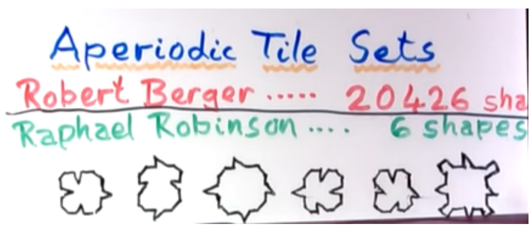
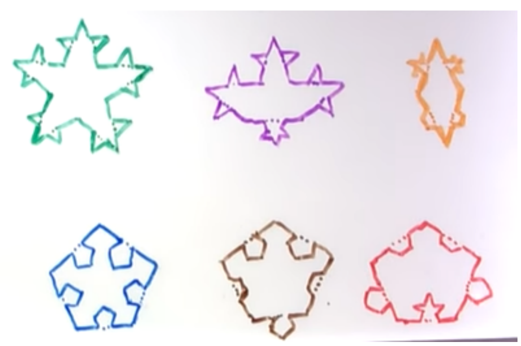
Forbidden symmetries

Sir Roger Penrose

(20 minutes in the middle of the talk)

Now there's another bit to the story, which is somewhat different. This was a set of tile shapes (I won't go into the whole story, it was quite a long interesting story to do with mathematical logic) and in the course of this story, an American mathematician called Robert Berger discovered a set of tooth 20426 shapes which would only tile the plane, tile the entire plane, but only in a way which never repeated itself. And he needed, to do that, to show that the tiling problem is not computable, that was actually what he was trying to do. But that's not part of my talk here. But Raphael Robinson who's another American mathematician managed to reduce this number from 20426 just to 6, and so, this was pretty impressive. It had gone down his steps from some other people but this was his achievement to do it with 6.

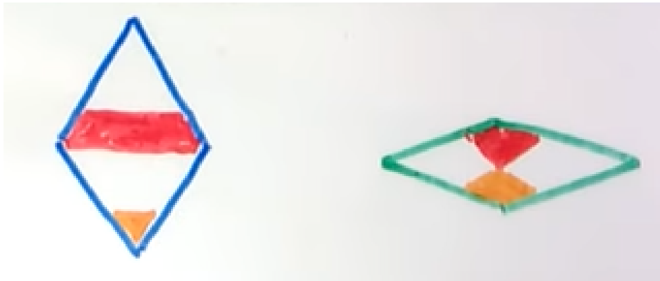


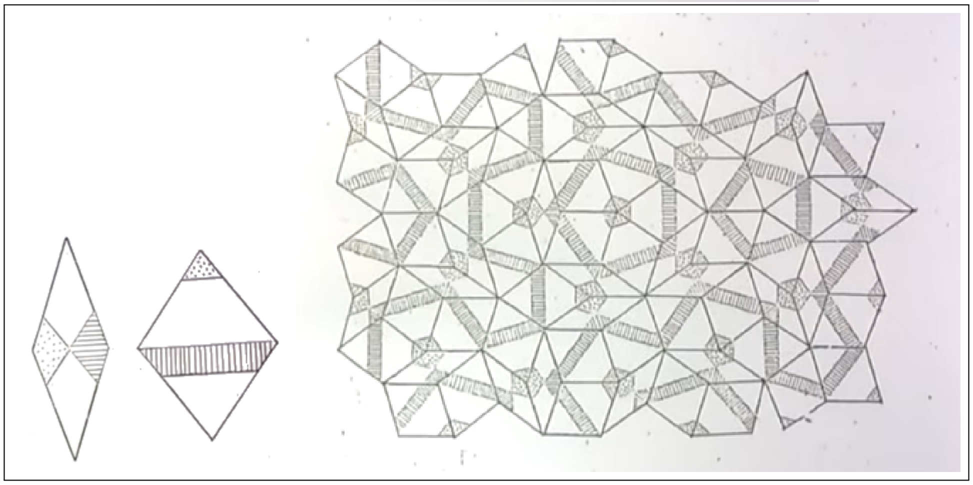


And I was talking to another American mathematician Sammy Cochin and he told me that Raphael Robertson was some do you like to get the numbers done small as possible. And I thought well 6, I see, I know I can do it with 5. Well, you see, these are the shapes that you get in the tiling pattern I was showing you here (you see, you'll get there the pentagons the rhombuses, the justice caps and the pentacores). But if you want to make a tiling problem which forces this arrangement, you can do it by putting little knobs and notches on the pieces, and that forces them to fit into that arrangement. That is I won't try to prove here but that's true. But you need three different versions of the pentagons, depending upon whether there's five other Pentagon's next to it only three or only two and that's the difference between the three kinds of pentagons. But the thing is you'll notice that this pentagon here[1] has that funny little thing there on the bottomfootnoteLike a little star. and this thing has that funny little thing and this is here, in two of them, so all you need to do is to take that one, glue it on there, and glue it twice on there, and you've only got five pieces. So I knew I could do it with five then I started fiddling around from it, and I realized you could do it with two ! *(Laughs)* And so that's the two and the way it works... Well, I had another version first, which I'll give it to you : you have to match the colors. You can do it with little knobs and not just like this as well, but if you just match the colors and here, we have a version which is pouring the uncovered, but you can see underneath is that tiling pattern.

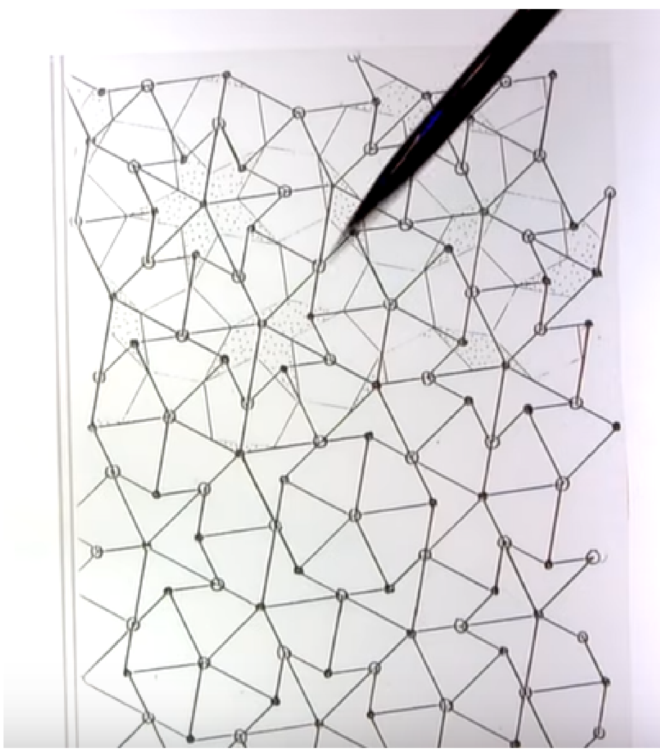
Transcription par Denise Vella-Chemla, octobre 2022, de la vidéo *Forbidden crystal symmetry in mathematics and architecture* visionnable ici : https://www.youtube-nocookie.com/embed/th3YMEamzmw, Royal Institution (RI) Event, 2013.

[1] The red one, on the left.

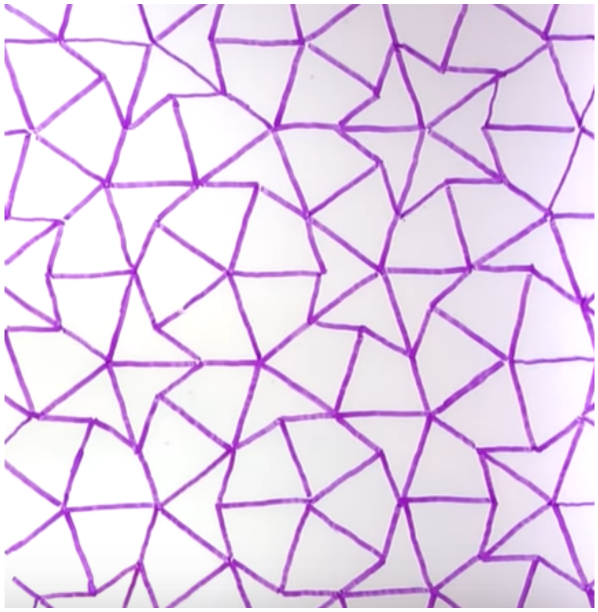




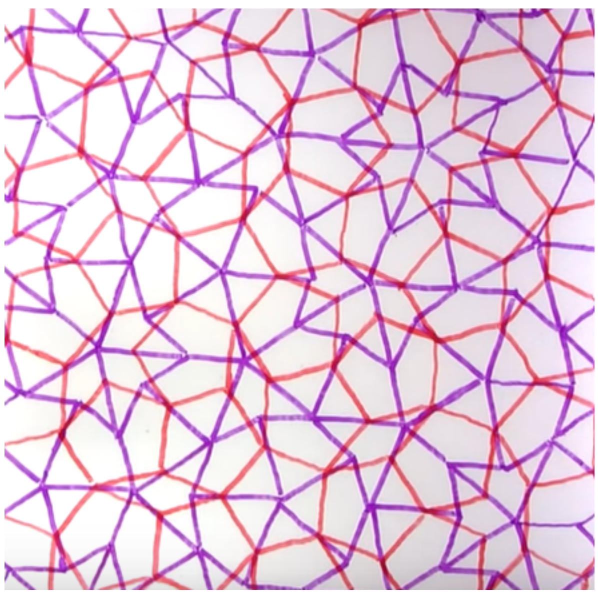
Well I'm going to show how these others are related to the pentagon pattern here, they're all part of the same story to make sure and keep these things in sight. And let's see something about the story. But first of all let me mention the other one which came first which is kites and darts, these are the kites and the darts down here, and at the top I hope you can see, you put the kites, you mark the kites and the darts ; each dart is marked the same way and each kite's mark the same way, and they fit together to form the tiling which I just showed you.



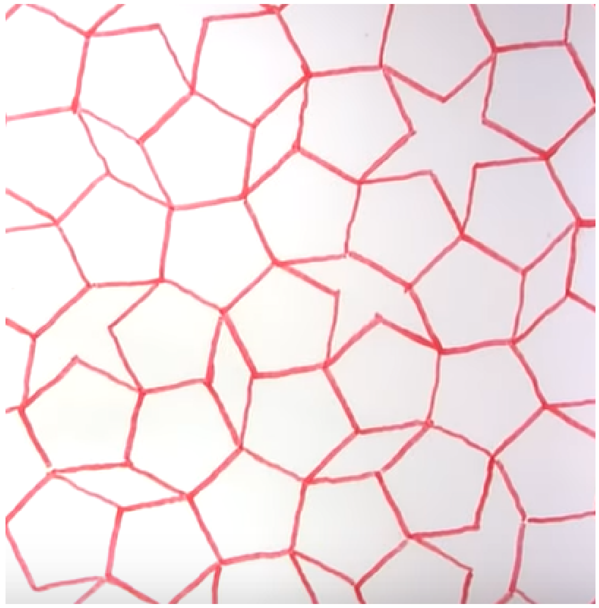
I think it would be better if I show you their kites and darts assembled



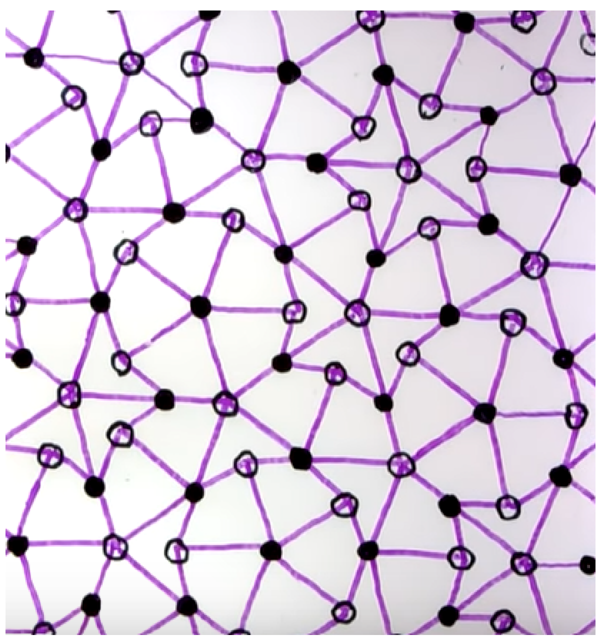
and here we have the pentagons.



So there we are, I think that's it, and if you look carefully, you'll see that every kite is marked the same way, and every dart is marked the same way. So this pattern of kites and darts is equivalent to this pattern :



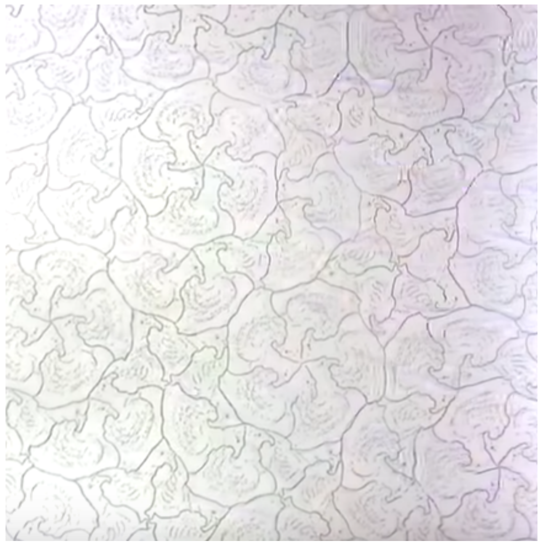
It produces this but only with only two shapes. So that's how you can do it with two. Of course I haven't put the notches and the knobs on it let me do that by putting whether actually I put knobs on here with black and white corners colors. So I can do that here.



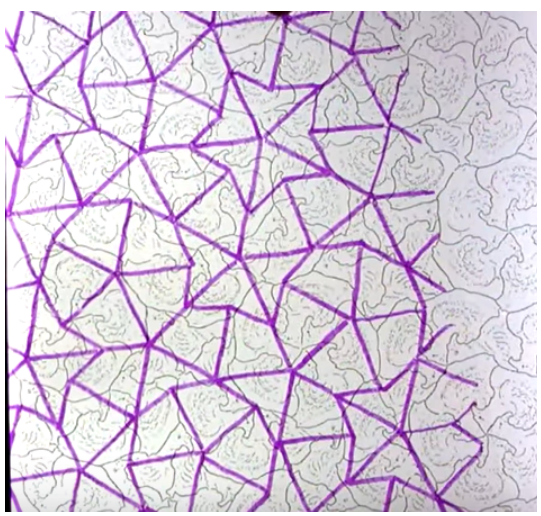
I just noticed that there is an opposite way from in that other picture but it doesn't matter. But here we have the kites and darts I'm now going to mark the corners and if you match those corners that the only way of putting them together is one of these non periodic arrangements.

I should perhaps make a point about this : does that mean that there's only one way of covering the entire plane ? Yes and no is the answer to that : strictly speaking mathematically, the answer is no, that is to say there are many ways of continuing to infinity. However in a certain finite sense, they're all same. That is to say if you find any two of these complete tilings of the plane, with the same shapes that is, and I take one of them and I take a region no matter how big, that's finite, I can find that, in the other one. So you can't tell ever which one you're in. It doesn't matter how big the region is, you can always find it in the other one, infinitely many times. okay that's not so hard to prove actually but curious. Okay.

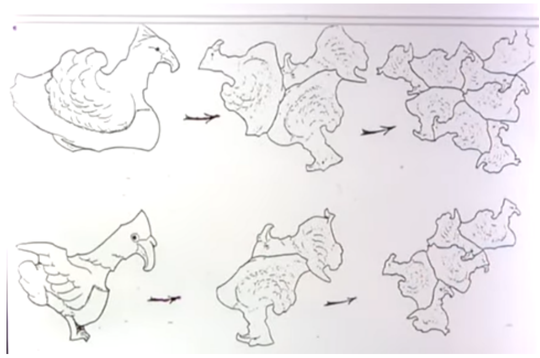
You can also do it by making a jigsaw puzzle instead of the notches and knob : jigsaw puzzle, okay, here's one : that's a sort of Escher-ization I would say.



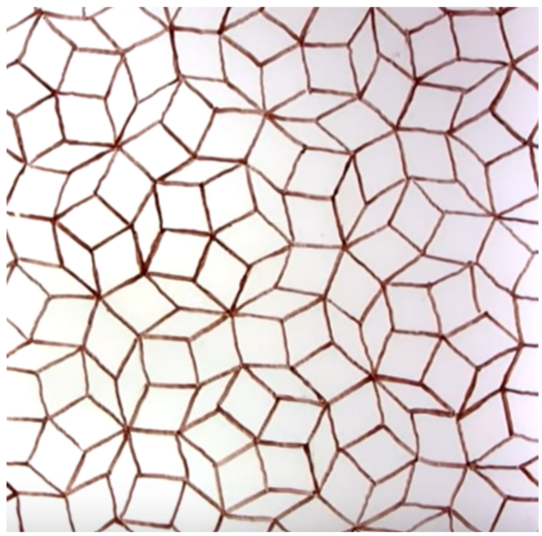
I'm not sure this will match on that, I didn't try to get them to match ; oh, that's not too bad, they might do but I haven't actually tried that. There are about the same, you see that if I concentrate on that bird there ; it's that one and that one, they won't go all the way around, but yes, there we are, these ones fit but I'd have to find a better place to match them.



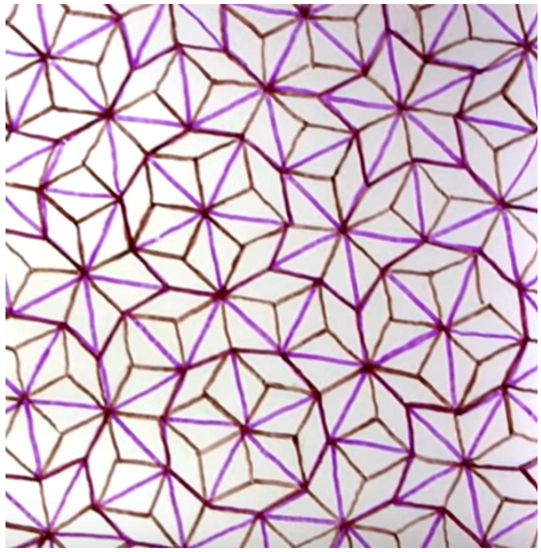
But as I said, any finite region which will be in tips etc, etc, but if you want to know how to do it, you do it this way, they're really kites and darts disguised, and then there's a hierarchy which you can infer *(Laughs)*.



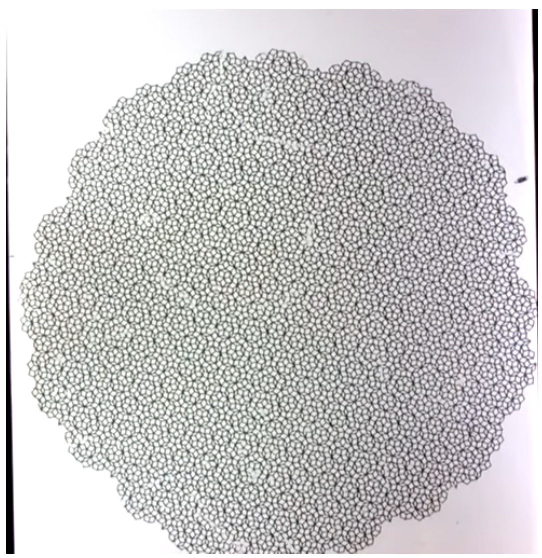
I won't go into that here, but just to show you those sort of things you can do. Now here is a rhombus pattern.



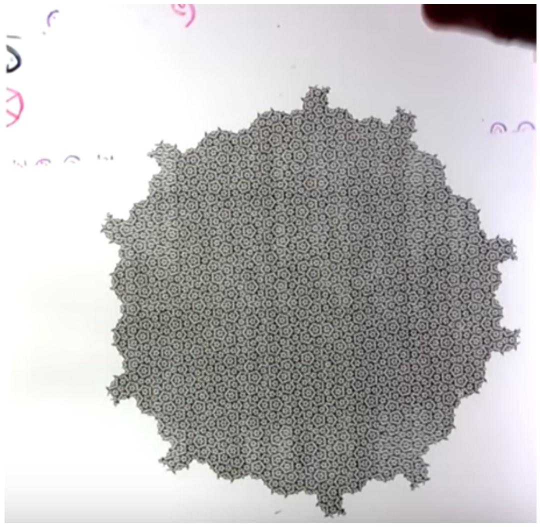
When people use it, they tend to do it without any adornments. I prefer it when you see where you are ; the trouble with this if you just take it not as a jigsaw puzzle (there are zillions of ways of doing of course because you can tile with either one, without using the other one, and there are many ways of doing that. So you have to have some rule about how you fit them together and the rule could just be well it's that pattern but that's not very economical. You could say the rule is that they have to match the stripes which I showed you before and so on there's also a close relationship between the kites and darts and the rhombus one :



You see that many of the lines are in common between the two. Okay, let's go back to the rhombus pattern again, and then "memorize that pattern" *(Laughs)* and here is a rather finer one.

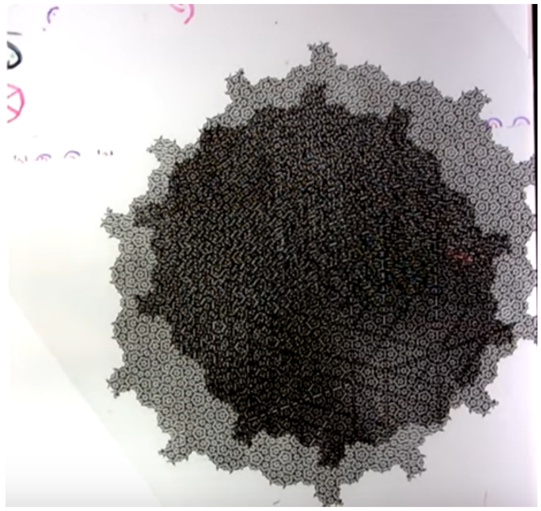


I hope you can see the rhombuses all right, I'm afraid there's some places where it's got a little worn. But that is indeed what it is. Of course you see I did these things were by hand, but of course now with the computer, people take over and they can make much more impressive pictures than I ever could. So that's a computer picture. I'm going to show you another computer picture you can go even finer than that.

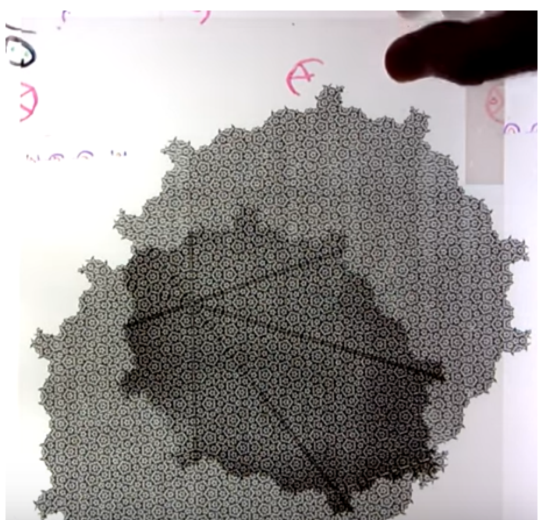


And now that *is* rhombus pattern ; I don't think you can quite see the rhombuses, but what you probably can see is moiré patterns.

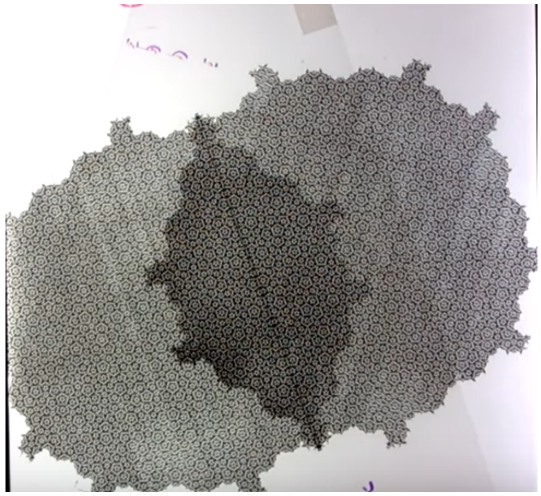
Now it's quite interesting because you can get one of them in the middle so it takes a little skill to do this, and I may have lost it because I yeah, you have to move the thing at right angles so the way where you want to move it, so it's tricky, I'm trying to get that spot in the middle.



Now you see various lines going across. Those lines are the places where the patterns differ ; where they agree are the spaces in between. Now there's another spot, let's try that one, I'm not sure I'm getting any better at it, but I'll try. I think that the lines are further well. I'm going to cheat now, I'm going to use some little guide marks at the edge. Now I'm not sure that I get it accurately enough, it's quite hard to do, ah yes you may see there, yeah !

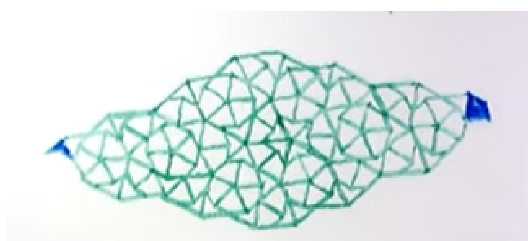


It agrees everywhere except along those lines there, and if I move it far enough away, I can make the patch where degrees as bigger as big as you like, I think I might just try one more here, whether I can actually find the right marks, I'm not sure, I'm up with the edge here to try and find it, let's hope that spreads it yep, it agrees everywhere, except that one line across the middle.



Except I'm not going to find it, you can do it this way I think, but then that's, I don't know, let me just match the things at the edge of the legend. Can you see it ? That's it, there we are, that's it, huh, okay, that's the demonstration, okay.

Now let me make a point ; I used to give lectures on these things a long time ago and in the 70s, and people would say "well doesn't that mean there's a sort of area of crystallography opening up ?" and I used to say, well yes in principle, that means you could imagine such things, but I couldn't imagine how nature would do it. Now there's a reason for saying that, and that is that the assembly can't be in the way you would imagine a crystal to be made, because this sort of classic crystal assembly is you have a little part that comes along and sort of sit in a little bridge one by one, and then they come on another layer, and another like that, that's a local assembly. Now you see the assembly is never local, and I knew this because suppose we have that pattern :

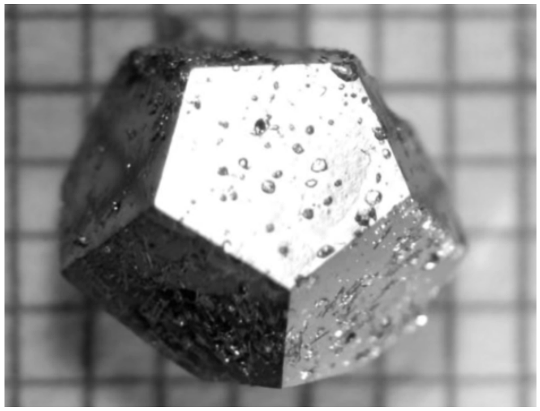


These are cats and dogs and that it is correctly assembled, you could continue that to infinity, if you put a dart there you can still continue it to infinity if I take that dart off, and I put a kite here, you can still continue that to infinity. But if I put a dart there *and* a kite there, it goes wrong just about there *(showing a point 10 cm at the bottom of the picture, and then, laughs of the audience).* So it's a non-local feature. And it seems puzzling how you could get crystalline type substances to grow if it is this kind of tiling. So I was sort of a bit skeptical that maybe you could have such a thing. Well I think I want to move now over to the PowerPoint images.

This is a design for a poster which was in the Mathematical Institute, where we're moving to this other building, and I don't think it's been put up in the new building yet, but the idea was to have some assembly of these tilings. These are all tilings that were given to me by people who manufactured individual tiles, the ones in the middle were given by a mathematician called Ron Graham, I think he just wanted to play with these ideas, and he put little knobs on them so that you could fit them together. You're not supposed to turn them over, otherwise you can do it the wrong way too. These are some of his, too, so you could see the kites and the darts. this kite and dart pattern is also that I have five different colors and the coloring of the pattern is uniquely determined by the tiling. I won't go into what it is but it makes that the pattern of colors is absolutely fixed by the tiling pattern, with certain rules about it. You may have been able to see things sort of jumping out at you when you look at the pattern. Here we have a rhombus version and here we have the rules about the hierarchy how they work.
I would probably not go into that in detail.

Here we have some amusement with I showed the birds to you those are the birds these are really the birds kites and darts, you can see them as individual birds the other side, well not the other side but the alternative way of coloring it, brings back the original pentagon tiling. You notice there's a little foreign creature in the middle, there's a dog there ; if you put one dog in this pattern, the entire pattern running out to infinity is completely unique. There's exactly one way of doing that is quite curious. It's a little bit hard to see what's going on with that pattern. Down in the corner here, we have some actual materials this is an actual quasi-crystal. I think they now call these things crystals instead of quasi-crystals, I'm not quite sure : notion of crystal has extended to include these things, but you see this beautiful regular dodecahedron, which is certainly not allowed in ordinary crystallography. And here we have the diffraction pattern these things were originally discovered by ? by looking at the diffraction patterns. And if you look carefully, you see that there are pentagons and things, in the diffraction pattern itself.

There is a bigger picture of that very beautiful run become very beautiful dodecahedron,



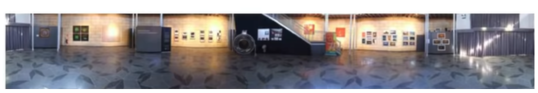
a regular dodecahedron and with very nice edges and corners. And it is believed that the atomic arrangements are of the kind that you've seen here. There are lots of versions of these atomic arrangements that you can have but I don't know in this particular case what arrangement this is ; it's only a three-dimensional version of course, I think Robert Ammann was the first person to produce a three-dimensional version of the pentagonal tilings but then various mathematicians discovered very sophisticated ways of generating these things by taking lattices and high dimension slicing them and projecting them, I don't want to go into all that here.

Let me see what the next is.

Okay, now let's see some architecture this was the first use I know of of any of these tilings. This was done quite soon after I produced them by a Japanese architect, I was really quite impressed with him because there are all sorts of things in its kites and darts it's little hard to see at first because they're decorated by certain arcs which was suggested by John Conway a mathematician to produce nice patterns. But here you have two darts, two kites and one dart, and there's a dart, five kites and so on. But the architect noticed when they put this thing up that there was a mistake in it and you insisted they took it down and corrected the mistake so I was impressed with that, that he cared about that carefully enough.

But it's a very intriguing and interesting pattern, I was quite flattered to see this thing produced and there's a younger version of me standing there. Now this story hall in Melbourne in Australia. They wanted me to be there at the opening, I wasn't able to go, I've visited later on, it's the most extraordinary place... I'm not sure what I think of it.*(Laughs)*

Gravity Discovery Centre, Perth WA, Australia.



You can go inside and you find more of these things all over the place. So it is based onthis rhombus tiling and quite correctly done I think how, but it's elaborated and all sorts of curious ways so it's quite a an interesting building, but let's move on.

Now here we have... This must be in Australia huh I have a crib here which tells me no this is the one in the Science Centre, I think so too two of these tilings in Perth : the Science Center, I think this one's the Science Center, that's right, and it's straightforward rhombus tilings and you see them edge on, and there are two different, the fat and thin rhombus is there, it's a bit hard to see them in this picture, but you get some feeling for the extent of them and that's the shot taken from above.

Gravity Discovery Centre, Perth WA, Australia.

Yes, this is the Science Center in a place near Perth, in Western Australia. very impressive, I think they're very nicely done.

St. John's Cambridge, Penrose Building

Photo: Paul Everest

Now this is the kites and darts, this is in my old college St. John's College in Cambridge, and I think Peter Goddard who was the Master at the time sort of liked the pun of the idea, because this is the entrance to the Penrose building if nothing to do with any member of my family, as far as I know, but it happens to be somebody called Penrose who designed it. And the piece of wood in the middle of the picture here is the door, it's the entrance to the library and this is a sort of circular pattern on the floor, and the door here swings round, there's a pillar up the middle there, and it swings around. So you can't really see the whole pattern all at once, but you can swing the door around and see different parts of it. Here we have just seen the pattern it looks like,

St. John's Cambridge, Penrose Building

Photo: Paul Everest

So it’s straightforward kites and darts, it looks very nice.

Now, this is Helsinki and I shown this.

Tiles in Helsinki

And I’m a little puzzled by it because it took me a little while to realize what it was. At first sight, it looks like something else but when you look carefully you see these triangles and that shape, they make a dart ; so the darts are broken down, let’s see if I can find one, here, here there’s a dart, that’s the nearest one, there’s a dart that is cut in two places to make smaller tiles I suppose they didn’t like their tile shapes to be convex, I don’t know if that’s what it was, but that the the kites are complete kites. It was a really quite big area tiled in this way, I don’t think I’ve ever actually seen it, but that’s in Helsinki, in Finland, okay.