paradigm, and all of understanding that we have of variability is thought in these terms.

Let’s do a little math excursion to try to understand how mathematicians have sought to formulate this that it is only a variable and how this formulation was dethroned by the quantum. When you ask a mathematician what is a real variable, he will say this: it is a set and an application of this set in real numbers. It may seem a little obscure, but that’s the standard answer. We can then point out to the mathematician that there are variables that take only discrete values, for example the age of a person, that will be expressed only as an integer, and others who take continuous variables. The two cases are quite different.

There can be no coexistence in mathematics between a discrete variable and a continuous variable. Indeed, a discrete variable takes only a countable number of values (we can enumerate them one by one) while a continuous variable takes a not countable number of values so the set where it takes its source cannot be the same as that associated with a discrete variable. It is a fact. The first wonder is that formalism of quantum mechanics that von Neumann has developed solves this paradox of the non-coexistence of the discrete and the continuous. It is resolved, as I said above, because Schrödinger found that the spectra were spectra of operators in the space of Hilbert. In this same Hilbert space, on the same stage in somehow, some operators will have a discrete spectrum, like the integers, and others a continuous spectrum, that is to say that they can take all the real values between zero and infinity. The only nuance is that the two operators cannot switch.

Thus, the formalism of the operators in Hilbert space solves the paradox.

This formalism provides the framework for non-commutative geometry.

And it is thanks to it that we will be able to try to understand the emergence of time.

Chapter 8

Length unit

How does this formalism generalize the geometry in such a way that it absorbs everything that quantum has brought us? This is where the link with physics absolutely appears fundamental.

At the time of the French Revolution, there were in France as many definitions of unit of length as cities, or almost. There must have been about a thousand. At the entrance of a village, we found something about a meter which defined the unit of length in use at this location. The fabric of a merchant be a multiple of this length to be able to be sold there. It was very annoying.

We then sought to unify the unit of length, for France in particular. Scientists were first asked to give one valid definition which is not dependent on the place. They reflected, and took the biggest object at their disposal: the Earth. They have considered the circumference and then defined the unit of length as being a portion of this circumference: the forty millionth part. Since it is impossible to directly measure the entire circumference of the Earth they used an angle by pointing some stars. They knew very precisely the angle having summits the center of the Earth, Dunkirk and Barcelona, it was their so easy to calculate the total length from the measurement of the distance between Barcelona and Dunkirk, which had to be measured directly. They sent a team of scientists, Delambre and Méchain, to make these measurements. The expedition was adventurous, the France and Spain being in conflict. While they were at the top of a hill, equipped with a telescope to make measurements by triangulation, they had a lot of trouble explaining to the soldiers enemies that they weren't spying on. But they had success. The result was a platinum bar supposed to be exactly the length of the forty millionth part of the Earth circumference. It was deposited near Paris, in Sèvres.

This unit of length, as such, was not very practical.

Difficult to measure a bed by comparing it to this bar. A lot of replicas were therefore made. And then it happened, in the 1920s, a completely fabulous phenomenon, a phenomenon that is the exact parallel of the transition from ordinary geometry to non-commutative geometry.

A physicist made very precise measurements by comparing the platinum bar with the wavelength of a spectral line of the krypton, and he realized that the unit of length... was changing length! It is very annoying to have a unit that is not stable! It was therefore decided, after a fairly long time, to use what had allowed to see the change as a new unit: the line krypton orange. But it was not practical. It would have been better to take a unit that is in the order of microwaves, which have been incidentally discovered by accident (people who worked on the radar noticed that their chocolate bar had melted). There is fortunately a chemical body, cesium, which has what is called a hyperfine transition: in the outer layer of a cesium atom, there are two states so close that their energies are very close too. This means that the correspondent transition, the one I was talking about Heisenberg, between the two energy levels indexed by two indices, is such that its frequency is very small and, therefore, that the associated wavelength is big. For this transition, we get a wavelength of about 3 centimeters, that is to say it takes about 33 times more to get 1 meter. We will therefore have a measuring instrument which will be able to make precise measurements on a fabulous routine basis.

There is now a commercially available device, based on this wavelength, which measures a length with twelve decimal places given.

On the other hand, if we want to unify the metric system throughout the Galaxy, this will be a problem because cesium is not necessarily present on other star systems. A chemical body with a sufficiently high atomic number is indeed produced only in supernovae, and even super-supernovae. I think that one day or another we will be able to base the unit of length, not on cesium, but on hydrogen or helium. Why ? Because they are present practically everywhere in the Universe.

Chapter 9

Infinitesimals

What is happening at the mathematical level? Exactly the same thing. The geometry was based by Riemann on a measurement of lengths which exactly corresponds to the way of measuring of Delambre and Méchain. It consists, when we take two points in a geometric space, to consider the shortest path between these two points. In doing so, we do not need to measure this length, that of the element of infinitesimal length, which we call ds, whose Riemann gives the formula only for the square, what we call ds^2 .

What we call geometry in the Riemannian form is so a geometry based on the element of infinitesimal length, which is expressed in the form of what is called g, μ, ν. Mathematical content doesn't matter, what should be remembered is that it is something extremely concrete and which corresponds exactly to the way of measuring of Delambre and Méchain.

In physics, we had to replace the paradigm of unit of length given by the standard meter by the spectral paradigm, which precisely corresponds to a spectrum. The way it happened in non-commutative geometry is exactly parallel: the infinitesimals have their place among operators in the space of Hilbert. Some operators are infinitesimal. They have a discrete spectrum, but which decreases towards zero, and corresponds exactly to the definition Newton gave of infinitesimals.

What is new is that infinitesimals can no longer switch with continuous variables. The crucial point is that there are an infinitesimal which is characteristic of a geometry. This infinitesimal was introduced by physicists when they founded field theory and quantum theory, this is what they call the "propagator" for fermions. Physicists therefore have developed in their theory an entity which is an operator in the space of Hilbert and who has all the properties to embody the element of infinitesimal length. We can clearly see the gain both in physics and in mathematics. In physics, this allows for a system of length measurement, based on the spectrum of hydrogen, which is really universal. We can exchange with a visitor from another stellar system without having to bring him to Sèvres for showing him the stallion.

In mathematics, it's exactly the same thing. When we takes the propagator of fermions as the length element of a geometry, as it doesn't switch with coordinates, since coordinates are of continuous value, it has the property of not being able to be located and to be present everywhere. It is no longer located somewhere. If it had switched with the coordinates, the fact that it is infinitesimal, it would have been somewhere. But the fact that it doesn't switch allows him to be everywhere.

This gives a new geometry of spectral nature, that is to say it manifests as a spectrum. It's very new since usually when we talk about a geometric space, we think of it as a whole with a distance, a structure, which are given to it locally. It's not like that a space of non-commutative geometry will manifest. He will manifest by its spectrum.

Chapter 10

The music of forms

A new phenomenon appears: this manifestation by a spectrum can be understood musically. If I take any shape, a drum, a sphere or any other, we know since the XIX$^{\text{th}}$ century thanks to Helmholtz that a range is associated to it. And since Mark Kac and his famous talk *"Can we hear the shape of a drum?"*, it is formalized in mathematics. What does this mean ? You might think that when you hit a drum, the sound product will always be the same. It is a serious mistake. In the XVIII$^{\text{th}}$ century, the vibrations of the drum were observed by putting sand underneath.

When the drum vibrates, the sand is concentrated where the vibration is the smallest. We thus observe that the vibration of the drum is of this or that shape depending on where it was struck.

Two parameters actually qualify this vibration exactly: how many oscillations if we start from the center towards the circumference? and how much when we go around the drum? If we know these two parameters, for example three oscillations from the center towards the circumference and four when we go around, the vibration is then perfectly defined. It will produce a particular frequency. The simplest vibration produces the lowest frequency. And, at each time the value of the parameters is increased, the vibration becomes more acute. We can calculate the frequency of these vibrations mathematically. We get a range. We finally realize that each form has a range. They are not always different. Some forms, different, however have the same range. So we lose a bit of information. But, basically, we essentially knows an object from its range.

A space is manifested by a range, and this is the starting point of non-commutative geometry. It includes space from its range. There is an additional invariant, which must be known to really understand the whole space: what are the agreements possible? A point in space gives more information than the scale, it gives an agreement on the notes of this one. If we know all agreements, we recognize the space.

When I found this result, I gave a lecture at the Collège de France on the link that there was between forms and music.

I called it “music of forms”. In preparing this conference, I asked myself a question: there are many different forms (sphere, disc, square, rectangle, etc.), would there be one who allows us to make music as we know it? I tried different forms and I realized that it was a disaster. For example, if you want to play Au clair de la Lune, none of the forms I have mentioned gives convincing results.

Why ? The ear is sensitive to multiplication by two. If you double the frequency of a note, the ear will hear something nice, a resonance between the two frequencies. And that corresponds to something very concrete: it's the transition to the octave.

The ear is also very sensitive to tripling.

As it is sensitive to doubling, we can also multiply by three halves, which is like playing a C and a G.

Now let's think a bit about it. To fall back on my feet, I would like to multiply by two enough times to be the same thing as to multiply by three a certain number of times. It is impossible, because when we take a power of two we always have an even number, and when we take a power of three, we always have an odd number. Which is true and amazing, and this is the basis of music as we know it is that multiply nineteen times twice, it's almost the same as multiplying twelve times by three. There is a better way to say it: the twelfth root of two is almost equal to the nineteenth root of three.

How does it manifest? This is what I call the guitar spectrum. On the neck of a guitar there are the frets, these lines perpendicular to the neck that produce a sound specific. They are not regularly spaced. It's not at all an arithmetic progression. These are the powers of this number: $2^{1/12}$. By taking this number, we get exactly the frets positions on the neck of a guitar.

The spectrum of the space we are looking for is therefore given to us by the guitar spectrum. Could it be a sphere of dimension 2? No, because mathematical theorems say how the range of a shape develops when we take more and more notes, and the dimension of the space we are looking at is intimately linked to the way notes develop. On the spectrum of the guitar, a little mathematical calculation indicates that the dimension of the corresponding space must be zero. More specifically, it must be smaller than any positive number. Therefore we cannot find it among the spaces we know. And it is a non-commutative space; this is called the non-commutative sphere, which was found by physicists. So we see where non-commutative geometry gives freedom to find spaces far more extraordinary than the one we have with ordinary spaces.

Danye Chéreau, my wife, Jacques Dixmier and I have written a book, *Le Spectre d'Atacama*, which evokes the spectrum received by the large Atacama Desert Observatory (ALMA: Atacama Large Millimeter Array) and who seeks to understand what this spectrum represents. And we still haven't understood. It is connected to one of the most difficult conjectures in mathematics. It's fascinating to see that a space as we know it produces a spectrum. But there are examples where space is perceived by its spectrum, and we don't know what this space is; it remains mysterious.

All this geometric side has been considerably developed and allowed us, with my collaborators Ali Chamseddine and Walter van Suijlekom, to understand why, in physical reality, there is not only gravitation, but also the standard model; why are there other forces of physics that appear naturally. In the context of non-commutative geometry, we may, by purely geometric reasoning, fall by miracle that it's easier to describe space-time, the geometric space in which we are, by non-commutative variables. This simpler way of describing it requires other forces beyond gravitation than pure gravity in this new space. These other forces correspond exactly to what we are measuring, i.e. the forces of the standard model.

It is a very elaborate theory and extremely satisfactory in aesthetic and conceptual level. A part is missing on which we are working: it exists at the level of what is called the first quantified, and does not yet reach the level of what is called the quantum gravitation, that is to say in which we would really quantify fields.

But back to the essence of quantum. It's something absolutely fundamental, which is not yet fully understood.

Chapter 11

The ticking of the Divine clock

We human beings reduce any variability to the passage of time. And we are still looking to write a story. It is one of the ways we understand things, and a story is of course written over time.

But when we try to understand quantum, we are faced with paradoxes. The most typical is what we calls in French entanglement. Einstein never accepted the quantum, although it was he who practically initiated it with the photoelectric effect. In one of his poems, Alfred Brendel tells Einstein arriving in heaven, realizing that God is playing dice and then asking for the address of hell. Because Einstein never accepted the random side of quantum and he gradually built a number of counterexamples, paradoxes. The first one gave rise to a wonderful story. Einstein imagines a Swiss cuckoo hanging from a spring. The cuckoo must emit a photon to a precise moment. Thus, it will indicate the exact time when it issued it. In weighing the cuckoo before and after the emission of the photon, we would know exactly its mass, and therefore that of the emitted photon, or rather its energy, since energy is equal to mass. So, as we will know the energy of the photon emitted with limitless precision, we will contradict Heisenberg's uncertainty principle, which says that we cannot know time and energy simultaneously. That gave place to an extraordinary episode, with a triumphant Einstein who explains its paradox. And Bohr who follows him with an absolutely confused mine because Einstein's argument seemed unstoppable to him. But the story did not end there.

What was Einstein's idea? Since we were going to measure cuckoo's mass (before and after), the gravitational constant would be involved in calculating mass from weight. So it was impossible for the Planck constant to stand out by itself as the principle of uncertainty requires it, it seemed impossible to eliminate the constant of gravitation! Bohr did not sleep overnight, and returned in the morning with a wonderful response. He showed that one obtained exactly the Heisenberg's uncertainty principle in using Einstein's theory of gravitation! According to this theory, and it has been measured since, time does not pass in the same way when we change altitude and this change involves again the gravitational constant. If we do the math, we realize that the two gravitational constants are eliminated. Only Planck's constant remains, and we get the principle of uncertainty as stated by Heisenberg.

This episode was a defeat for Einstein, but it didn't confessed defeated. Five or six years later, he wrote an article with Podolsky and Rosen, an article that was almost never cited at first, now its quotes are growing exponentially. It was the first time that quantum entanglement appeared. Einstein proposed to create two particles at a given location, these two particles having exactly, by conservation of the moment, opposite moments. These particles propagate. We measure the position of one and the moment of the other. As they are causally separated, these two measures are independent. So we get a contradiction with the principle of uncertainty since, on the one hand, we measures the position, on the other side, the speed and by symmetry we deduct position and speed for both: it's won.

This paradox is much deeper and much more difficult to eliminate than the one Bohr had solved. Basically, what people say, is that when you take a measurement on one side, there is an action, which Einstein called *spooky action at a distance*, which affects the other side almost instantly, and therefore goes much faster than the speed of light. Alain Aspect has had experiences that showed that the action spreads at least ten thousand times faster than the speed of light, which is incredible.

We think we are solving the paradox by saying that, if there is indeed an action, we cannot transmit information with it, but we're still hungry. What I claim is that the reason of this paradox is that we try to write a story in relation to the time. And when we try to write a story over time, we necessarily get a contradiction. Why ? Because the fact that one is outside the cone of light shows that there is no causal relationship between the two measures, this means that, depending on the benchmark that we are going to take, one event occurs before the other or the other before the one. There can therefore be no causal relationship between both. It is strictly impossible. What I claim is that the real meaning of quantum entanglement is that the quantum hazard on one side and the quantum hazard on the other are not completely independent.

The basic idea that has not yet been realized is to try to understand how the quantum hazard generates the passage of time. In *The Quantum Theater*, our first book with Danye Chéreau and Jacques Dixmier, we had found a sentence which should be remembered simply because it well expresses the problem: "The hazard of quantum is the ticking of the Divine clock". That means that true variability does not come from the passage of time, but from this fantasy, this constant imagination, the quantum. It's here that things vary, and time is just an emerging phenomenon.

We should have a much more precise philosophical reflection, much deeper, which would say that the hazard of quantum is not completely random, not completely independent, when one takes distant points, but that there are going to be correlations between the hazard of quantum at one point and the hazard at another point when there is quantum entanglement. We should be able to define correctly the quantum hazard, and we have the mathematical tool which allows us to make time emerging. This tool is what I found in my thesis from what the two Japanese men had done.

So we have the tools. But it lacks a philosophical reflection, exercise little appreciated by physicists in our time. Back to Einstein, Dirac, Heisenberg, Schrödinger, philosophical reflection was an essential ingredient of the discipline. For instance, Heisenberg and Dirac had the extraordinary chance to do a boat trip from California to Japan and they were able to chat indefinitely. Our time is very crowded by all kinds of external disturbances. We no longer know boredom which was fundamental in creative power. We now know the extraordinary success of Einstein's theory or quantum theory. And there is still stagnation. We live in a period where we are constantly disturbed, by fifteen daily mails or such report to make. We no longer have time to be bored, and we no longer have the will to do it. *Le Spectre d'Atacama* is a praise of boredom, in a way. The hero of the book is faced with this spectrum that he doesn't understand and, instead of staying at the observatory, he fled to the far south to live on an island almost deserted for a while. And he manages to find this fundamental state of the soul, which is that of boredom.

It is a difficult state to appreciate these days. It must be admitted that the CNRS is one of the rare institutions that allows find. So Vincent Lafforgue, who just had a very big price, does exactly that. He is able to isolate himself to think about a problem for years, practically in underwater state. This ability gives it the necessary depth to make great discoveries. It is a miracle that the CNRS allows. It is a very different system from that of ERC (European Research Council) or NSF in the United States. Young people are there constantly writing articles, and must constantly show that they are productive. It is a perversion which has for consequence of creating scientific feudalities and which does not authorize the diversity. We must preserve this incredible chance which allows some to isolate themselves and find this state of fundamental research so important, so creative and so impossible to appreciate, to judge in the short term. What's terrible about these selection methods on project is that researchers are asked to say, in advance, what they will find. It's ridiculous, in physics like in math. If we knew what we were going to find, discipline would lose interest. Which is really interesting, which is really exciting, it's just to look at a problem and then, at the bend of a path, to find something that we were absolutely not waiting.

I recently had to give a talk at the Collège de France on the mathematical language. I was wondering what I was going to talk about. And then finally I chose to talk about Morley's theorem. Morley has found this result by accident. He was looking for more complicated things, and he fell on it! The wording is very simple.

We take any triangle. We cut each of its angles in three equal parts. And then we intersect the lines two by two corresponding. Morley's theorem says that the triangle obtained in such a way is equilateral.

It's a shame to corset research in a more and more administrative straight jacket, because, ultimately, it encourages researchers to confine themselves to small problems in which they can make small strides, and in no way favors large discoveries.

Posts by Alain Connes in the blog http://noncommutativegeometry.blogspot.com

Sunday, February 25, 2007
Real and Complex
I would like to discuss the "next entry" in the parallel texts that Masoud was presenting in his post.

On the function theory side we are talking about "real and complex variables". A perfect book to get introduced to that is "real and complex analysis" by W. Rudin (McGraw-Hill). It is a classic and remains one of the best entrance doors to the subject. What one learns is the constant interplay between the "real variable" techniques such as the Lebesgue integral, differentiability almost everywhere, etc., and the "complex variable" techniques. There is a saying of André Weil like "The complex world is beautiful, the real world is dirty". One might then be tempted to ignore the "real world" and only work in the complex variable set-up where "any" function is holomorphic and hence infinitely differentiable etc... That's fine, and one can go some distance with that, except that most of the deep results in complex analysis do rely on real analysis. Now what about the next entry in the parallel text ? It is

Complex variable........................	Operator on Hilbert space
Real variable...............................	Self-adjoint operator

where I have slightly rewritten the previous entry

Functions $f : X \to C$	Operators on Hilbert space

of Masoud's post to stress that the right column gives an ideal model for what the loose notion of a "variable" is... The set of values of the variable is the spectrum of the operator, and the number of times a value is reached is the spectral multiplicity. Continuous variables (operators with continuous spectrum) coexist happily with discrete variables precisely because of non-commutativity of operators.

The holomorphic functional calculus gives a meaning to $f(T)$ for all holomorphic functions f on the spectrum of T, and a deep result controls the spectrum of $f(T)$. The really amazing fact is that while for general operators T in Hilbert space the only functions $f(z)$ that can be applied to T are the holomorphic ones (on the spectrum of T), the situation changes drastically when one deals with self-adjoint operators : for $T = T^*$ the operator $f(T)$ makes sense for any function f ! You can take a pencil and draw the graph of a function, it does not need to be continuous... nor even piecewise continuous, just anything you can name will do... (at the technical level the only requirement on f is that it is universally measurable but nobody can construct explicitly a function which does not fulfill this condition !)... Moreover a bounded operator is a function of T (ie is of the form $f(T)$) if and only if it shares all the symmetries of T (ie if it commutes with all operators that commute with T).

I remember that, at a very early stage of my encounter with mathematics, it is this very fact that convinced me of the power of the Hilbert space techniques in close relation with the adjoint operation $T \to T^*$. This was enough to resist the temptation of starting directly in the "complex world" of algebraic geometry which was attracting most beginners at that time, following the aura of Grothendieck, who described so well his first encounter with that world : "Je me rappelle encore de cette impression saisissante (toute subjective certes), comme si je quittais des steppes arides et revêches, pour me retrouver soudain dans une sorte de "pays promis" aux richesses luxuriantes, se multipliant à l'infini partout où il plaît à la main de se poser, pour cueillir ou pour fouiller..."

AC said...
"anonymous" the point is that the class of arbitrary functions (real analysis) is that which operates on self-adjoint operators, while only the holomorphic ones operate on general operators T. The case of normal operators ($[T, T^*] = 0$) is just "two real variables" and has nothing to do with complex analysis. When the class of functions is smaller you expect more properties, but that does not mean that it is "easier" (rather the opposite) so the analogy is not backwards...
February 26, 2007 at 8 :42 AM

Wednesday, March 7, 2007
Le rêve mathématique
I guess one possible use of a blog, like this one, is as a space of freedom where one can tell things that would be out of place in a "serious" math paper. The finished technical stuff finds its place in these papers and it is a good thing that mathematicians maintain a high standard in the writing style since otherwise one would quickly loose control of what is proved and what is just wishful thinking. But somehow it leaves no room for the more profound source, of poetical nature, that sets things into motion at an early stage of the mental process leading to the discovery of new "hard" facts.

Grothendieck expressed this in a vivid manner in *Récoltes et semailles* : "L'interdit qui frappe le rêve mathématique, et à travers lui, tout ce qui ne se présente pas sous les aspects habituels du produit fini, prêt à la consommation. Le peu que j'ai appris sur les autres sciences naturelles suffit à me faire mesurer qu'un interdit d'une semblable rigueur les aurait condamnées à la stérilité, ou à une progression de tortue, un peu comme au Moyen Age où il n'était pas question d'écornifler la lettre des Saintes Ecritures. Mais je sais bien aussi que la source profonde de la découverte, tout comme la démarche de la découverte dans tous ses aspects essentiels, est la même en mathématique qu'en tout autre région ou chose de l'Univers que notre corps et notre esprit peuvent connaitre. Bannir le rêve, c'est bannir la source - la condamner à une existence occulte".

I shall try to involve on the post of Masoud about tilings and give a heuristic description of a basic qualitative feature of noncommutative spaces which is perfectly illustrated by the space T of Penrose tilings of the plane. Given the two basic tiles : the Penrose kites and darts (or those shown in the pictures), one can tile the plane with these two tiles (with a matching condition on the colors of the vertices) but no such tiling is periodic. Two tilings are the same if they are carried into each other by an isometry of the plane. There are plenty of examples of tilings which are not the same.

The set T of all tilings of the plane by the above two tiles is a very strange set because of the following : "Every finite pattern of tiles in a tiling by kites and darts does occur, and infinitely many times, in any other tiling by the same tiles".

This means that it is impossible to decide locally with which tiling one is dealing. Any pair of tilings can be matched on arbitrarily large patches and there is no way to tell them apart by looking only at finite portions of each of them. This is in sharp contrast with real numbers for instance since if two real numbers are distinct their decimal expansions will certainly be different far enough. I remember attending quite long ago a talk by Roger Penrose in which he superposed two transparencies with a tiling on each and showed the strange visual impression one gets by matching large patches of one of them with the other... he expressed the intuitive feeling one gets from the richness of these "variations on the same point" as being similar to "quantum fluctuations". A space like the space T of Penrose tilings is indeed a prototype example of a noncommutative space. Since its points cannot be distinguished from each other locally one finds that there are no interesting real (or complex) valued functions on such a space which stands apart from a set like the real line R and cannot be analyzed by means of ordinary real valued functions. But if one uses the dictionary one finds out that the space T is perfectly encoded by a (non-commutative) algebra of q-numbers which accounts for its "quantum" aspect. See this book http://alainconnes.org/docs/book94bigpdf.pdf for more details. In a comment to the post of Masoud on tilings the question was formulated of a relation between aperiodic tilings and primes. A geometric notion, analogous to that of aperiodic tiling, that indeed corresponds to prime numbers is that of a Q-lattice. This notion was introduced in our joint work with Matilde Marcolli and is simply given by a pair of a lattice L in R together with an additive map from Q/Z to QL/L. Two Q-lattices are commensurable when the lattices are commensurable (which means that their sum is still a lattice) and the maps agree (modulo the sum). The space X of Q-lattices up to commensurability comes naturally with a scaling action (which rescales the lattice and the map) and an action of the group of automorphisms of Q/Z by composition. Again, as in the case of tilings, the space X is a typical noncommutative space with no interesting functions. It is however perfectly encoded by a noncommutative algebra and the natural cohomology (cyclic cohomology) of this algebra can be computed in terms of a suitable space of distributions on X, as shown in our joint work with Consani and Marcolli.

There are two main points then, the first is that the zeros of the Riemann zeta function appear as an absorption spectrum (ie as a cokernel) from the representation of the scaling group in the above cohomology, in the sector where the group of automorphisms of Q/Z is acting trivially (the other sectors are labeled by characters of this group and give the zeros the corresponding L-functions).

The second is that if one applies the Lefschetz formula as formulated in the distribution theoretic sense by Guillemin and Sternberg (after Atiyah and Bott) one obtains the Riemann-Weil explicit formulas of number theory that relate the distribution of prime numbers with the zeros of zeta. A first striking feature is that one does not even need to define the zeta function (or L-functions), let alone its analytic continuation, before getting at the zeros which appear as a spectrum. The second is that the Riemann-Weil explicit formulas involve rather delicate principal values of divergent integrals whose formulation uses a combination of the Euler constant and the logarithm of 2π, and that exactly this combination appears naturally when one computes the operator theoretic trace, thus the equality of the trace with the explicit formula can hardly be an accident. After the initial paper an important advance was done by Ralf Meyer who showed how to prove the explicit formulas using the above functional analytic framework (instead of the Cauchy integral).

This hopefully will shed some light on the comment of Masoud which hinged on the tricky topic of the use of noncommutative geometry in an approach to RH. It is a delicate topic because as soon as one begins to discuss anything related to RH it generates some irrational attitudes. For instance I was for some time blinded by the possibility to restrict to the critical zeros, by using a suitable function space, instead of trying to follow the successful track of André Weil and develop noncommutative geometry to the point where his argument for the case of positive characteristic could be successfully transplanted. We have now started walking on this track in our joint paper with Consani and Marcolli, and while the hope of reaching the goal is still quite far distant, it is a great incentive to develop the missing noncommutative geometric tools. As a first goal, one should aim at translating Weil's proof in the function field case in terms of the noncommutative geometric framework. In that respect both the paper of Benoit Jacob and the paper of Consani and Marcolli that David Goss mentionned in his recent post open the way. I'll end up with a joke inspired by the European myth of Faust, about a mathematician trying to bargain with the devil for a proof of the Riemann hypothesis. This joke was told to me some time ago by Ilan Vardi and I happily use it in some talks, here I'll tell it in French which is a bit easier from this side of the atlantic, but it is easy to translate...

La petite histoire veut qu'un mathématicien ayant passé sa vie à essayer de résoudre ce problème se décide à vendre son âme au diable pour enfin connaître la réponse. Lors d'une première rencontre avec le diable, et après avoir signé les papiers de la vente, il pose la question "L'hypothèse de Riemann est-elle vraie ?" Ce à quoi le diable répond "Je ne sais pas ce qu'est l'hypothèse de Riemann" et après les explications prodiguées par le mathématicien "hmm, il me faudra du temps pour trouver la réponse, rendez-vous ici à minuit, dans un mois". Un mois plus tard le mathématicien (qui a vendu son âme) attend à minuit au même endroit... minuit, minuit et demi... pas de diable... puis vers deux heures du matin alors que le mathématicien s'apprête à quitter les lieux, le diable apparaît, trempé de sueur, échevelé et dit "Désolé, je n'ai pas la réponse, mais j'ai réussi à trouver une formulation équivalente qui sera peut-être plus accessible !".

Tuesday, March 20, 2007

Time

I will try to describe in loose terms the steps that lead to the emergence of time from noncommutativity in operator algebras. This hopefully will answer the questions of Paul and Sirix (at least in parts) and of Urs.

First I'll explain the basic formula due to Tomita that associates to a state L a one parameter group of automorphisms. The basic fact is that one can make sense of the map $x \to s(x) = LxL^{-1}$ as an (unbounded) map from the algebra to itself and then take its complex powers s^{it}. To define this map one just compares the two bilinear forms on the algebra given by $L(xy)$ and $L(yx)$. Under suitable non-degeneracy conditions on L both give an isomorphism of the algebra with its dual linear space and thus one can find a linear map s from the algebra to itself such that $L(yx) = L(xs(y))$ for all x and y.

One can check at this very formal level that s fulfills $s(ab) = s(a)s(b) : L(abx) = L(bxs(a)) = L(xs(a)s(b))$.

Thus still at this very formal level s is an automorphism of the algebra, and the best way to think about it is as $x \to LxL^{-1}$ where one respects the cyclic ordering of terms in writing $Lyx = LyL^{-1}Lx = LxLyL^{-1}$. Now all this is formal and to make it "real" one only needs the most basic structure of a noncommutative space, namely the measure theory. This means that the algebra one is dealing with is a von Neumann algebra, and that one needs very little structure to proceed since the von Neumann algebra of an NC-space only embodies its measure theory, which is very little structure. Thus the main result of Tomita (which was first met with lots of skepticism by the specialists of the subject, was then succesfully expounded by Takesaki in his lecture notes and is known as the Tomita-Takesaki theory) is that when L is a faithful normal state on a von Neumann algebra M, the complex powers of the associated map $s(x) = LxL^{-1}$ make sense and define a one parameter group of automorphism s_L of M.

There are many faithful normal states on a von Neumann algebra and thus many corresponding one parameter groups of automorphism s_L. It is here that the two by two matrix trick (Groupe modulaire d'une algèbre de von Neumann, C. R. Acad. Sci. Paris, Sér. A-B, 274, 1972) enters the scene and shows that in fact the groups of automorphism s_L are all the same modulo inner automorphisms!

Thus if one lets $\mathrm{Out}(M)$ be the quotient of the group of automorphisms of M by the normal subgroup of inner automorphisms one gets a completely canonical group homomorphism from the additive group R of real numbers $\delta : R \to \mathrm{Out}(M)$ and it is this group that I always viewed as a tantalizing candidate for "emerging time" in physics.

Of course it immediately gives invariants of von Neumann algebras such as the group $T(M)$ of "periods" of M which is the kernel of the above group morphism. It is at the basis of the classification of factors and reduction from type III to type II + automorphisms which I did in June 1972 and published in my thesis (with the missing III_1 case later completed by Takesaki).

This "emerging time" is non-trivial when the noncommutative space is far enough from "classical" spaces. This is the case for instance for the leaf space of foliations such as the Anosov foliations for Riemann surfaces and also for the space of Q-lattices modulo scaling in our joint work with Matilde Marcolli.

The real issue then is to make the connection with time in quantum physics. By the computation of Bisognano-Wichmann one knows that the s_L for the restriction of the vacuum state to the local algebra in free quantum field theory associated to a Rindler wedge region (defined by $x_1 > \pm x_0$) is in fact the evolution of that algebra according to the "proper time" of the region. This relates to the thermodynamics of black holes and to the Unruh temperature. There is a whole literature on what happens for conformal field theory in dimension two. I'll discuss the above real issue of the connection with time in quantum physics in another post.

AC said... Urs : Yes, what happens in fact is that for any quantum system with infinitely many degrees of freedom the hamiltonian H does not belong to the algebra of observables. Thus the corresponding automorphisms are not inner. To see what happens it is simplest to take the case of a system of spins on a lattice. The algebra of observables is the inductive limit of the finite tensor products of matrix algebras one for each lattice site. The hamiltonian H is, even in the simplest non-interacting case, an infinite sum of the hamiltonians associated to each lattice site. Thus it does not belong to the algebra of observables and the corresponding one parameter group is not inner(both in the norm closure ie the C^*-algebra, and in the weak closure)... In QFT the situation is entirely similar and has of course infinitely many degrees of freedom from the start...
March 26, 2007 at 8 :53 AM

Dirac and integrality
In the first paper on "second quantization", namely the paper of Dirac called "The quantum theory of the emission and absorption of radiation" the process of second quantization is introduced and is related again to "integrality". This time it is not the Fredholm index that is behind the integrality but the following simple fact : if an operator a satisfies $[a, a^*] = 1$, then the spectrum of a^*a is contained in xN, the set of positive integers (as follows from the equality of the spectra of aa^* and a^*a except possibly for 0)... Second quantization is obtained simply by replacing the ordinary complex numbers a_j which label the Fourier expansion of the electromagnetic field by non-commutative variables fulfilling $[a_j, a_j^*] = 1$... (more precisely the 1 is replaced by $\hbar\nu$ where ν is the frequency of the Fourier mode). This example shows of course that integrality and non-commutativity are deeply related... While the Fredholm index is a good model of relative integers (positive or negative), the aa^* for $[a, a^*] = 1$ is a good model for positive integers...

Wednesday, April 25, 2007
Another very striking recent development was described in the talk of U. Haagerup on his joint work (I think it is with Magdalena Musat but am not sure, the paper is not out yet) on the classification of factors modulo isomorphism of the associated operator spaces. He gave an amazing necessary and sufficient condition for the class of the hyperfinite III_1 factor : that the flow of weights admits an invariant probability measure. (One knows that this holds for the von Neumann algebra of a foliation with non-zero Godbillon-Vey class). This special case suggests that the general necessary and sufficient condition should be the “commensurability” of the flow of weights, and the idea of Mackey of viewing an ergodic flow as a “virtual subgroup” of the additive group R should be essential in developing the appropriate notion of “commensurability” for ergodic flows. I was off at the beginning of the week for a short sobering trip in Sweeden (Atiyah’s “Witten” talk always has a sobering effect) and heard a really interesting talk by Nirenberg which suggests that the Holder exponent 1/3 which enters as the limit of regularity for the winding number formula of Kahane corresponds to the $3 = 2 + 1$ of the periodicity long exact sequence in cyclic cohomology. There is yet another conference taking place the whole week in paris, organized by Vincent Rivasseau.

Saturday, May 26, 2007
The Ascona meeting on Pauli and Jung
Thanks to Masoud for keeping the blog alive in the middle of all these trips to conferences. The last one I attended ended today and was dedicated to Pauli’s philosophical ideas. It was quite interesting and gave me a great occasion to get a better knowledge of these ideas. An interesting talk was given by Rafael Nunez “Where does mathematics come from ? Pauli, Jung, and contemporary cognitive science” with a brave attempt to dismiss platonism (and in particular Pauli’s view) using “contemporary cognitive science”. The talk was very entertaining, in particular on the representation of the future as relative motion as in expressions of the form “winter is coming” or “we are arriving at the end of the year”. Or when the infamous Chilian dictator, after the coup, said successively : “Communism has taken us at the edge of the abyss” and “today we took a big step forward”. Unfortunately I missed the talk of Arthur Miller “When Pauli met Jung - and what happened next”... but I could talk to him directly and got very interested with these images Jung was showing to Pauli after hearing his dreams. I had to give a rather improvised talk on Wednesday morning (about the nature of mathematical reality and also the relations with physics) and barely made it in time, since my plane to Milan had been canceled the day before (strike of Air Control) and I had to go there by train. This took the whole day and the only possible way to reach Ascona in time was to rent a car in Milan and drive there in the middle of the night. I did it since I really hate to accept giving a talk somewhere and not be able to make it at the last moment, but there was definitely a kind of “Pauli effect” making it quite difficult to reach the place in time...

Jürg Fröhlich was prevented to come for family reasons and the talk on Pauli’s work was given by Harald Atmanspacher who replaced Fröhlich at the last minute. What was really striking in this meeting was that all talks were followed by long and passionate discussions which usually lasted for almost half an hour and one could learn a lot just because there was so much interaction.

AC said...
Dear Nic, yes and this was a very nice point in the talk. I completely share this view that the past is the only thing we control and in fact I believe we just keep trying to rearange the past in order to cope with whatever the present gives us...

May 30, 2007 at 10 :09 AM
AC said...
Anonymous, the lecture of Arthur Miller was based on a book which is scheduled for publication next year. So one will have to wait for that one.

As far as I know it is true that Grothendieck has been writing a lot about dreams, but my information is not reliable and the only source is the “Grothendieck Circle” website at http://www.grothendieckcircle.org/ from which you can get better information.
May 30, 2007 at 4 :41 PM

AC said...
Dear Lieven
Thanks for your comment, it is quite difficult to “maintain” the blog just Masoud and me. We really

welcome comments like yours and would be happy to have you as a "guest" blogger.
One basic concern I have is to try and bridge the gap between the various aspects of NCG. I really like the purely algebraic aspect (and did a bit of work with Michel Dubois-Violette on that). Both the purely algebraic noncommutative geometry as well as the more "operator theoretic" differential noncommutative geometry are mature enough now not to be frightened to interact more openly. When a theory is at its beginning I believe it is important to leave it a chance to grow by itself and "protect" it somehow, but obviously time is ripe now for a broader perspective and attitude. That's pretty much what is going on with the new Journal and the organized meetings such as the Newton Institute program of last fall or precisely the Chicago meeting. Since I was there only briefly it is difficult for me to write a full report but I'll do what I can (after doing that for the vanderbilt meeting) and will try to get a "volunteer" to give a better account than my partial one (because of the small number of talks I could attend, being quite tired after the many classes I had to give in Vanderbilt).
June 6, 2007 at 8 :11 PM

Tuesday, July 3, 2007
Noncommutative spacetime
As I explained in a previous post, it is only because one drops commutativity that, in the calculus, variables with continuous range can coexist with variables with countable range. In the classical formulation of variables, as maps from a set X to the real numbers, we saw above that discrete variables cannot coexist with continuous variables.

The uniqueness of the separable infinite dimensional Hilbert space cures that problem, and variables with continuous range coexist happily with variables with countable range, such as the infinitesimal ones. The only new fact is that they do not commute.

One way to understand the transition from the commutative to the noncommutative is that in the latter case one needs to care about the ordering of the letters when one is writing. As an example, use the "commutative rule" to simplify the following cryptic message I received from a friend : "Je suis alençonnais, et non alsacien. Si t'as besoin d'un conseil nana, je t'attends au coin annales. Qui suis-je ?"

It is Heisenberg who discovered that such care was needed when dealing with the coordinates on the phase space of microscopic systems.

At the philosophical level there is something quite satisfactory in the variability of the quantum mechanical observables. Usually when pressed to explain what is the cause of the variability in the external world, the answer that comes naturally to the mind is just : the passing of time. But precisely the quantum world provides a more subtle answer since the reduction of the wave packet which happens in any quantum measurement is nothing else but the replacement of a "q-number" by an actual number which is chosen among the elements in its spectrum. Thus there is an intrinsic variability in the quantum world which is so far not reducible to anything classical. The results of observations are intrinsically variable quantities, and this to the point that their values cannot be reproduced from one experiment to the next, but which, when taken altogether, form a q-number. Heisenberg's discovery shows that the phase-space of microscopic systems is noncommutative inasmuch as the coordinates on that space no longer satisfy the commutative rule of ordinary algebra. This example of the phase space can be regarded as the historic origin of noncommutative geometry. But what about spacetime itself ? We now show why it is a natural step to pass from a commutative spacetime to a noncommutative one.

The full action of gravity coupled with matter admits a huge natural group of symmetries. The group of invariance for the Einstein-Hilbert action is the group of diffeomorphisms of the manifold and the invariance of the action is simply the manifestation of its geometric nature. A diffeomorphism acts by permutations of the points so that points have no absolute meaning.

The full group of invariance of the action of gravity coupled with matter is however richer than the group of diffeomorphisms of the manifold since one needs to include something called "the groupof gauge transformations" which physicists have identified as the symmetry of the matter part. This is defined as the group of maps from the manifold to some fixed other group, G, called the "gauge group", which as far as we known is : $G = U(1).SU(2).SU(3)$. The group of diffeomorphisms acts on the group of gauge transformations by permutations of the points of the manifold and the full group of symmetries of the action is the semi-direct product of the two groups (in the same way, the Poincaré group which is the invariance group of special relativity, is the semi-direct product of the group of translations by the group of Lorentz transformations). In particular it is not a simple group (a simple group is one which cannot be decomposed into smaller pieces, a bit like a prime number cannot be factorized into a product of smaller numbers) but is a "composite" and contains a huge normal subgroup.

Now that we know the invariance group of the action, it is natural to try and find a space X whose group of diffeomorphisms is simply that group, so that we could hope to interpret the full action as pure gravity on X. This is the old Kaluza-Klein idea. Unfortunately this search is bound to fail if one looks for an ordinary manifold since by a mathematical result, the connected component of the identity in the group of diffeomorphisms is always a simple group, excluding a semi-direct product structure as that of the above invariance group of the full action of gravity coupled with matter. But noncommutative spaces of the simplest kind readily give the answer, modulo a few subtle points. To understand what happens note that for ordinary manifolds the algebraic object corresponding to a diffeomorphism is just an automorphism of the algebra of coordinates ie a transformation of the coordinates that does not destroy their algebraic relations. When an involutive algebra A is not commutative there is an easy way to construct automorphisms. One takes a unitary element u of the algebra ie such that $uu^* = u^*u = 1$. Using u one obtains an automorphism called inner, by the formula $x \to uxu^*$.

Note that in the commutative case this formula just gives the identity automorphism (since one could then permute x and u^*). Thus this construction is interesting only in the noncommutative case. Moreover the inner automorphisms form a subgroup denoted $\mathrm{Int}(A)$ which is always a normal subgroup of the group of automorphisms of A.

In the simplest example, where we take for A the algebra of smooth maps from a manifold M to the algebra of matrices of complex numbers, one shows that the group $\mathrm{Int}(A)$ in that case is (locally) isomorphic to the group of gauge transformations ie of smooth maps from M to the gauge group $G = PSU(n)$ (quotient of $SU(n)$ by its center). Moreover the relation between inner automorphisms and all automorphisms becomes identical to the exact sequence governing the structure of the above invariance group of the full action of gravity coupled with matter.

It is quite striking that the terminology coming from physics : internal symmetries agrees so well with the mathematical one of inner automorphisms. In the general case only automorphisms that are unitarily implemented in Hilbert space will be relevant but modulo this subtlety one can see at once from the above example the advantage of treating noncommutative spaces on the same footing as the ordinary ones. The next step is to properly define the notion of metric for such spaces and we shall indulge, in the next post, in a short historical description of the evolution of the definition of the "unit of length" in physics. This will prepare the ground for the introduction to the spectral paradigm of noncommutative geometry.

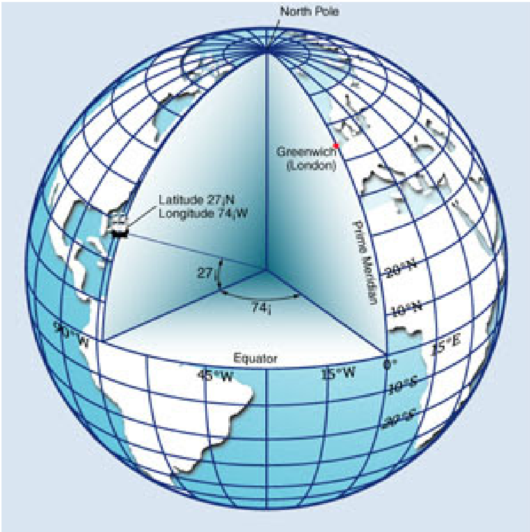
AC said...
Guy on the street, just try to permute some letters and get 4 times a name which is not so difficult to guess... what can you come up with starting with "non alsacien" for instance ?
July 4, 2007 at 6 :59 PM

AC said...
Dear Fabien
Your question is pertinent. The role of the finite space is now much better understood from the very recent papers with A. Chamseddine : "*Why the Standard Model*" (https://arxiv.org/pdf/0706.3688.pdf) and "*A dress for SM the beggar*" (https://pdfs.semanticscholar.org/765c/4b648502de8f1628258f79ca7bc7e61fe3fc.pdf) which are on the hep-th arXiv. My intention is to use this blog, this summer holidays, to explain their content in details, but one step at a time. So far I just wanted to explain why it is natural to consider NC spacetimes and not be so dependent on the "point-set" commutative view of spaces. So even if one cannot exclude that the finite space F is, as you suggest, a "remnant of a shrunk down ancestral continuous space" it will give us a lot more freedom to drop the dependence on the commutative view.
July 4, 2007 at 7 :10 PM

AC said...
I believe that the extension of the "symplectic" framework to the NC world is simply the notion of the first order term in a deformation of the NC-algebra. This is quite clear in the commutative case where a symplectic structure (or more generally Poisson structure) is just the first term in the expansion of the

deformed product. Thus it is a semi-classical form of the deformation. In the NC case there are many examples where it is natural to use a similar starting point for deformations (for instance in Rankin-Cohen brackets generalized to deformations of NC projective structures).
July 8, 2007 at 11 :35 AM

Tuesday, July 10, 2007
A brief history of the metric system



The next step is to understand what is the replacement of the Riemannian paradigm for noncommutative spaces. To prepare for that, and using the excuse of the summer holidays, let me first tell the story of the change of paradigm that already took place in the metric system with the replacement of the concrete “mètre-étalon” by a spectral unit of measurement.

The notion of geometry is intimately tied up with the measurement of length. In the real world such measurement depends on the chosen system of units and the story of the most commonly used system - the metric system - illustrates the difficulties attached to reaching some agreement on a physical unit of length which would unify the previous numerous existing choices. As is well known, the United States are one of the few countries that are not using the metric system and this lack of uniformity in the choice of a unit of length became painfully obvious when it entailed the loss of a probe worth 125 million dollars just because two different teams of engineers had used the two different units (the foot and the metric system). In 1791 the French Academy of Sciences agreed on the definition of the unit of length in the metric system, the “mètre”, as being the ten millionth part of the quarter of the meridian of the earth. The idea was to measure the length of the arc of the meridian from Barcelone to Dunkerque while the corresponding angle (approximately 9.5°) was determined using the measurement of latitude from reference stars. In a way this was just a refinement of what Eratosthenes had done in Egypt, 250 years BC, to measure the size of the earth (with a precision of 0.4 %).

Thus in 1792 two expeditions were sent to measure this arc of the meridian, one for the Northern portion was led by Delambre and the other for the southern portion was led by Méchain. Both of them were astronomers who were using a new instrument for measuring angles, invented by Borda, a French physicist. The method they used is the method of triangulation and of concrete measurement of the “base” of one triangle. It took them a long time to perform their measurements and it was a risky enterprize. At the beginning of the revolution, France entered in a war with Spain.

Just try to imagine how difficult it is to explain that you are trying to define a universal unit of length when you are arrested at the top of a mountain with very precise optical instruments allowing you to follow all the movements of the troops in the surrounding.

Both Delambre and Méchain were trying to reach the utmost precision in their measurements and an important part of the delay came from the fact that this reached an obsessive level in the case of Méchain. In fact when he measured the latitude of Barcelone he did it from two different close by locations, but found contradictory results which were discordant by 3.5 seconds of arc. Pressed to give his result he chose to hide this discrepancy just to “save the face” which is the wrong attitude for a Scientist. Chased from Spain by the war with France he had no second chance to understand the origin of the discrepancy and had to fiddle a little bit with his results to present them to the International Commission which met in Paris in 1799 to collect the results of Delambre and Méchain and compute the “mètre” from them. Since he was an honest man obsessed by precision, the above discrepancy kept haunting him and he obtained from the Academy to lead another expedition a few years later to triangulate further into Spain. He went and died from malaria in Valencia. After his death, his notebooks were analysed by Delambre who found the discrepancy in the measurements of the latitude of Barcelone but could not explain it. The explanation was found 25 years after the death of Méchain by a young astronomer by the name of Nicollet, who was a student of Laplace. Méchain had done in both of the sites he had chosen in Barcelone (Mont Jouy and Fontana del Oro) a number of measurements of latitude using several reference stars. Then he had simply taken the average of his measurements in each place. Méchain knew very well that refraction distorts the path of light rays which creates an uncertainty when you use reference stars that are close to the horizon. But he considered that the average result would wipe out this problem.

What Nicollet did was to ponder the average to eliminate the uncertainty created by refraction and, using the measurements of Méchain, he obtained a remarkable agreement (0.4 seconds ie a few meters) between the latitudes measured from Mont Jouy and Fontana del Oro. In other words Méchain had made no mistake in his measurements and could have understood by pure thought what was wrong in his computation. I recommend the book of Ken Adler (*The measure of all things : the seven-year odyssey and hidden error that transformed the world*, eds Little, Brown et compagny, 2003, ou *Mesurer le monde*, Flammarion, 2005) for a nice account of the full story of the two expeditions.

In any case in the meantime the International commission had taken the results from the two expeditions and computed the length of the ten millionth part of the quarter of the meridian using them. Moreover a concrete platinum bar with approximately that length was then realized and was taken as the definition of the unit of length in the metric system. With this unit the actual length of the quarter of meridian turns out to be 10 002 290 rather than the aimed for 10 000 000 but this is no longer relevant. In fact in 1889 the reference became another specific metal bar (of platinum and iridium) called “mètre-étalon”, which was deposited near Paris in the pavillon de Breteuil. This definition held until 1960.

Already in 1927, at the seventh conference on the metric system, in order to take into account the inevitable natural variations of the concrete called “mètre-étalon”, the idea emerged to compare it with a reference wave length (the red line of Cadmium). Around 1960 the reference to the called “mètre étalon” was finally abandoned and a new definition of the unit of length in the metric system (the “mètre) was adopted as 1650763.73 times the wave length of the radiation corresponding to the transition between the levels 2p10 and 5d5 of the Krypton 86Kr.

In 1967 the second was defined as the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of Caesium-133. Finally in 1983 the “mètre” was defined as the distance traveled by light in 1/299792458 second. In fact the speed of light is just a conversion factor and to define the “mètre” one gives it the specific value of $c = 299792458$ m/s. In other words the “mètre” is defined as a certain fraction $9192631770/299792458 \sim 30.6633...$ of the wave length of the radiation coming from the transition between the above hyperfine levels of the Caesium atom.

The advantages of the new standard of length are many. First by not being tied up with any specific location, it is in fact available anywhere where Caesium is. The choice of Caesium as opposed to Helium or Hydrogen which are much more common in the universe is of course still debatable, and it is quite possible that a new standard will soon be adopted involving spectral lines of Hydrogen instead of Caesium. See this paper of Bordé for an update http://christian.j.borde.free.fr/ChB.pdf.

While it would be difficult to communicate our standard of length with other extraterrestrial civilizations if they had to make measurements of the earth (such as its size) the spectral definition can easily be encoded in a probe and sent out. In fact spectral patterns provide a perfect “signature” of chemicals, and a universal information available anywhere where these chemicals can be found, so that the wave length of a specific line is a perfectly acceptable unit, while if you start thinking a bit you will find out that we would be unable to just tell where the earth is in the universe...

Coordinates ? yes but whith respect to which system ? One possibility would be to give the sequence of redshifts to nearby galaxies, and in a more refined manner to nearby stars but it would be quite difficult to be sure that this would single out a definite place.

AC said...
Dear Apprenticing Physicist. First I do not know any real evidence of the variation in time of the constants of nature. Just look for instance at
http://www-cosmosaf.iap.fr/Cste%208%20mai%202004%20html.htm

In fact Dirac's large number idea of 1937, that was predicting a variation of G with time was not validated by experiment and one has an experimental upper bound of the order of $dG/G < 4\ 10^{-11}$ per year. Thus I am not sure that there is any convincing experimental evidence yet for what you say. If there were it would be interesting to discuss more precisely of which constant we are talking. For instance the spectral unit of length which I discussed in the post depends on the Rydberg constant and hence on the electron mass. In NCG that mass is tied up to the inverse size of the finite geometry F that will be discussed later in this blog. At least what can be said is that, in NCG, the "atomic units" are intimately tied up to the geometry of the finite space F while the astronomical units are tied up with the manifold M. The NCG model of space-time is the product M times F, but nothing prevents the geometry of F to vary over M.
July 13, 2007 at 10 :27 AM

Tuesday, July 17, 2007
Non Standard stuff
I am not sure I really know how to make use of a "blog" like this one. Recently I had to write a sollicited paper describing the perspective on the structure of space-time obtained from the point of view of noncommutative geometry. At first I thought that I could just be lazy and after the paper was written (it is available here https://alainconnes.org/docs/shahnlong.pdf) just use pieces of it to keep this blog alive during the summer vacations. However, when trying to do that, I realized that it was better (partly because of the impractical use of latex in the blog) to first make the paper available and then tell in the blog the additional things one would not "normally" write in a paper (even a non-technical general public paper such as the above). I am not keen on turning the blog into a place for controversies since it is unclear to me that one gains a lot in such discussions. The rule seems to be that, most often, people have prejudices against new stuff mostly because they don't know enough and take the lazy attitude that it is easier to denigrate a theory than to try and appreciate it. I am no exception and have certainly adopted that attitude with respect to supersymmetry or string theory. A debate will usually exhibit the strong opinions of the various sides and it is rare that one witnesses a real change taking place. So much for the "controversy" side. However I do believe that there are some points that can be quite useful to know and which, provided they are presented in a non-polemic manner can help a lot to avoid some pitfalls. I will discuss as an example the two notions of "infinitesimals" that I know and try to explain the relevance of both. This is not a "math paper" but rather an informal discussion.

When I was a student in Ecole Normale about 40 years ago, I fell in love with a new math topic called "nonstandard analysis" which was advocated by A. Robinson. Being a student of Gustave Choquet at that time, I knew a lot about ultrafilters. These maximal filters were (correct me if I am wrong) discovered by H. Cartan during a Bourbaki workshop. At that time Cartan had no name for the new objects but he had found the remarkable efficiency they had in any proof where a compactness and choice arguments were needed. So (this I heard from Cartan) the name he was using was "boum" ! ! ! Of course he knew that it gave a one line proof of the existence of Haar measure (boum...). And also that because of uniqueness of the latter it was in fact proving a rather strong convergence statement on the counting functions that approximate the Haar measure. He wanted to make sure, and wrote in a Compte-Rendu note the full details of a direct geometric argument proving the expected convergence. From ultrafilters to ultraproducts is an easy step. And I got completely bought by ultraproducts when I learnt (around that time) about the Ax-Cochen theorem : the ultraproduct of p-adic fields is isomorphic to the ultraproduct of local function fields with the same residue fields. Thus I started trying to work in that subject and obtained, using a specific class of ultrafilters called "selective", a construction of minimal models in nonstandard analysis. They are obtained as ultraproducts but the ultrafilters used are so special that, for instance, in order to know the element of the ultrapower of a set X, one does not need to care about the labels : the image ultrafilter in X is all that is needed. I wrote a paper explaining how to use ultraproducts and always kept that tool ready for use later on. I used it in an essential manner in my work on the classification of factors. So much for the positive side of the coin. However, quite early on I had tried in vain to implement one of the "selling adds" of nonstandard analysis, namely that it was finally giving the promised land for "infinitesimals". In fact the adds came with a specific example : a purported answer to the naive question "what is the probability "p" that a dart will land at a given point x of the target" in playing a game of darts. This was followed by 1) the simple argument why that positive number "p" was smaller than epsilon for any positive real epsilon 2) one hundred pages of logic 3) the identification of "p" with a "non-standard" number...

At first I attributed my inability to concretely get "p" to my lack of knowledge in logics, but after realizing that the models could be constructed as ultraproducts this excuse no longer applied. At this point I realized that there is some fundamental reason why one will never be able to actually "pin down" this "p" among non-standard numbers : from a non-standard number (non-trivial of course) one canonically deduces a non-measurable character of the infinite product of two element groups (the argument is simpler using a non-standard infinite integer "n", just take the map which to the sequence a_n (of 0 and 1) assigns its value for the index "n"). Now a character of a compact group is either continuous or non-measurable. Thus a non-standard number gives us canonically a non-measurable subset of $[0, 1]$. This is the end of the rope for being "explicit" since (from another side of logics) one knows that it is just impossible to construct explicitely a non-measurable subset of $[0, 1]$!

It took me many years to find a good answer to the above naive question about "p". The answer is explained in details here. It is given by the formalism of quantum mechanics, which as explained in the previous post on "infinitesimal variables" gives a framework where continuous variables can coexist with infinitesimal ones, at the only price of having more subtle algebraic rules where commutativity no longer holds. The new infinistesimals have an "order" (an infinitesimal of order one is a compact operator whose characteristic values μ_n are a big O of $(1/n)$. The novel point is that they have an integral, which in physics terms is given by the coefficient of the logarithmic divergence of the trace. Thus one obtains a new stage for the "calculus" and it is at the core of noncommutative differential geometry.

In Riemannian geometry the natural datum is the square of the line element, so that when computing the distance $d(A, B)$ between two points one has to minimize the integral from A to B along a continuous path of the square root of $g_{\mu\nu} dx^\mu dx^\nu$. Now it is often true that "taking a square root" in a brutal manner as in the above equation is hiding a deeper level ofunderstanding. In fact this issue of taking the square root led Dirac to his famous analogue of the Schrödinger equation for the electron and the theoretical discovery of the positron. Dirac was looking for a relativistic invariant form of the Schröodinger equation. One basic property of that equation is that it is of first order in the time variable. The Klein-Gordon equation which is the relativistic form of the Laplace equation, is relativistic invariant but is of second order in time.

Dirac found away to take the square root of the Klein-Gordon operator using Clifford algebra. In fact (as pointed out to me by Atiyah) Hamilton had already written the magic combination of partial derivatives using his quaternions as coefficients and noted that this gave a square root of the Laplacian. When I was in St. Petersburg for Euler's 300'th, I noticed that Euler could share the credit for quaternions since he had explicitly written their multiplication rule in order to show that the product of two sums of 4 squares is a sum of 4 squares.

So what is the relation between Dirac's square root of the Laplacian and the above issue of taking the square root in the formula for the distance $d(A, B)$. The point is that one can use Dirac's solution and rewrite the same geodesic distance $d(A, B)$ in the following manner : one no longer measures the minimal length of a continuous path but one measures the maximal variation of a function : ie the absolute value of the difference $f(A) - f(B)$. Of course without a restriction on f this would give infinity, but one requires that the commutator $[D, f]$ of f with the Dirac operator is bounded by one.

Here we are in our "quantized calculus" stage, so that both the functions on our geometric space as well as the Dirac operator are all concretely represented in the same Hilbert space H. H is the Hilbert space of square integrable spinors and the functions act by pointwise multiplication. The commutator $[D, f]$ is the Clifford multiplication by the gradient of f so that when the function f is real, its norm is just the Sup norm of the gradient. Then saying that the norm of $[D, f]$ is less than one is the same as asking that f be a Lipschitz function of constant one ie that the absolute value of $f(A) - f(B)$ is less than $d(A, B)$ where the latter is the geodesic distance. For complex valued functions one only gets an inequality, but it suffices to show that the maximum variation of such f gives exactly the geodesic distance : ie we recover the geodesic distance $d(A, B)$ as Sup $f(A) - f(B)$ for norm of $[D, f]$ less than one.

Note that D has the dimension of the inverse of a length, ie of a mass. In fact in the above formula for distances in terms of a supremum the product of "f" by D is dimensionless and "f" has the dimension of a length since $f(A) - f(B)$ is a distance.

Now what is the intuitive meaning of D ? Note that the above formula measuring the distance $d(A, B)$ as a supremum is based on the lack of commutativity between D and the coordinates "f" on our space. Thus there should be a tension that prevents D from commuting with the coordinates. This tension is provided by the following key hypothesis "the inverse of D is an infinitesimal".

Indeed we saw in a previous post that variables with continuous range cannot commute with infinitesimals, which gives the needed tension. But there is more, because of the fundamental equation $\mathbf{ds} = 1/D$ which gives to the inverse of D the heuristic meaning of the line element. This change of paradigm from the $g_{\mu\nu}$ to this operator theoretic **ds** is the exact parallel of the change of the unit of length in the metric system to a spectral paradigm. Thus one can think of a geometry as a concrete Hilbert space representation not only of the algebra of coordinates on the space X we are interested in, but also of its infinitesimal line element **ds**. In the usual Riemannian case this representation is moreover irreducible. Thus in many ways this is analogous to thinking of a particle as Wigner taught us, ie as an irreducible representation (of the Poincaré group).

AC said...

Dear Arivero, thanks ! nice that Tao and myself are discussing the same thing. In fact my first compte rendu note (1970) was about selective ultrafilters and minimal nonstandard models constructed using ultraproducts. My belief is that the two points of view on infinitesimals are the reflection of the nuances existing already at the beginning of the invention of the calculus. The non-standard numbers in the sense of logics or ultrafilters are very close to the point of view of Leibniz.

July 17, 2007 at 7 :30 PM

AC said...

Dear Alon Levy

Yes, of course I am willing to put the post about nonstandard analysis on the carnival blog, thanks,

Alain

August 19, 2007 at 1 :57 PM

Friday, August 3, 2007

Paul's Seventies

I am just back from a very nice event around Paul Baum's seventies, which took place in Warsaw last Monday, thanks in particular to Piotr Hajac. I have known Paul since the summer of 1980 when we first met in Kingston. I really had, when I first met him, the impression of meeting l'"Homologie en personne". The more I got to know him through our very long collaboration, the better I enjoyed his clarity of mind and his relentless quest for simplicity and beauty. In many ways he succeeds indoing something very difficult, which Grothendieck advised in *Récoltes et Semailles*, namely to keep "une innocence enfantine" in front of mathematics. The dinner on monday night was comparable in intensity to the memorable one in Martin Walter's place, in Boulder, for Paul's sixties when the team Paul Baum - Raoul Bott forced Martin to search (again and again) his cellar for more bottles of wine to keep up with their drinking ability !.

Raoul Bott died in December 2005. Not long before, Paul went all the way to California to visit him and they talked together for an entire day. This type of faithfulness in friendship and understanding of what really matters, is an attitude towards life which Paul has and which I truly admire.

Monday, August 13, 2007
Harmonic mean
This "post" is mainly an attempt to see if one can manage to use formulas in the blog and discuss some real stuff somehow. The formulas should be really visible, a bit like with transparencies. So as a pretext, I'll start by discussing an issue related to the basic Ansatz :

$$\mathbf{ds} = D^{-1}$$

which gives the operator theoretic line element in terms of the Dirac operator in the general framework of "metric" noncommutative geometry. The kernel of the operator D is finite dimensional and one takes **ds** to vanish on that kernel. As was already discussed here, the knowledge of D gives back the metric. Moreover the noncommutative integral, in the form of the Dixmier trace, gives back the volume form. Thus the integral of a function f in dimension n is simply given by

$$⨍ f|\mathbf{ds}|^n$$

where the "cut" integral is the Dixmier trace ie the functional that assigns to an infinitesimal of order one the coefficient of the logarithmic divergency in the series that gives the sum of its eigenvalues.

I will not try to justify the heuristic definition of the line element any further. It is more interesting to put it to the test, to question it, and I will discuss an example of an issue which left me perplex for quite sometime but has a pretty resolution.

The point is to understand what happens when one takes the product of two noncommutative geometries. One gets the following relation for the squares of the corresponding Dirac operators :

$$D^2 = D_1^2 + D_2^2$$

where we abuse notations by removing the tensor product by the identity operator that normally goes with each of the operators D_j. Now this relation is quite different from the simple Pythagorean relation of the classical line elements whose square simply add up and it thus raises the question of reconciling the above Ansatz with the simple formula of addition of the squares of the Dirac operators. More generally, one can consider a bunch of NC spaces with Dirac operators D_μ and combine them as follows : One starts with a positive matrix of operators in Hilbert space :

$$g = (g^{\mu\nu}) \in M_n(\mathcal{L}(\mathcal{H}))$$

and one extends the above formula giving D^2 for a product of two spaces and forms the following sum :

$$D^2 = \sum_{\mu,\nu} D_\mu g^{\mu\nu} D_\nu^*$$

we make no commutativity hypothesis and even drop the self-adjointness of D_μ which is not needed. We want a formula for the inverse of the square of D ie for :

$$\mathbf{ds}^2 = D^{-2}$$

in terms of the inverse matrix :

$$g^{-1} = (g_{\mu\nu})$$

which plays a role similar to the $g_{\mu\nu}$ of Riemannian geometry, and of the operators

$$dz^\mu = D_\mu^{-1}$$

where the notation with z stresses the fact that we do not even assume self-adjointness of the various D_μ.

It sounds totally hopeless since one needs a formula for the inverse of a sum of noncommuting operators. Fortunately it turns out that there is a beautiful simple formula that does the job in full generality. It is reminiscent of the definition of distances as an infimum. It is given by :

$$\langle \xi, \mathbf{ds}^2 \xi \rangle = \operatorname{Inf} \sum_{\mu,\nu} \langle dz^\mu \xi^\mu, g_{\mu\nu} dz^\nu \xi^\nu \rangle$$

The infimum is taken over all decompositions of the given vector as a sum :

$$\sum_{\mu} \xi^{\mu} = \xi$$

Note that this formula suffices to determine the operator $\mathbf{ds}^2$ completely, since it gives the value of the corresponding positive quadratic form on any vector in Hilbert space. The proof of the formula is not difficult and can be done by applying the technique of Lagrange multipliers to take care of the above constraint on the free vectors ξ^{μ}.

AC said...
Dear Christophe
What I did was to embed small images inside the text, first I wrote a pdf file and then I extracted small portions of the pdf using adobe professional and saving them as jpeg.
September 7, 2007 at 4 :51 PM

Wednesday, September 5, 2007
News on K-front
Today the editorial board of the new Journal of K-theory put out a public statement, which we reproduce below : STATEMENT OF THE EDITORS OF THE "JOURNAL OF K-THEORY"

After several public statements and news articles regarding the Springer journal "K-theory" (KT), and the new "Journal of K-Theory" (JKT) to be distributed by Cambridge University Press (CUP), the mathematical community has become aware of ongoing changes. On behalf of the entire Editorial Board of the new JKT, we want to give as precise a picture of the situation as we can at the moment, especially to the authors. It is very important to us that the authors should not suffer as a result of the transition. Those authors who submitted papers to KT before August 2007, regardless of whether the paper has already been accepted or is just awaiting review, have three choices : 1) Choose another journal. 2) Maintain submission with KT for final review if necessary and publication if accepted. 3) Transfer their article to the new JKT. All authors who have not yet done so should please notify Professor Bak on the one hand, Professors Lueck and Ranicki on the other hand, about their choice, as soon as possible. For those who opt for choice 2, Professors Lueck and Ranicki have promised to take over the remaining editorial duties. We can guarantee that the authors who choose option (3) will have a smooth transition, with their articles progressing as if there has been no change. We will also do everything we can to help those who choose options (1) and (2). In particular, if the authors instruct us, we will be happy to forward to the journals of their choice the full information regarding the status of their articles. In 2004, because of growing dissatisfaction with Springer, the editorial board of KT authorized Prof. Anthony Bak, the Editor in Chief, to begin negotiations with other publishers. The editorial board was unhappy with the poor quality of the work done by Springer, for example the huge number of misprints in the published version of the articles, the long delay in publication and the high prices Springer was charging. The negotiations came to a conclusion in 2007. A new journal, entitled "Journal of K-theory" (JKT) will commence publication in late 2007. It will be printed by Cambridge University Press. Papers will appear earlier online, as "forthcoming articles". The title of JKT is currently owned by a private company. This situation is only meant as a temporary solution to restart publication of K-theory articles as soon as possible. It is the Board's intention to create a non-profit academic foundation and to transfer ownership of JKT to this foundation, as soon as possible, but no later than by the end of 2009, a delay justified by many practical considerations. This shift towards more academic control of journals is not new. We follow here a path opened by Compositio Mathematica, Commentarii Mathematici Helvetici, and others (see for instance the interesting paper of Gerard van der Geer which appeared in the Notices of the AMS in May 2004). We believe that such changes can help keep prices low. We trust in Prof. Bak's leadership for the launching of JKT and forming, together with the editorial board, the foundation to house the Journal. The statutes of the foundation will provide democratic rules governing the future course and development of the journal, including the election of the managing team. We hope to have provided a fair picture of the current situation, and we plan to issue another public statement when new developments come up. In case of further questions, please contact any of the signatories. Let us conclude from a broader perspective : The editorial board is committed to secure the journal's quality and long-term sustainability. Signatures : A. Bak, P. Balmer, S. Bloch, G. Carlsson, A. Connes, E. Friedlander, M. Hopkins, B. Kahn, M. Karoubi, G. Kasparov, A. Merkurjev, A. Neeman, T. Porter, J. Rosenberg, A. Suslin, Guoping Tang, B. Totaro, V. Voevodsky, C. Weibel, Guoliang Yu.

AC said...
Dear Peter
First I (ac) heard about this episode and checked its accuracy only now. If I had known earlier I would of course have done something.

What needs to be done for sure is to remove from all librairies the copy of the Journal with that terrible "data corruption". As you say it is the responsibility of Springer to do that. They have been able to make the author sign an "excuse" as if any sensible person could believe that it was the intention of the author to have his paper appearing in that corrupted form!!!

Only Springer has the means to recall the volume and to replace it, and I am writing to them to ask that they do it. I don't see what Kreck has to do with that, except if you consider that he represents Springer. The present crisis has its origin in innumerable printing problems since 2003-2004 (less serious than the above one fortunately), but just to quote examples nilpotent → impotent everywhere in a paper, 4 pages missing in the middle of a paper, the author of a ten pages paper spending more than a year of back and forth proofs with the publisher before getting a sensible version etc etc...

This resulted in permanent complaints of authors to Bak and no solution was found except to move to another publisher. The only sensible thing to do at this point is to move ahead, get the new JKT on the rails making sure that it will be ran in a democratic manner and stop all this sterile quarelling.
September 12, 2007 at 4 :31 PM

AC said... Good news!
I just got a positive answer from Catriona Byrne from Springer concerning the corrupted issue of K-theory which was discussed in this blog. I had sent an email asking if the corrupted volume could be destroyed and fortunately the answer is "yes" : "We agree. This is actually in the works right now. The corrected issue will have a covering note asking librarians to destroy the original issue, and pointing out that the online version is correct."
September 13, 2007 at 7 :10 PM

AC said...
Update
I am really grateful to Catriona Byrne for her understanding of the situation and her help in removing the corrupted copy of K. I realise that the phrase :
"They have been able to make the author sign an "excuse" as if any sensible person could believe that it was the intention of the author to have his paper appearing in that corrupted form!!!" could be misunderstood and I would rather say "They have been able to publish an "excuse" of the author, as if any sensible person could believe that it was the intention of the author to have his paper appearing in that corrupted form!!!"

What I meant of course is that in my opinion the responsability for missing the "typos", and in particular the disordered alphabetic listing of references, should be shared and not entirely endorsed by the author. What happened to the author in that case is something that could happen to any of us : some "data corruption" occured at some point in the publication process and he missed the typo in the proofreading process. Good typesetters ask us to update old bibliographical references and of course they double check things like the alphabetic ordering.
September 16, 2007 at 9 :12 AM

AC said...
A Scholarly Society Makes a Logical - and Symbolic - Move to Cambridge U. Press
By JENNIFER HOWARD
In scholarly-journal publishing, as in marriage, love can have very little to do with one's decision to stay committed to a partner.

Lately, scholarly societies have been tempted to make alliances with well-heeled suitors. A commercial outfit like Springer or Wiley-Blackwell commands vast global marketing and distribution networks; a specialized nonprofit publisher can offer publishing platforms and services that university presses may find hard to match. And such assets often help seal the deal.

The American Anthropological Association, for instance, announced in September that it would leave the

University of California Press for Wiley-Blackwell (The Chronicle, September 19).

And this year the American Astronomical Society abandoned the University of Chicago Press for IOP Publishing, part of the nonprofit Institute of Physics (The Chronicle, May 18).

But another society, the Association for Symbolic Logic, has reversed the trend and decided to ditch a commercial publisher for a university press. It has severed its ties to Springer, which owns and publishes the Journal of Philosophical Logic, a journal edited by the association, and formed an alliance with Cambridge University Press.

Together the association, which is known as ASL, and the press will start the Review of Symbolic Logic as a successor to the Springer-owned journal. Revenue from the new journal will be shared between the parties, while the association will retain editorial control.

All of the ASL editors of the Springer journal are switching over to the new journal, which will make its debut in June 2008, taking its place alongside the association's two other publications, The Journal of Symbolic Logic and The Bulletin of Symbolic Logic. (The group typesets those publications itself, and the American Mathematical Society handles printing and mailing.) Dues-paying members of the symbolic-logic group will receive the journal as a benefit of membership.

Those involved with the new Review say it will be broader in scope than its predecessor. The association brings together logicians who work in mathematics, philosophy, linguistics, computer science, cognitive science, and other fields. It envisions its new journal as a meeting place for work in several complementary areas, with an emphasis on philosophical logic and its applications, the history of philosophy of logic, and the philosophy and methodology of mathematics. Penelope Maddy, president of the association and a professor of logic and the philosophy of science at the University of California at Irvine, points to a number of "growth industries" that the Review will spotlight – for instance, how scholars in computational linguistics, game theory and decision theory, and cognitive science apply the tools and methods of philosophical logic. "It's all about logic," she says, "but you can come at logic from very different disciplinary perspectives."

"Different perspectives" would be a kind way to sum up the association's relationship with Springer. Thanks to the vicissitudes of corporate mergers, Springer is the latest in a line of commercial publishers to own the Journal of Philosophical Logic, which was founded in 1972. The association has edited it since 1987.

Working with Springer was a headache almost from the start, says G. Aldo Antonelli, coordinating editor of the journal and chairman of the department of logic and the philosophy of science at Irvine. Papers went missing, typesetting went awry. "Authors were up in arms," he says. The editors would submit clean manuscripts and "get page proofs back that were full of typos and errors."

The association even tried to buy the journal from Springer, but its offer was rebuffed. So in 2006, when Cambridge signed on to handle book projects for the group, talk quickly turned to a new journal as well.

Charles Erkelens, editorial director for the humanities at Springer, plays down the troubles in the relationship. "There has been an occasional article where things have gone wrong and we've fixed them again, but I have no bad relations with ASL in any way," he says.

The Journal of Philosophical Logic has done well for Springer, and the company will continue to publish it, with a new editorial board. "It's fine for philosophical logic to have more outlets for people to publish in," Mr. Erkelens says. "I still think the Journal of Philosophical Logic will remain the most important of those."

David Tranah, editorial director of mathematical sciences at Cambridge University Press, was matchmaker for both the books program and the new journal. Commercial publishers like Springer "have been vigorously courting learned societies," he says, but often "what they require is more than they can offer." Cambridge has vowed not to be so demanding. "We do not insist on ownership, we do not insist on retaining copyright," he says. "We want to explore possibilities for them. It's a different sort of partnership."

The union may be a meeting of minds, but both partners stand to gain in financial terms as well. Previously "we were putting in all this work and Springer was making pots of money," says Charles Steinhorn, the association's secretary-treasurer, who is a professor of mathematics at Vassar College. If Cambridge's calculations are correct, he says, "we should be able to support new scholarly activities" with the extra income - a graduate fellowship, perhaps, or research support. Meanwhile Cambridge has an incentive to be active on the journal's behalf, spreading the word through its networks of editors and marketers. The association's officers say they're over the moon. "We were nowhere near this with Springer," Ms. Maddy says. "Assuming the Review does as well as we think it will do, this is a great boon to the organization."

September 29, 2007 at 5 :01 PM
Anonymous said...
The reference for the story above is Chronicle of Higher Education, September 27, 2007.
October 5, 2007 at 10 :18 AM

Tuesday, September 18, 2007
Les motifs - ou le coeur dans le coeur
It is with this fascinating title that A. Grothendieck presents in *Récoltes et Semailles* (cfr. Promenade à travers une oeuvre ou l'Enfant et la Mère) the subject of motives : the deepest of the twelve research themes around which he developed his "long-run" research program that literally revolutionized the field of algebraic geometry in the decade 1958-68. Motives were envisaged as the "heart of the heart" of the new geometry (arithmetic geometry) that Grothendieck invented following a scientific strategy based on the introduction of a series of new concepts organized on a progressive level of generality : starting with schemes, topos and sites then continuing with the yoga of motives and motivic Galois groups and finally introducing anabelian algebraic geometry and Galois-Teichmuller theory.

If the notions of scheme and topos were the two crucial ideas which constituted the original driving force in the development of this new geometry – Grothendieck was evidently fascinated by the concepts of geometric point, space and symmetry – it is only with the notion of a motive that one eventually captures the deepest structure, the heart of the profound identity between geometry and arithmetic.

Grothendienck wrote very little about motives. The foundations are documented in his unpublished manuscript "Motifs" and were discussed on a seminar at the Institut des Hautes Études Scientifiques, in 1967. We know, by reading *Récoltes et Semailles*, that he started thinking about motives in 1963-64. J.P. Serre has included in his paper "Motifs" an extract from a letter that Grothendieck wrote to him in August 1964 in which he talks (rather vaguely, in fact) of the notions of motive, fiber functor, motivic Galois group and weights.

Motives were introduced with the ultimate goal to supply an intrinsic explanation for the analogies occurring among the various cohomological theories for algebraic varieties : they were expected to play the role of a universal cohomological theory (the motivic cohomology) and also to furnish a linearization of the theory of algebraic varieties, by eventually providing (this was Grothendieck's viewpoint) the correct framework for a successful attack to the Weil's Conjectures on the zeta function of an algebraic variety over a finite field.

Unlike in the framework of algebraic topology where the standard cohomological functor is uniquely characterized by the Eilenberg-Steenrod axioms in terms of the normalization associated to the value of the functor on a point, in algebraic geometry there is no suitable cohomological theory with integers coefficients, for varieties defined over a field k, unless one provides an embedding of k into the complex numbers. In fact, by means of such mapping one can form the topological space of the complex points of the original algebraic variety and finally compute the Betti (singular) cohomology. This construction however, does in general depend upon the choice of the embedding of k in the field of complex numbers. Moreover, Hodge cohomology, algebraic de-Rham cohomology, etale l-adic cohomology furnish several examples of different cohomology functors which can be simultaneously associated to a given algebraic variety, each of which supplying a relevant information on the topological space.

Grothendieck theorized that this plethora of different cohomological data should be somewhat encoded systematically within a unique and more general theory of cohomological nature that acts as an internal "liaison" between algebraic geometry and the collection of available cohomological theories. This is the idea of the "motif", namely the common reason behind this multitude of cohomological invariants which governs and controls systematically all the cohomological apparatus pertaining to an algebraic variety or more in general to a scheme.

The original construction of a category M of (pure) motives over a field k starts with two preliminary considerations. The first consideration is that M should be the target of a natural contravariant functor connecting the category C of smooth, projective algebraic varieties over a field k to M. Such functor should map an object X in C to its associated motive $M(X)$. The second consideration is that this functor should, by construction, factor through any particular cohomological theory.

Now, keeping in mind this goal, one thinks about the axiomatizing process of a cohomological theory in algebraic geometry. This is done by introducing a contravariant functor $X \to H(X)$ from C to a graded abelian category, where the sets of morphisms between its objects form K-vector spaces (K is a field of characteristic zero, that for simplicity, I fix here equal to the rationals). One also would like that any correspondence $V \to W$ (an algebraic cycle in the cartesian product $V \times W$ that can be view as the graph of a multi-valued algebraic mapping) induces contravariantly, a mapping on cohomology and that the target category is suitably defined so that it contains among its objects any "Weil cohomological theory", namely a cohomology which satisfies among other axioms Poincaré duality and Künneth formula.

This preliminary disquisition helps one in formalizing the construction of the category of motives by following a three-steps procedure. One wishes to enlarges the category C in a precise way with the hope to produce also an abelian category. The three steps are shortly resumed as follows.

(1) One moves from C to a category with the same objects but where the sets of morphisms are the equivalence classes of rational correspondences. Here, the natural choice of the equivalence relation is the numerical equivalence relation as it is the coarsest one among the possible relations between algebraic cycles which can be seen to induce, via the cohomological axioms of any Weil cohomological theory, well-defined homomorphisms in cohomology.

(2) One enlarges the collection of objects of the category defined in (1), by formally adding kernels and images of projectors. This step is technically referred to as the "pseudo-abelian envelope" of the category defined in (1) and it is motivated by the expectation to define an abelian category of motives in which for instance, the Künneth formula can be applied.

(3) Finally, one considers the opposite of the category defined in (2).

Now, after having diligently applied all this abstract machinery, one would like to see a fruitful application of these ideas, in the form, for instance, of the proof of a major conjecture. However, one also perceives quite soon that a successful application of the yoga of motives is subordinated to a thorough knowledge of the theory of algebraic cycles, since the construction of the category M is centered on the idea of enlarging the sets of morphisms by implementing the notion of correspondence. It is for this reason that the Standard Conjectures (cohomological criteria for the existence of interesting algebraic cycles) were associated, since the beginning, to the theory of motives as they seem to play the "conditio sine qua non" a theory of motives has a concrete and successful application.

However, in order to put the Standard Conjectures in the right perspective and to avoid perhaps, an over-estimation of their importance, one should also record that Y. Manin gave in 1968, the first interesting application of these ideas on motives by producing an elegant proof of the Riemann-Weil hypothesis for non-singular three-dimensional projective unirational varieties over a finite field, without appealing to the Standard Conjectures. Moreover, we also know that the Weil's Conjectures have been proved by P. Deligne in 1974 without using neither the theory of motives nor the Standard Conjectures.

Almost forty years have passed since these ideas were informally discussed in the "Grothendieck's circle". An enlarged and in part still conjectural theory of mixed motives has in the meanwhile proved its usefulness in explaining conceptually, some intriguing phenomena arising in several areas of pure mathematics, such as Hodge theory, K-theory, algebraic cycles, polylogarithms, L-functions, Galois representations etc. Very recently, some new applications of the theory of motives to number-theory and quantum field theory have been found or are about to be developed, with the support of techniques supplied by noncommutative geometry and the theory of operator algebras.

In number-theory, a conceptual understanding of the interpretation proposed by A. Connes of the Weil explicit formulae as a Lefschetz trace formula over the noncommutative space of adèle classes, requires the introduction of a generalized category of motives which is inclusive of spaces which are highly singular from a classical viewpoint. Several questions arise already when one considers special types of zero-dimensional noncommutative spaces, such as the space underlying the quantum statistical dynamical system defined by J.B. Bost and Connes in their paper “Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking” (Selecta Math. (3) 1995). This space is a simplified version of the adèle classes and it encodes in its group of symmetries, the arithmetic of the maximal abelian extension of the rationals. A new theory of endomotives (algebraic and analytic) has been recently developed in “Noncommutative geometry and motives : the thermodynamics of endomotives” (to appear in Advances in Mathematics). The objects of the category of endomotives are noncommutative spaces described by semigroup actions on projective limits of Artin motives (these are among the easiest examples of pure motives, as they are associated to zero-dimensional algebraic varieties). The morphisms in this new category generalize the notion of (algebraic) correspondences and are defined by means of etale groupoids to account for the presence of the semigroup actions.

An open and interesting problem is connected to the definition of a higher dimensional theory of noncommutative motives and in particular the set-up of a theory of noncommutative elliptic motives and modular forms. A suitable generalization of the yoga of motives to noncommutative geometry has already produced some interesting results in the form, for example, of an analog in characteristic zero of the action of the Weil group on the etale cohomology of an algebraic variety. It seems quite exciting to pursue these ideas further : the hope is that the motivic techniques, once suitably transferred in the framework of noncommutative geometry may supply useful tools and produce even more substantial applications than those obtained in the classical commutative context.

1 comment :
AC said...
Dear Katia
Thanks for this beautiful post. Your question was left unanswered for sufficiently long now, and I’ll try (why not) to give some answer in a coming post. Of course it will be some (partial) answer from my own point of view and as such it will have zero pretence to being “the” answer.
October 4, 2007 at 1 :21 PM

Wednesday, October 31, 2007
HEART BIT 1
Katia’s last post ended with a provocative question motivated by Grothendieck’s description in *Récoltes et Semailles* of the “heart of the heart” of arithmetic geometry, namely the theory of motives. Her question was formulated like this :
——— What is the “heart of the heart” of noncommutative geometry ? ———
I’ll try to explain here that there is a definite “supplément d’âme” obtained in the transition from classical (commutative) spaces to the noncommutative ones. The main new feature is that “noncommutative spaces generate their own time” and moreover can undergo thermodynamical operations such as cooling, distillation etc.

This opens up completely new ways of handling geometric spaces and our work with Matilde Marcolli and Katia Consani is just one example of potential applications to number theory. It is closely related to the Riemann zeta function and is very close in spirit to Grothendieck’s ideas on motives so that it is not out of place in the present discussion of Katia’s question. The story starts by a qualitative distinction between spaces which comes from the classification (by von Neumann) of noncommutative algebras in types I, II and III. The commutative spaces are all of type I. When encoding a space X by an algebra A of (complex valued) functions on X one uses some structure on X to restrict the class of functions (e.g. to smooth functions on a smooth space) and the above distinction between types uses the coarsest possible structure which is the measure theory. The corresponding algebras (called von Neumann algebras) are quite simple to characterize abstractly : they are commutants in Hilbert space of some unitary representation. Since one can take the direct sum of algebras A and B, one can mix algebras of different types.

More precisely any von Neumann algebra decomposes uniquely as an integral of algebras which cannot be decomposed further and are called factors. A factor is a von Neumann algebra whose center is as small as it can be, namely is reduced to the complex numbers. The factors of type I are Morita equivalent to the complex numbers, and thus a type I factor really corresponds to the classical notion of “point” in a space X.

To understand geometrically what factors of type II and III look like, it is useful to describe the (von

Neumann) algebra A associated to the leaf space of a foliated manifold : (V, F). An element T of A assigns to each leaf an operator in the Hilbert space of square integrable functions on the leaf, and it makes sense to say that T is bounded, measurable, or zero almost everywhere. The algebraic operations are done leaf per leaf, and the algebra of bounded measurable elements modulo the negligible ones is a von Neumann algebra. The simplest example corresponds to the foliation whose leaf space is the noncommutative torus. It is the foliation of the two torus by the equation "$dy = a\ dx$" in flat coordinates. The corresponding von Neumann algebra is a factor when "a" is irrational and this factor is not of type I but of type II. To obtain type III examples one can take any codimension one foliation whose Godbillon-Vey invariant does not vanish. The integrable subbundle F defining a codimension one foliation is the orthogonal of a one form v and integrability gives dv as the wedge product of v by a one form w. The Godbillon-Vey invariant is the integral over V of the wedge product of w by dw when V is compact oriented of dimension three. In essence the form w is the logarithmic derivative of a transverse volume element and the GV invariant is an obstruction to finding a holonomy invariant tranverse volume element ie one which does not change when one moves along a leaf keeping track of the way the nearby leaves are developing.

More generally the factors of type II are those which possess a trace and those of type III are those which are neither of type I nor of type II. In the foliation context, a holonomy invariant tranverse volume element allows one to integrate the ordinary trace of operators and this yields a trace on the von Neumann algebra of the foliation.

Until the work of the Japanese mathematician Minoru Tomita, very few positive results existed on type III factors. The key result of Tomita is that a cyclic and separating vector v for a factor A in a Hilbert space H generates a one parameter group of automorphisms of A by the following recipee :

one considers the modulus square $S * S$ of the closable operator S which sends xv to $S(xv) = x * v$ for any x in A, and then raises it to the purely imaginary power "it". Tomita showed that the resulting unitary operator normalizes A and hence defines an automorphism of A. One obtains in this way a one parameter group of automorphisms of A associated to the choice of a cyclic and separating vector v. He also showed that the phase J of the above closable operator S yields an antiisomorphism of A with its commutant A' which coincides with JAJ. In his account of Tomita's work, Takesaki characterized the relation between the state defined by the cyclic and separating vector v and the one parameter group of automorphisms of Tomita as the Kubo-Martin-Schwinger (KMS) condition, which had been formulated in C^*-algebraic terms by the physicists Haag, Hugenholtz and Winnink.

The key result of my thesis (in 1972) is that the class modulo inner automorphisms of the Tomita automorphism group is in fact independent of the choice of the (faithful normal) state that is used in its construction. Needless to say it is this uniqueness that allows to define invariants of factors. The simplest is the subgroup $T(A)$ of R which is formed of the periods, namely the set of times t for which the corresponding automorphism is inner. This, together with the spectral invariant $S(A)$, led me to the classification of type III factors into subtypes III_s for s in $[0, 1]$ and the reduction from type III to type II and automorphisms done in my thesis except for the case III_1 which was later completed by Takesaki. All of this goes back to the beginning of the seventies and will suffice for this first heart beat. It is only the beginning of a long saga which is far from over hopefully, and whose main theme is this mysterious generation of an intrinsic "time" that emerges from the noncommutativity of a von Neumann algebra. Exactly as manifolds come with a natural "smooth" measure class, a noncommutative space X generally gives rise to a von Neumann algebra A which encodes the natural measure class on X. It is thus a totally new feature of the noncommutative world that the corresponding time evolution is well defined and gives a canonical homomorphism :

$$\delta : \mathbb{R} \to \mathrm{Out}(A)$$
$$1 \to \mathrm{Int}(A) \to \mathrm{Aut}(A) \to \mathrm{Out}(A) \to 1$$

where the second line gives the definition of the group of outer automorphisms $\mathrm{Out}(A)$ of A as the quotient of the group $\mathrm{Aut}(A)$ of automorphisms by the normal subgroup $\mathrm{Int}(A)$ of inner automorphisms (which are obtained by conjugating by a unitary element of the algebra A).

AC said...
Dear Urs
What you need to understand is that all the interesting stuff here occurs when the number of degrees of freedom involved is infinite. A typical example is quantum statistical mechanics (such as a spin system

on a lattice). Systems occuring in quantum field theory, in the examples related to prime numbers are all involving infinitely many degrees of freedom and are most often of type III. Very simple quantum mechanical systems are of type I, of course and the deeper structure does not appear there. It has nothing to do with anomalies.

AC said...
Dear Urs
The time evolution is as "canonical" as it can be since any noncommutative algebra has inner automorphisms. Moreover one can show that the time evolution belongs to the center of $\mathrm{Out}(A)$!

If you take very simple examples as the lattice case you will find that an inner automorphism essentially ony affects what happens on finitely many lattice sites. In a simple translation invariant product situation, the hamiltonian (which generates the time evolution we are talking about) is an infinite sum of contributions of lattice sites and its essence is unaltered by a perturbation coming from finitely many terms in the sum. It is the fact that the sum is infinite and does not belong to the algebra of observables that creates the type III behavior.

You can slightly perturb this time evolution by an inner automorphism but its overall global action on the algebra of observables will remain essentially unaltered, since it will only be changed on finitely many of the degrees of freedom. Put in other words this "time evolution" of the algebra is taking place overall, on all degrees of freedom, whereas inner automorphisms only control a total of finitely many such degrees of freedom !

You need to carefully study various examples, including foliations, the set of primes, or the case of QFT to appreciate what is going on... (and I need to get some sleep at this point)...
October 31, 2007 at 10 :16 PM

AC said...
Dear Urs
It is not really nicely spelled out anywhere, so the best is to understand the basic idea in an example without entering in the technicalities. Consider the spin system on an infinite lattice.
The algebra of observables is the inductive limit of the finite dimensional algebras that come from tensor products of matrix algebras over finite subsets of the lattice. By construction these only involve finitely many lattice sites at a time. Thus an inner automorphism - since it is implemented by a unitary element of the algebra - really only "sees" finitely many degrees of freedom.
November 1, 2007 at 5 :49 PM

AC said...
Dear Anonymous
The space-time which allows to recover the Standard Model coupled to gravity is of type I, since it is the product of a manifold M by a finite space F ie a space whose algebra of coordinates is finite dimensional. It is not at this level that we expect to get "emergent time" but rather at the level of the algebra of observables in QG. The origin of this idea comes from Carlo Rovelli who - completely independently from the KMS story - had found by reflecting about basic philosophical issues in QG that the "time we feel" (as opposed to a time coordinate in space-time) should be of thermodynamical nature and should be tied up to a thermal state : the heat bath of the relic photon radiation which breaks naturally Lorentz invariance. The real thing now is to put one's hands on a good model for an algebra of spectral observables in QG. Some ingredients towards this are explained at the end of our forthcoming book with Matilde Marcolli. But I'd rather tell the story in one of the coming "heart beats" rather than explain it in a comment...
November 2, 2007 at 2 :42 PM

AC said...
Dear Anonymous
I don't like to be too negative in my comments. Li's paper is an attempt to prove a variant of the global trace formula of my paper in Selecta. The "proof" is that of Theorem 7.3 page 29 in Li's paper, but I stopped reading it when I saw that he is extending the test function h from ideles to adeles by 0 outside ideles and then using Fourier transform (see page 31). This cannot work and ideles form a set of measure 0 inside adeles (unlike what happens when one only deals with finitely many places).
July 3, 2008 at 7 :50 AM

AC said... 1/8/8

As an epilogue of this long-overview articles on the Workshop at Vanderbilt University we would like to thank all the speakers for their spontaneous and generous participation and for sharing their ideas with us about the field with one element and the new connection with NCG. We also would like to thank all the participants for coming to the talks and patiently listening to the discussions which were at times intense and certainly "very alive" and stimulating...

Monday, August 4, 2008
IRONY

In a rather ironical manner the first Higgs mass that is now excluded by the Tevatron latest results is precisely 170 GeV, namely the one that was favored in the NCG interpretation of the Standard Model, from the unification of the quartic Higgs self-coupling with the other gauge couplings and making the "big desert" hypothesis, which assumes that there is no new physics (besides the neutrino mixing) up to the unification scale. My first reaction is of course a profound discouragement, mixed with an enhanced curiosity about what new physics will be discovered at the LHC.
I'll end with these verses of Lucretius :
Suave, mari magno turbantibus aequora ventis,
e terra magnum alterius spectare laborem ;
non quia vexari quemquamst jucunda voluptas,
sed quibus ipse malis careas quia cernere suave est.

[Pleasant it is, when over a great sea the winds trouble the waters,
to gaze from shore upon another's tribulation :
not because any man's troubles are a delectable joy,
but because to perceive from what ills you are free yourself is pleasant.]

Tuesday, February 21, 2012
Galois

This is just a very short post for those interested in a basic talk about Galois, his relations with the French mathematicians of his time, and a general introduction to the "theory of ambiguity". The talk is in French, available at http://www.alainconnes.org/fr/videos.php
Do not forget to click on the "HD" symbol on the screen to get a better quality of video..

Saturday, June 9, 2012
SAD NEWS

It is with profound sadness that we learn about the sudden death of Jean Louis Loday who fell by accident off his sailing boat on June 6th. We loose an outstanding mathematician with so many great achievements and a wonderful friend of many years.

Friday, August 10, 2012
A DRESS FOR THE BEGGAR ?

Since 4 years ago I thought that there was an unavoidable incompatibility between the spectral model and experiment. I wrote a post in this blog to explain the problem, on August 4 of 2008, as soon as the Higgs mass of around 170 GeV was excluded by the Tevatron. Now 4 years have passed and we finally know that the Brout-Englert-Higgs particle exists and has a mass of around 125 Gev.
In the meantime the problem of this discrepancy in the Higgs mass seemed very hard to resolve and this certainly slowed down quite a bit the interest in the spectral model since there seemed to be no easy way out and whatever one would try would not succeed in lowering the Higgs mass. The reason for this post today is that this incompatibility has now finally been resolved in a fully satisfactory manner in a joint work with my collaborator Ali Chamseddine, the paper is now on arXiv at http://fr.arxiv.org/pdf/1208.1030
What is truly remarkable is that there is no need to modify the spectral model in any way, it had already the correct ingredients and our mistake was to have neglected the role of a real scalar field which was already present and whose couplings (with the Higgs field in particular) were already computed in 2010 as one can see in http://fr.arxiv.org/pdf/1004.0464
This completely changes the perspective on the spectral model, all the more because the above scalar field has been independently suggested by several groups as a way for stabilizing the Standard Model in spite of the low experimental Higgs mass. So, after this fruitful interaction with experimental results, it is fair to conclude that there is a real chance that the spectral approach to high energy physics is on the right track for a geometric unification of all known forces including gravity.

A few words about the picture, the metaphor of the Standard Model as a beggar with a diamond inits pocket was suggested by Daniel Kastler a long time ago, so this explains the character on the right. The character on the left wears the symbols of NCG, ingredients of spectral nature which allow one to reconstruct the geometry from gravitational observables such as the spectrum of the Dirac operator, and to write down the action of the Standard Model coupled to gravity.

Tuesday, October 30, 2012
THE MUSIC OF SPHERES
The title of this post, the music of spheres, refers to a talk *The music of shapes* https://www.dailymotion.com/video/xuiyfo which I gave in Lille, on the 26th of September, on the occasion of a joint meeting with the Fields Institute. The talk is an introduction to the spectral aspect of noncommutative geometry and its implications in physics.

The starting point is the naive question “Where are we ?”, or how is it possible to communicate to aliens our position in the Universe. This question leads, in the Riemannian framework of geometry, to that of determining a complete set of geometric invariants, both for a space and for a point in a space. The theme of Mark Kac, “Can one hear the shape of a drum ?” associates to a shape its musical scale which is the spectrum of the square root of the Laplacian, or better of the Dirac operator. After illustrating this familiar theme by many concrete examples we give a hint of the additional invariant which allows one to recover the geometric picture, namely the CKM invariant, and illustrate it, in a simplified form, in the simplest possible example of isospectral but non congruent shapes.

What about the relation with music ? One finds quickly that music is best based on the scale (spectrum) which consists of all positive integer powers q^n for the real number $q = 2^{1/12} \sim 3^{1/19}$. Due to the exponential growth of this spectrum, it cannot correspond to a familiar shape but to an object of dimension less than any strictly positive number. As explained in the talk, there is a beautiful space which has the correct spectrum : the quantum sphere of Poddles, Dabrowski, Sitarz, Brain, Landi et all. Its spectrum consists of a slight variant of the q^j where each appears with multiplicity $O(j)$. (See the original paper of Dabrowski and Sitarz
arXiv:math/0209048 (Banach Center Publications, 61, 49-58, 2003)
for the precise formula, and the paper of Brain and Landi
arXiv:math/1003.2150 for a variant and the many references to the mathematicians involved, my apologies to each of them for not puting the list here.)

We experiment in the talk with this spectrum and show how well suited it is for playing music. The new geometry which encodes such new spaces, is then introduced in its spectral form, it is noncommutative geometry, which is then confronted with physics. There the core is the spectral Standard Model of A. Chamseddine and the author which goes back to 1996. We tell the tale of the resilience of this model in its successive confrontations with experiments. Both the start and the end part of the talk are unusual. The previous talk was a talk by Alain Aspect on his recent experiments, with his collaborators, confirming experimentally the "delayed choice" Gedankenexperiment of John Wheeler. So the very beginning of my talk refers to Aspect's point about the subtelty of the concept of "reality" implied by the quantum. The thesis which I defend briefly is that the total lack of control that we have on the outcome of a quantum experiment (we control only the probabilities), is a "variability" which is more primordial than the classicalvariability governed by the passing of time (on which we have no control either). I also explain briefly why time will emerge from the quantum variability. The end part, in the question session, is also unusual, it is a long answer to a question which was posed by Alain Aspect.

The three speakers, Lille 9/26 : E. Ghys, A.Aspect, A. Connes
Update : The talk of Alain Aspect is now also available at the conference website
https://www.youtube.com/watch?v=vqEg4VnoCmc.

Sunday, November 9, 2014
PARTICLES IN QUANTUM GRAVITY
The purpose of this post is to explain a recent discovery that we did with my two physicists collaborators Ali Chamseddine and Slava Mukhanov. We wrote a long paper *Geometry and the Quantum : Basics* https://arxiv.org/abs/1411.0977 which we put on the arXiv, but somehow I feel the urge to explain the result in non-technical terms. The subject is the notion of particle in Quantum Gravity. In particle physics there is a well accepted notion of particle which is the same as that of irreducible representation of the Poincaré group. It is thus natural to expect that the notion of particle in Quantum Gravity will involve irreducible representations in Hilbert space, and the question is "of what ?".

What we have found is a candidate answer which is a degree 4 analogue of the Heisenberg canonical commutation relation $[p, q] = ih$. The degree 4 is related to the dimension of space-time. The role of the operator p is now played by the Dirac operator D. The role of q is played by the Feynman slash of real fields, so that one applies the same recipe to spatial variables as one does to momentum variables. The equation is then of the form $E(Z[D, Z]^4) = \gamma$ where γ is the chirality and where the E of an operator is its projection on the commutant of the gamma matrices used to define the Feynman slash.

Our main results then are that :

1) Every spin 4-manifold M (smooth compact connected) appears as an irreducible representation of our two-sided equation.

2) The algebra generated by the slashed fields is the algebra of functions on M with values in $A = M_2(H) \oplus M_4(C)$, which is exactly the slightly noncommutative algebra needed to produce gravity coupled to the Standard Model minimally extended to an asymptotically free theory.

3) The only constraint on the Riemannian metric of the 4-manifold is that its volume is quantized, which means that it is an integer (larger than 4) in Planck units.

The result 1) is a consequence of deep results in immersion theory going back to the work of Smale, and also to geometric results on the construction of 4-manifolds as ramified covers of the 4-sphere, where the optimal result is a result of Iori and Piergallini asserting that one can always assume that the ramification occurs over smooth surfaces and with 5 layers in the ramified cover. The dimension 4 appears as the critical dimension because finding a given manifold as an irreducible representation requires finding two maps to the sphere such that their singular sets do not intersect. In dimension n the singular sets can have (as a virtue of complex analysis) dimension as low as $n-2$ (but no less) and thus a general position argument works if $(n-2)+(n-2)$ is less than n, while $n=4$ is the critical value.

The result 2) is a consequence of the classification of Clifford algebras. When working in dimension 4, the sphere lives in five dimensional Euclidean space and to write its equation as the sum of squares of the five coordinates one needs 5 gamma matrices. The two Clifford algebras $Cliff(+,+,+,+,+)$ and $Cliff(-,-,-,-,-)$ are respectively $M_2(H)+M_2(H)$ and $M_4(C)$. Thus taking an irreducible representation of each of them yields respectively $M_2(H)$ and $M_4(C)$.

The result 3) comes from the index formula in noncommutative geometry. One shows that the degree 4 equation implies that the volume of the manifold (which is defined as the leading term of the Weyl asymptotics of the eigenvalues of the Dirac operator) is the sum of two Fredholm indices and is thus an integer. It relies heavily on the cyclic cohomology index formula and the determination of the Hochschild class of the Chern character. The great advantage of 3) is that, since the volume is quantized, the huge cosmological term which dominates the spectral action is now quantized and no longer interferes with the equations of motion which as a result of our many years collaboration with Ali Chamseddine gives back the Einstein equations coupled with the Standard Model.

The big plus of 2) is that we finally understand the meaning of the strange choice of algebras that seems to be privileged by nature : it is the simplest way of replacing a number of coordinates by a single operator. Moreover as the result of our collaboration with Walter van Suijlekom, we found that the slight extension of the SM to a Pati-Salam model given by the algebra $M_2(H) \oplus M_4(C)$ greatly improves things from the mathematical standpoint while moreover making the model asymptotically free ! (see Beyond the spectral standard model, emergence of Pati-Salam unification.) To get a mental picture of the meaning of 1), I will try an image which came gradually while we were working on the problem of realizing all spin 4-manifolds with arbitrarily large quantized volume as a solution to the equation.
"The Euclidean space-time history unfolds to macroscopic dimension from the product of two 4-spheres of Planckian volume as a butterfly unfolds from its chrysalis."