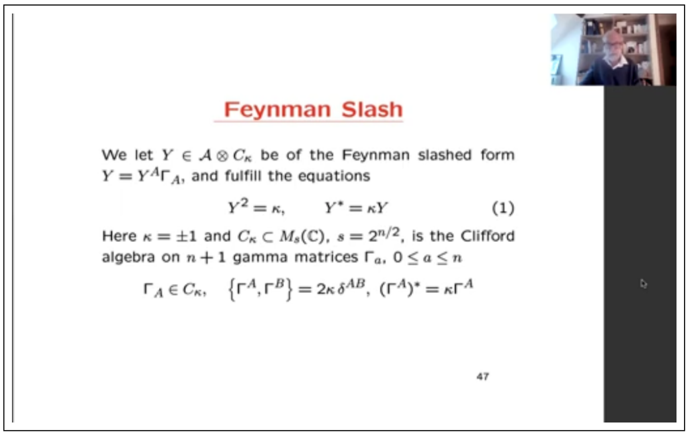
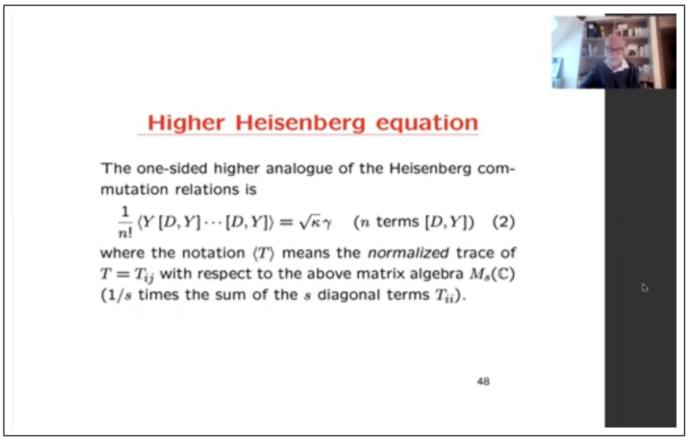
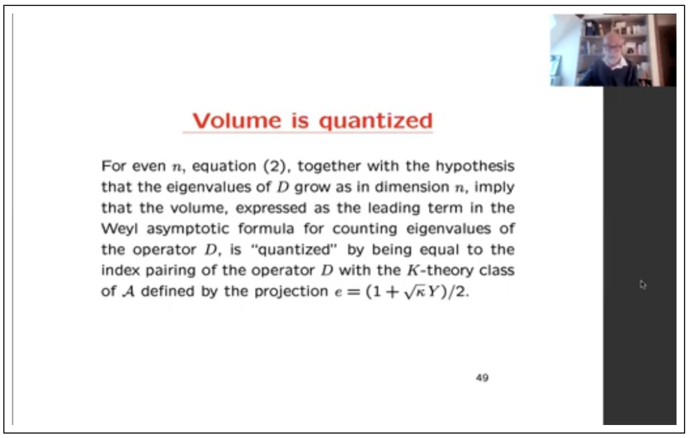


So, with Chamseddine and Mukhanov, we first started with a Feynman slash, we wrote down this higher Heisenberg equation,

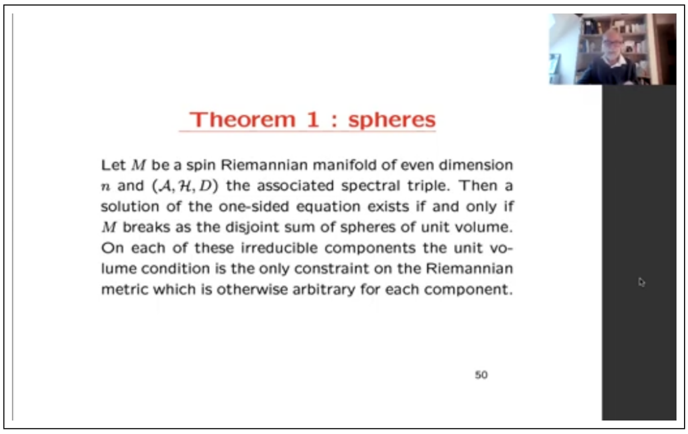


which resembles the equation for the circle, except that now, the commutator D commutator Y is raised to the power, which is the dimension of the space. So we wrote down this equation and we investigated this equation. And one of the first things that we found

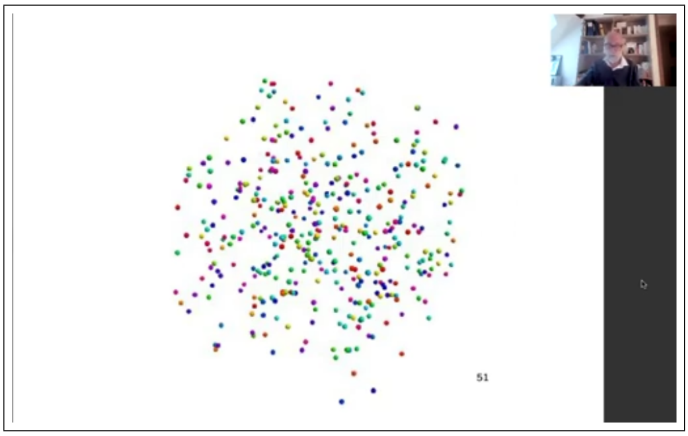


is that this equation, exactly like, in the case of the circle, it was giving you the length 2π, well, it quantizes the volume. So the volume which is given by the growth of the eigenvalues, or if you want, by the logarithmic divergencies of the trace of the right power, is quantized.

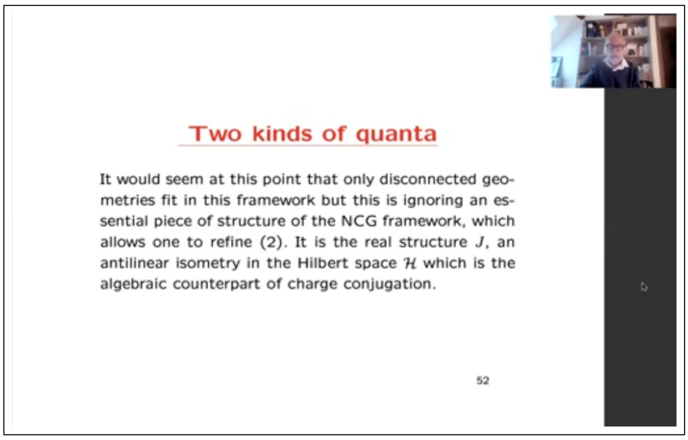
So this is the first thing.



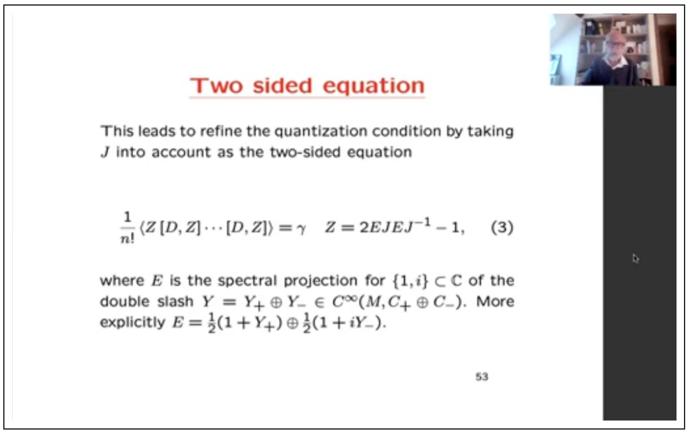
But then, we were a little bit disappointed because what we found is that when you have a solution to this equation, then, automatically, the solution, the manifold, will break as a disjunction of some spheres of unit volume. And if you work in physical units, you find that this unit volume is like the Planck volume.



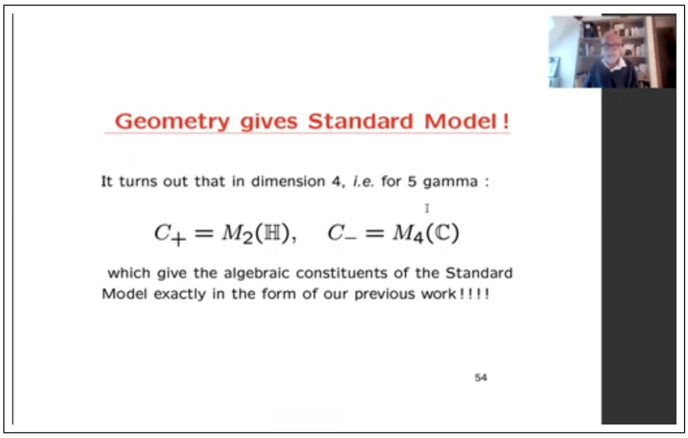
So at this point we were quite disappointed because we said "ok look. Space-time, euclidean or not, it doesn't look like that : it's not a union of spheres, very tiny little spheres." But we have forgotten the essential piece of structure, which is the J, which is charge conjugation, the real structure J.



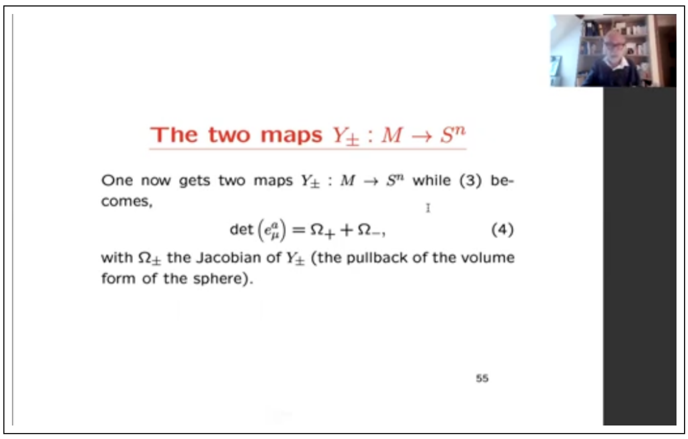
And when you incorporate the real structure J, what you find is that it automatically forces you to refine the higher Heisenberg equation. And because of this issue, KO-dimension 6, and so on and so forth, what do you find ? You find that you are forced to refine the equation by involving the J



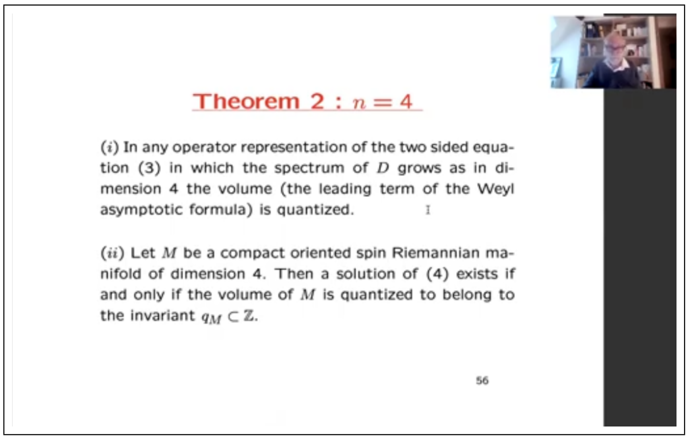
and the J is now involved by passing the projection coming from Y to the commutant. The equation becomes this equation. Now, what really came out of the blue is that all of this, of what I'm saying now, was inspired by the wish of trying to present the geometry in the simplest possible way, having this kind of pairing between the Dirac, and what you obtain by assembling the coordinates into a single operator.



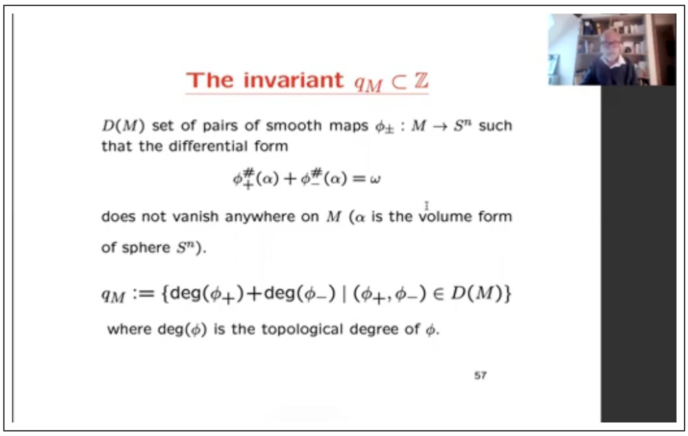
And now we looked at exactly what are the needed Clifford algebras in order to obtain this equation, to obtain the solution of this equation. We looked at the table of Clifford algebras, to find, in the case of dimension 4, so when you take 5 Gamma matrices, then you find that in order to write this, you have two Clifford algebras which appear, irreducibly. And the first one gives you in fact $M_2(\mathbb{H}) + M_2(\mathbb{H})$ but because you want to take an irreducible piece, you have $M_2(\mathbb{H})$. And the second is $M_4(\mathbb{C})$ here and they appear all together. They appear if you want as the sum of these two pieces C_+ and C_-.



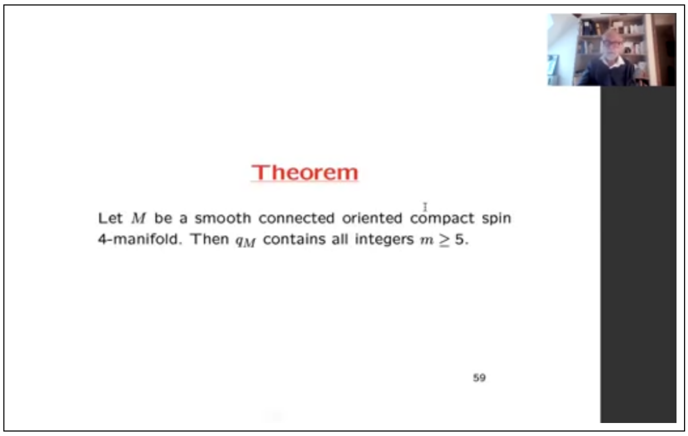
So in fact, out of the purely geometric problem, we found exactly the algebra that was as a sort of put by hand, you know, in our previous work, as a kind of bottom-up story. This was quite amazing but then of course, we had to go further, and we had to prove that we could obtain all possible spin-manifolds, from this construction and no longer a disjoint union of spheres. So what happens is that instead of having a single map from the manifold M to the sphere, and you know, because the sphere is simply connected in higher dimensions, you couldn't escape M to be itself a collection of spheres, but now you have two maps



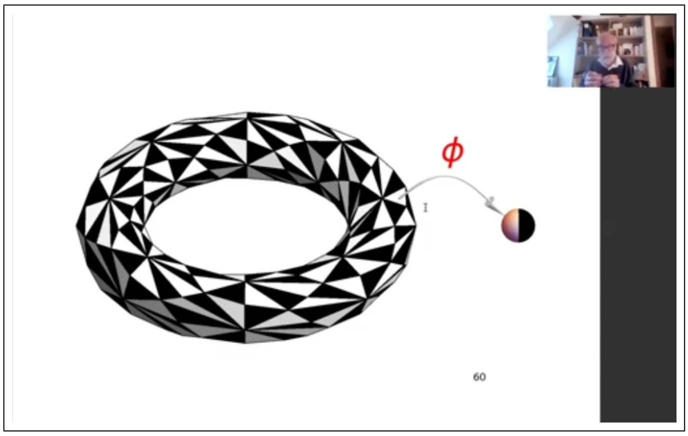
Y_+ and Y_- to the sphere, and the only condition is that when you pullback the volume form of the sphere by plus and by minus, it's not that individually they don't vanish, no, it's that their sum never vanishes. Their sum has to define a differential form that never vanishes, not individually, which of course is not possible, unless you are a sphere.
So very quickly, we obtained two results : we obtained the fact that the volume was quantized, I will come back briefly to that, but we obtain a much more precise fact, that is that if you take a compact oriented riemannian spin-manifold of dimension 4, then a solution of this equation exists if and only if the volume is quantized to belong to a certain invariant,



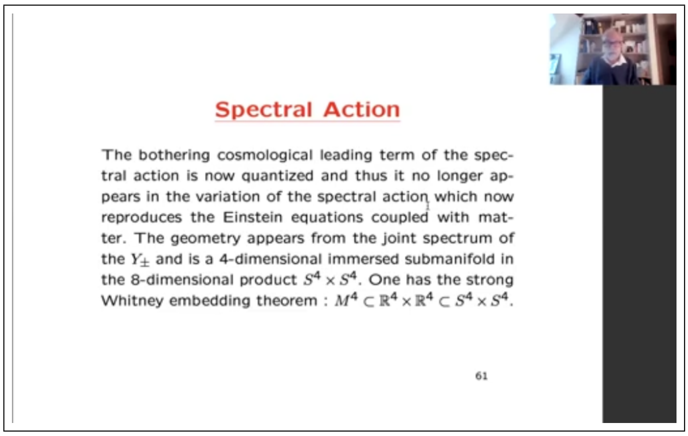
and this invariant is simply the sum of the degrees of this map ϕ_+ and ϕ_- which fulfills the condition that when you pullback the volume, you get something that doesn't vanish. Now after a lot of work, a lot of geometric work, which was using the existence of ramified covers of the sphere and alsousing the full power of the immersion theory which goes back to Smale, Milnor and Poenaru, in fact, a theorem of Poenaru that you have an open oriented manifold of dimension n, then you can immerse it in $\mathbb{R}^n$. Then we were able to prove that, in the case of dimension 4, for any spin-manifold, this invariant will contain all integers n bigger than 4. The case of dimension 2 and 3 is much easier by general transversality arguments but the case where $n = 4$ is much more difficult.



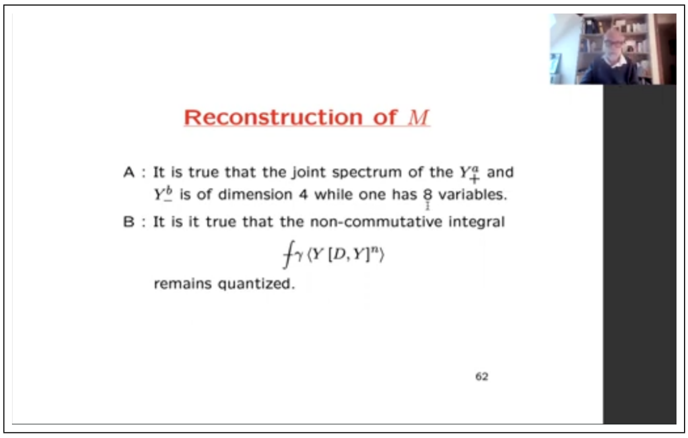
What happens is that now you can obtain any spin-manifold of any arbitrary volume. If you take a smooth connected oriented compact 4-manifold, then this invariant contains all integers bigger than 5 and what does it mean ?



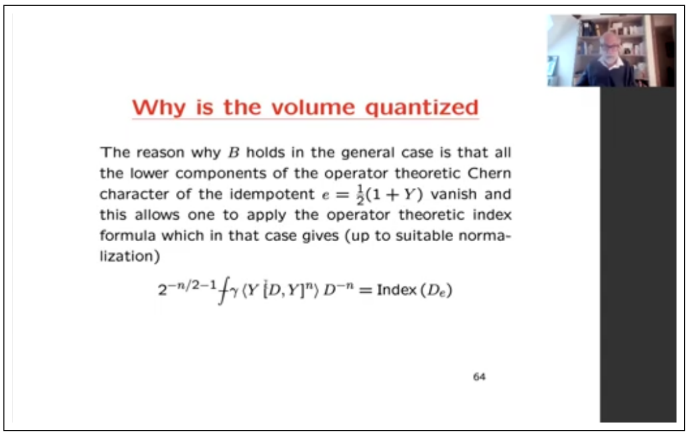
It means that you can sort of obtain this manifold from two little spheres of Planck size but of course, the manifold itself will sort of develop, and it will develop to arbitrary size. This is why we entitled the paper that we wrote with Ali Chamseddine and Mukhanov "*Quanta of Geometry*" because it's really what is going on. There are little quanta that mesh together to form this huge manifold. What happens also is that,



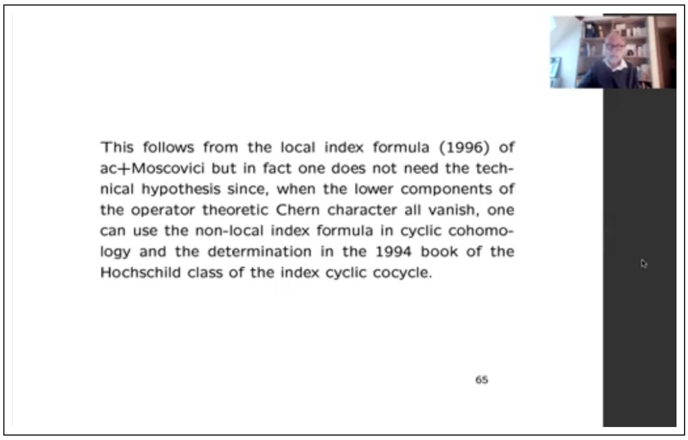
now because the volume is quantized, when you write down the spectral action as we had written with Ali, as I said, in the spectral action, what you have is that there is a cosmological term which is huge, and which is quite bothering. But now, because the volume is quantized, this cosmological term, which is the leading term of the spectral action, plays no role, when you write down the variational equation. And so, when you write down the variational equation, you really reproduce the Einstein action coupled with matter. The geometry is reconstructed as a joint spectrum, and it's a 4-dimensional sub-manifold of this 8-dimensional product of two very little spheres,



and there are rather general facts which are key, in order to do this reconstruction : there is the fact that the joint spectrum will be of dimension 4, this relies on a deep result of Dan Voiculescu, and also, it relies on the fact that the index theorem, will tell you that the volume will remain quantized.

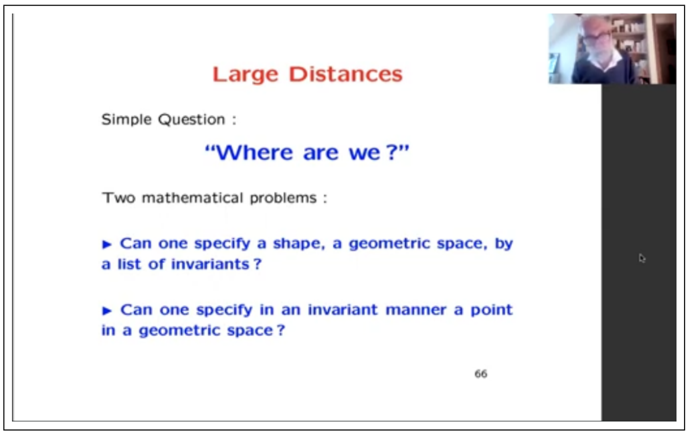


The fact that the volume remains quantized follows from the fact that you have the Heisenberg higher condition which gives you that these quantities will be equal to gamma, so gamma squared is 1, so they cancel on. But this is also an index. So the reason why it is an index relies on an index theorem that we proved

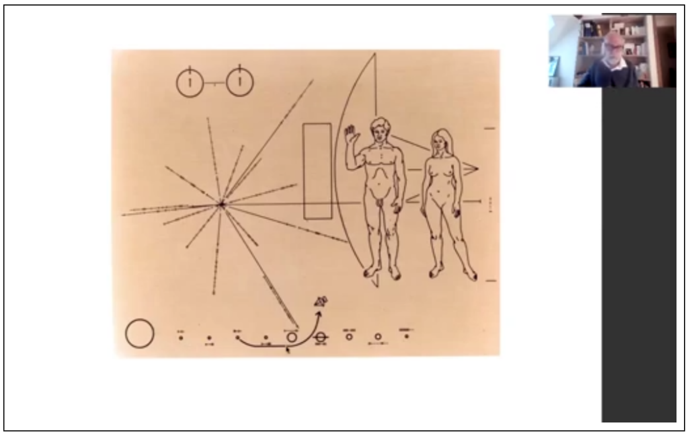


with Henri Moscovici back in 1996, but in fact, one can use a less general result, because it turns out that the components of the Chern characters of the Y automatically vanish, the lower components. So in fact, one doesn't need, if you want, in cyclic cohomology, the full understanding of the index, one just needs the understanding of the Hoschildt class of the index. Of course, this is very instrumental, in proving this result. So I hope I have convinced you... So I just want to add one thing, that some physicists will dismiss this, because at some point, we had made a wrong prediction, which was about the Higgs mass, but there is a very interesting story, which is that with Ali Chamseddine, we wrote a survey paper in 2010 in which we were explaining the theory, and in that paper we had a scalar field, which in fact we ignored, when we did the renormalization group calculations. This scalar field was opering as coefficients for the neutrino and so on, okay, this paper is published in 2010, now what happens is that in 2012, Ali wrote to me an email and he told me "you know, it's amazing because there are three independent groups of physicists who have shown that if you add a scalar field to the Standard Model, then you can recover the stability of the Higgs scattering parameter, the positivity of the Higgs scattering parameter at unification, which is exactly what was, if you want, contradicting our prediction, the fact that it was no longer positive in the usual model. I couldn't believe my eyes, I didn't believe Ali, and I checked and all the signs were correct, our scalar field was exactly the right one. So that, it could correct the prediction and make it compatible with the actual value of the Higgs mass. So the model is not at all disproved by this.

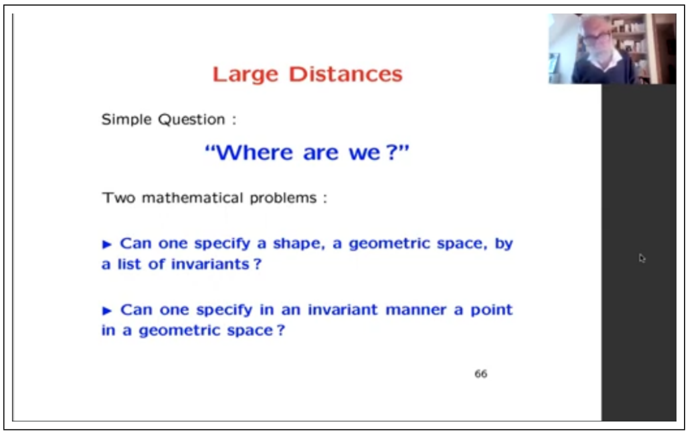
So now let me come to large distances.



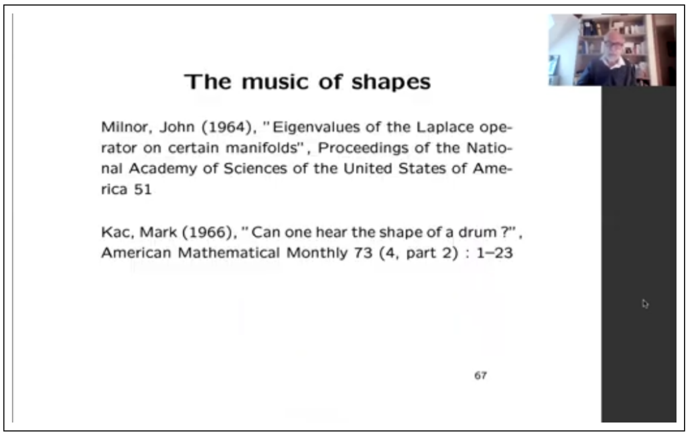
Riemann was very very careful in his inaugural talk, to distinguish between large distances and small distances. So I hope I made the point about small distances and now, when you look at large distances, I want to explain that the spectral point of view is equally relevant. For that, I will ask a simple question, which is "Where are we ?". By this I mean, how can we try to specify the Earth, if we send for instance a probe in outer space, how can we specify where this probe is coming from.



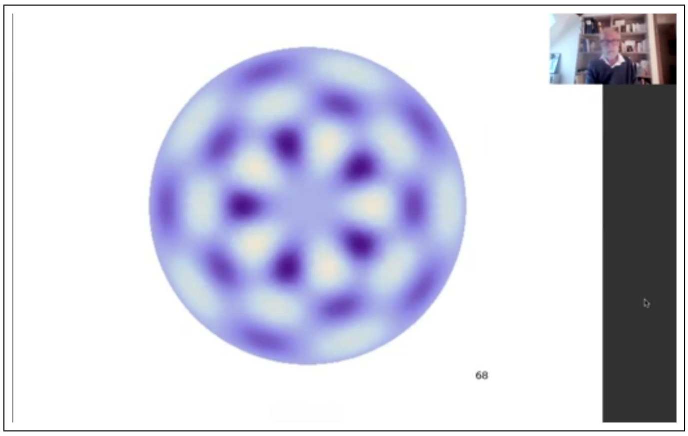
Of course, you can show the solar system, with our planet, you can show what we look like, but there is something that is much closer to the answer I want to explain, and which is this picture, when you have all those straight lines which all group in the same point, and on each of them you have a frequency, which is indicated.



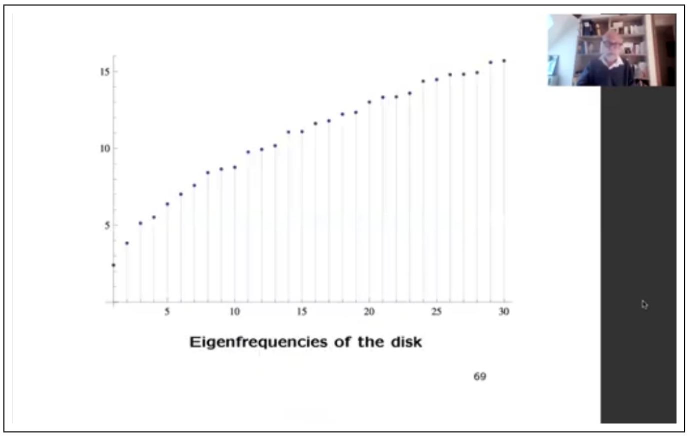
Now this gives birth to two mathematical problems, the first one is "can one specify a geometric shape by a list of invariants ?" So if you try to specify the universe or whatever space, by giving a chart, coordinates system, this is ridiculous, because if for instance you give what is the coordinates system, you have to specify the origin, so the question is completely circular. So what you have to do is first you have to specify the shape, by geometric invariants, by a list of invariants, and then, "can now specify in an invariant manner a point in the geometric space ?". So these are two mathematical problems, and the answer relies on two papers.



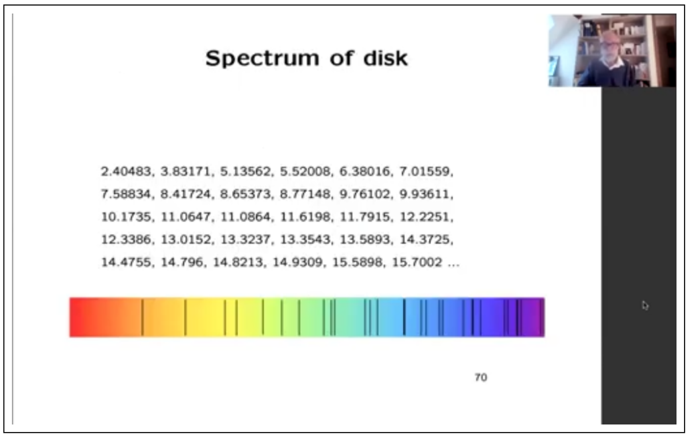
There is paper of Milnor in 1964 : he showed that when you take a space, when you take the eigenvalues, when you take the spectrum, the eigenvalues of the Laplace operator or of the Dirac operator, that doesn't matter for this, then it turns out that this is not a complete invariant of the geometry. He exhibited two spaces in dimension 16, that had the same eigenvalues and the reason is just modular forms and theta functions. And then, there is another paper, which is by Mark Kac in 1966 which is "can one hear the shape of a drum ?".



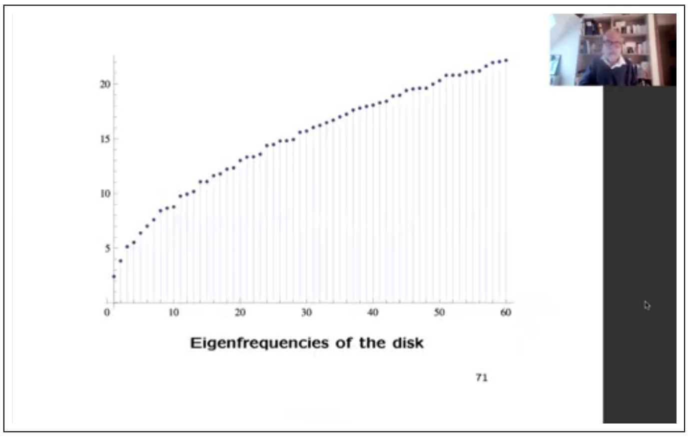
So if you take a drum, it will vibrate, it will have many forms of vibration, wihch depend on how many variations you have when you go around, and how many vibrations there are when you sort of go from the center to the external of the drum ; so it has sequences of eigenvalues.

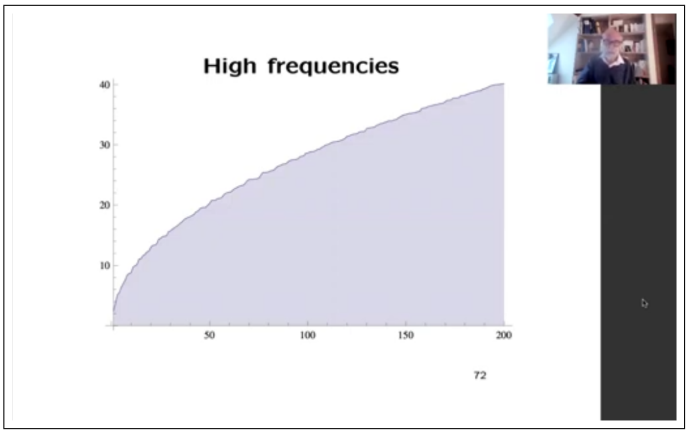


they grow,

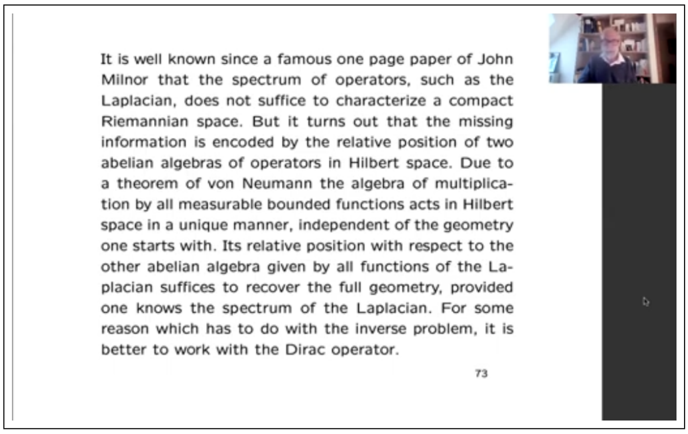


they are computable as zeros of Bessel functions, they form a kind of spectrum, and when you look at them for higher and higher frequencies, they form a parabola.

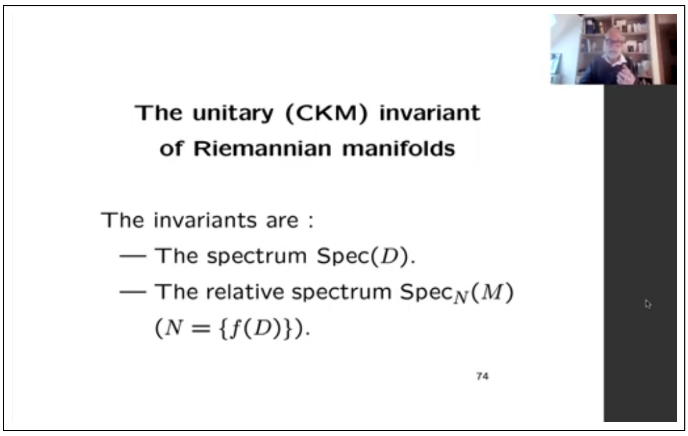




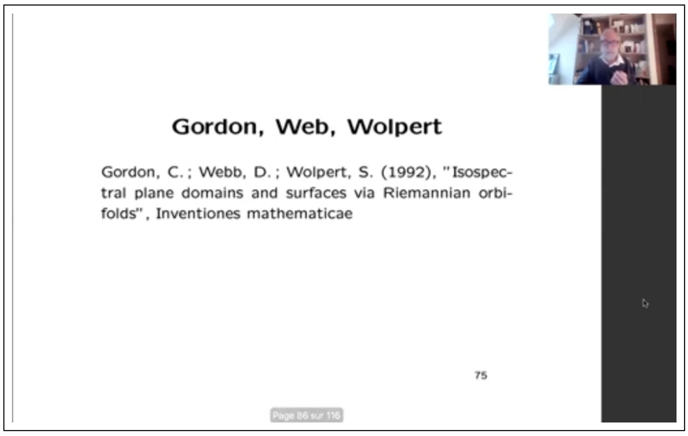
And this parabola indicates that you are handling a form of dimension 2. This is a result of Hermann Weyl which will be quite instrumental later.



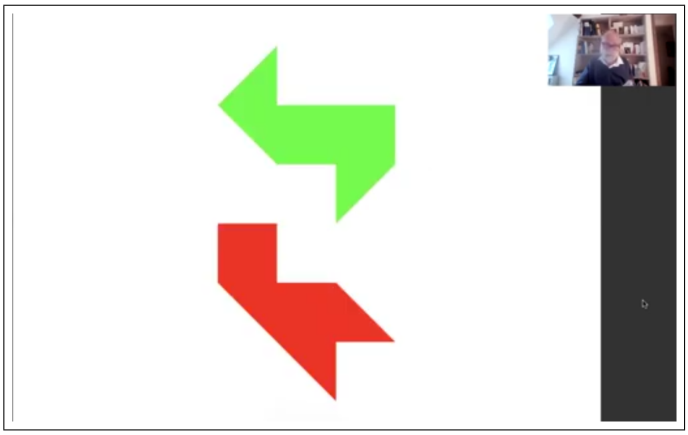
Now, the answer which I want to explain is what is missing when you only have the spectrum, when you only have what they will call the scale if you want, because if you could do music with the shape, you would have a scale which will be forced upon you and which will be the very specific frequencies which are given there. Now it turns out that the missing information that you need, that you are missing in order to reconstruct the space, the geometry with all its properties, is in fact given by the relative position of two abelian algebras of operators in Hilbert space. There is of course the Dirac operator, that by its spectrum is uniquely embeddable in Hilbert space, but there is another, and this comes from a theorem of von Neumann, because the result of von Neumann proves that if you take two manifolds of the same dimension, it turns out that... von Neumann algebra multiplicated by functions, by measurable functions on the manifolds... they are isomorphic in their action... but not only isomorphic as algebras but also isomorphic by the way they are acting on the Hilbert space. So if you want the pair which is given by the algebra and the Hilbert space is unique. The pair which is given by the Hilbert space and the Dirac operator is given by the spectrum. The only thing you are missing is what is their relative position. And this relative position let me to define



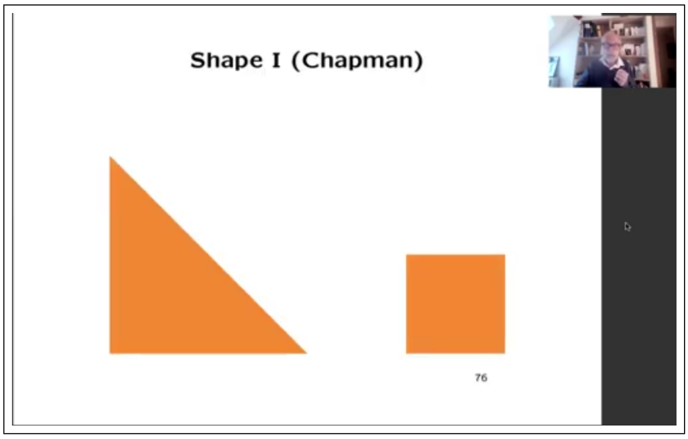
an invariant, which is rather subtle to define, but which I can illustrate very simply on an example, which I called the CKM invariant and the reason for which I called it CKM is because of Cabibbo-Kobayashi-Maskawa who are using a similar invariant when they define their... you know, the thing that was actually breaking the CP, you know, in the Standard Model. So the invariants are given by the spectrum of the Dirac operator, but by something which is like giving the possible chords, on this spectrum.



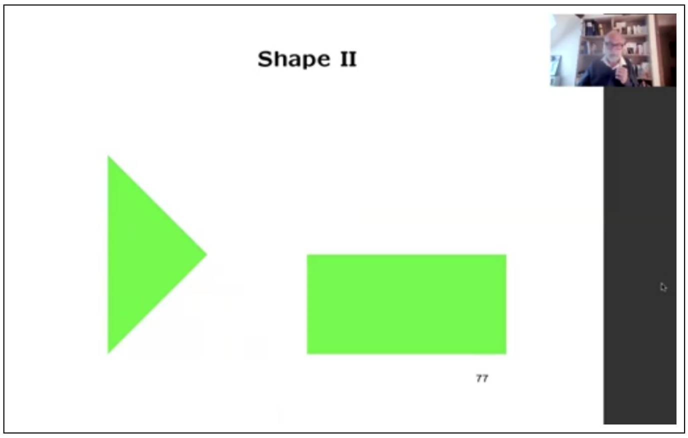
And to illustrate this, I will show you this on a very simple, of course very naive example, what it is. And for that, I will work in dimension 2, because thanks to the work of Gordon, Web and Wolpert for instance, one has beautiful example, of isospectral shapes, in dimension 2.



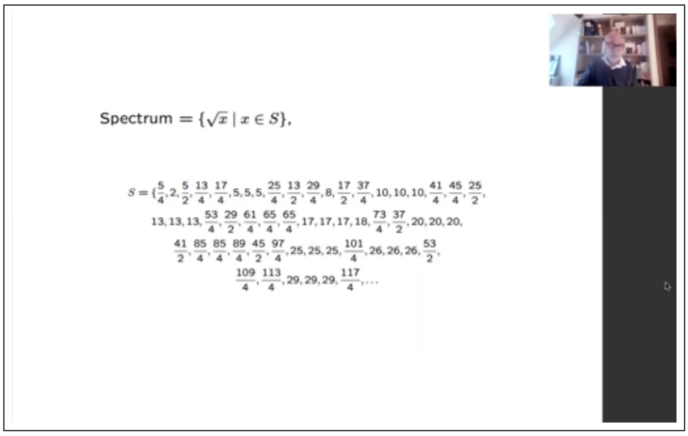
So these two shapes for instance, they have exactly the same spectrum, and of course, they are not the same, here you have this protube of the little square, that doesn't appear in the other shape. So there is another example which I will use which is due to Chapman, and the two shapes I will use are not connected.



So the first shape is this union, of this isosceles triangle and this little square,

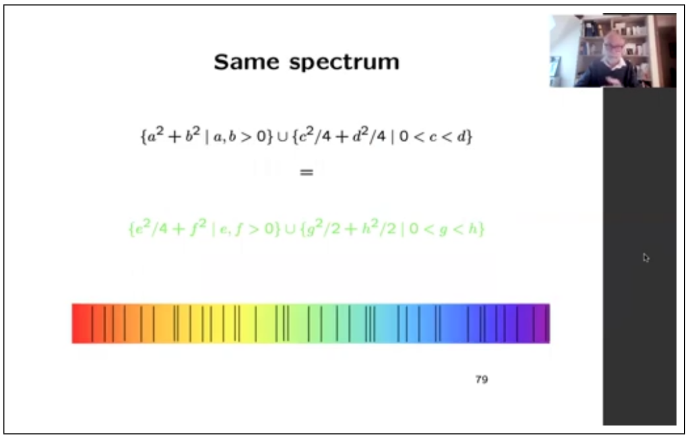


and the second shape is the union of this isosceles triangle with this little rectangle, now it turns out that when you compute, you'll find out that these two shapes have

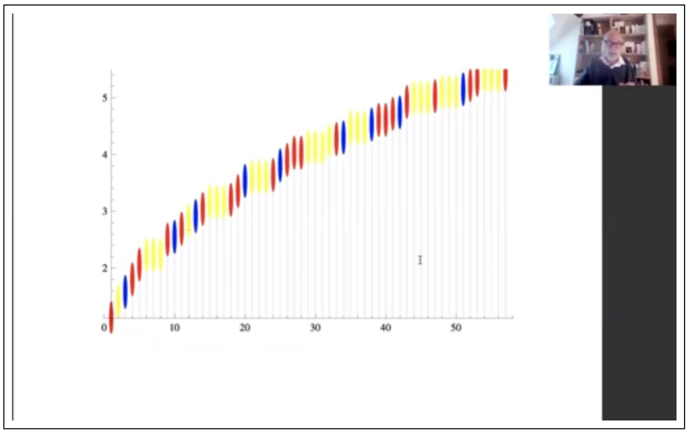


exactly the same spectrum. Each of them is disconnected, but they have the same spectrum. The fact that they are disconnected will help me to show you what is going on.

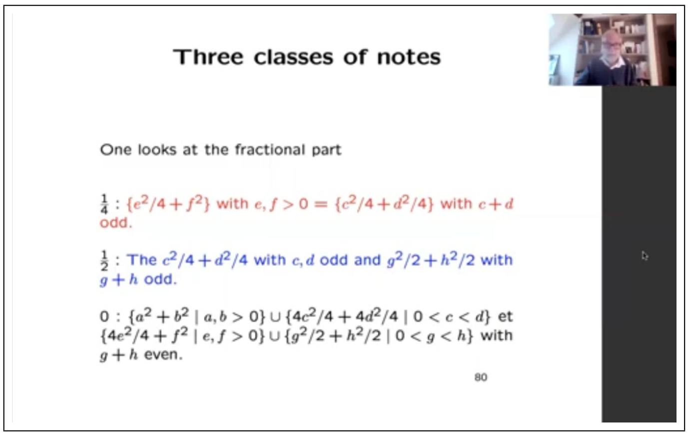
So when you compute the spectrum, you find, when you write the squares of the spectral lines, they are three types, there are three types of nodes in the scale : there are nodes which are of fractional parts 1/4, there are nodes which have 1/2 as fractional part, and there are nodes which are full integers.



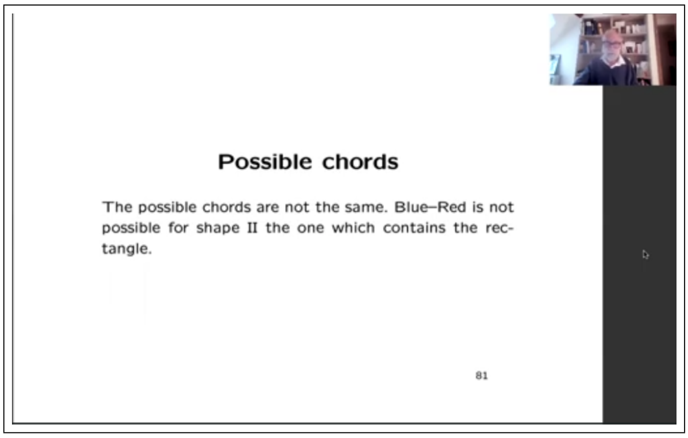
So in fact, this forms a kind of a scale like this, where you are like on a piano, you have the black and the white keys, here you have the blue, the red and the yellow. So you have three types of notes. Now as I said

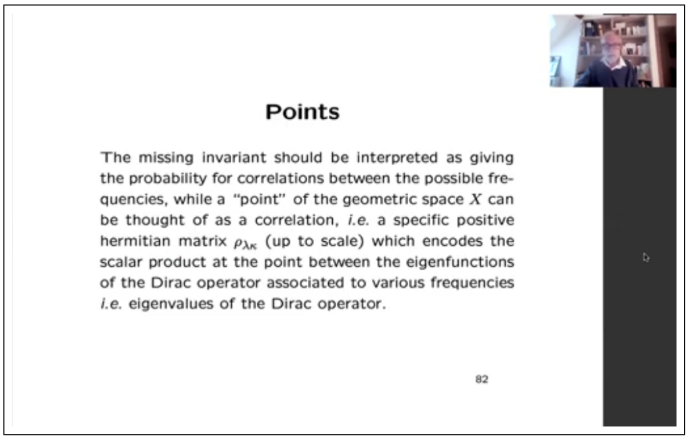


they have the same spectrum, spectrum looks like this, there are these three classes of nodes,

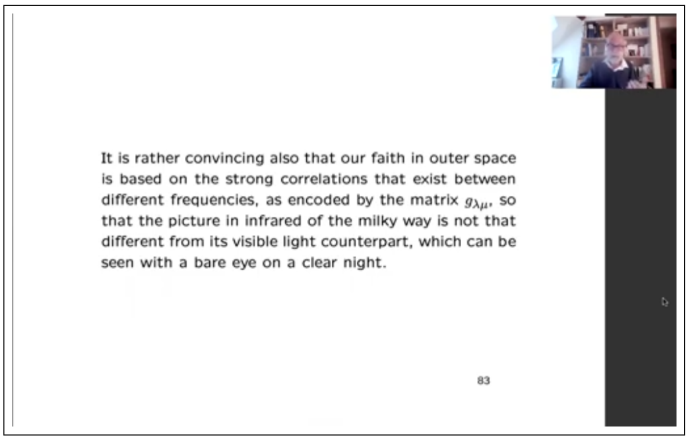


and now, the two shapes are not the same, why ? Because the possible chords are not the same. What you find out, you have to think a bit, is that the chord blue-red is not possible for shape II, the one which contains the rectangle, but that this chord blue-red is possible for shape number I. Okay you have to define what you mean by a chord, and so on.



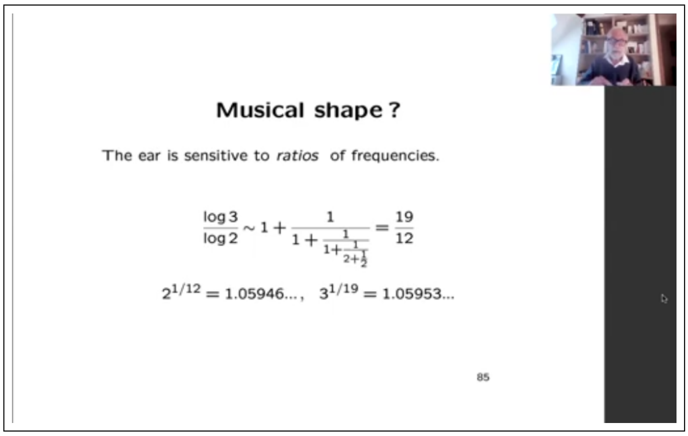


But what it means in general is that the idea of a point also emerges from this type of thinking. The invariant is quite difficult, quite delicate to define. What emerges also is the idea of a point. The idea is that a point in a geometric space should be thought of as a correlation. In fact, it's given there as a specific Hermitian metrics, but what it encodes is the scalar product at the point of the eigenfunctions of the Dirac operator, but what it encodes if you want is the correlation between various frequences.



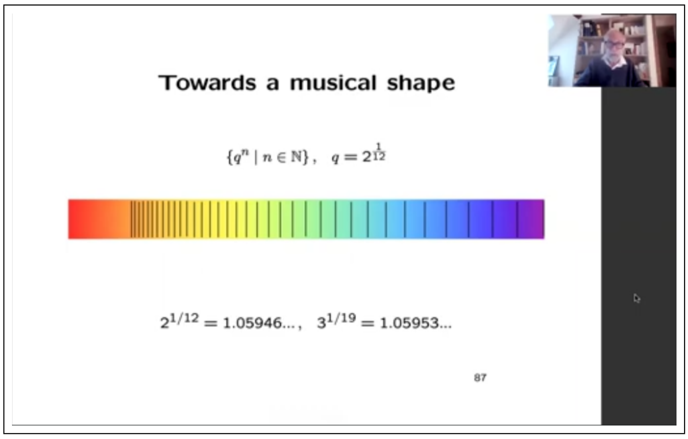
And this is very convincing since our faith in the existence of the outer space is based on the strong correlation which exists between different frequences. For instance, when we look at the milky way, we can look at it in visible light, but we can also look at it in other frequences like X-ray, or infrared, and so on. And it's crucial that all these various pictures that we get in different frequences are actually correlated to each other.

So this is what this additionnal invariant is telling. Now to make a little break, when I was playing with these various shapes and with their scales, I was wondering "is there one that would allow us to do music as we like it ?", like the notes like on a piano. And of course, you have to know the minimal amount of music which is that the ear is sensitive not to adding one, like you would get in an arithmetic progression, not at all, the ear is sensitive to ratios of frequences : if you multiply a frequency by 2, it's like when you play on a piano, you play an A, and now if you play the same A one octave up, you are in fact just doubling the frequency, and the ear is very sensitive to that. Now it's also sensitive to multiplication by 3,

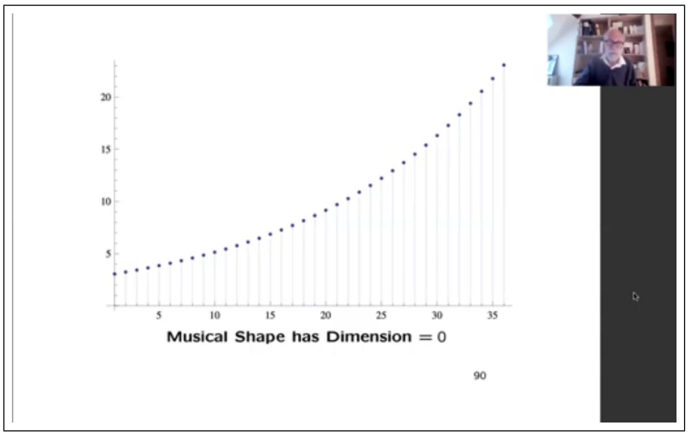


and one mathematical fact which is extremely used in music is the fact that when you look at 2^{19}, it's almost 3^{12}. Of course, they can be equal, because one is even and the other is odd, but what it means is that if you take $\frac{\log 3}{\log 2}$, it's very closed to $\frac{19}{12}$. In fact $\frac{19}{12}$ appears in the continuous fractions expansion. So in fact, the twelve-th root of 2 is very closed to the nineteen-th root of 3, and it turns out that the correct musical shape is the one that you can see on a guitar. You see, when you look at a guitar,

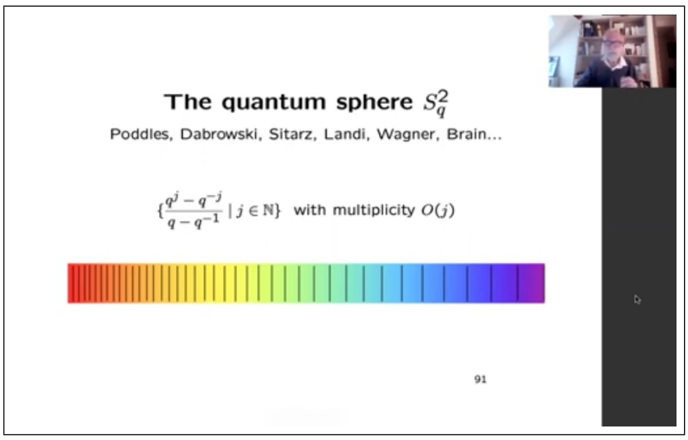
you will find out that the frettes on the guitar which are like here, they do not form at all an arithmetic progression, they are not equally spaced, no. If you think a bit, you have to think, and then you compare it, you make some measurements and so on, you find that these frettes are exactly the powers of this number q which is $2^{1/12}$.



And the spectrum we are looking at, this musical shape, what it should be is exactly what happens with the frettes, namely it should be the powers of this number. Now, you can look at the shapes we know, we are used to, and try to find this spectrum. You get nowhere : if you get the sphere

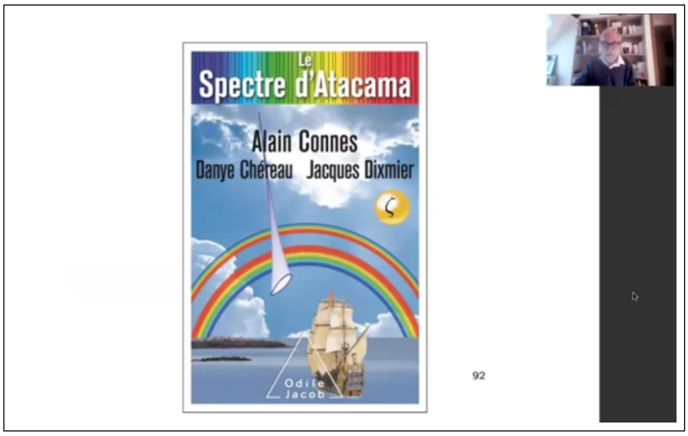


for instance, the 2-sphere, well of course, you know, the high frequencies look like a parabola, but you get nowhere, why ?
You get nowhere because this musical shape when you look at the spectrum, it grows exponentially fast, of course, because it's a geometric series. So when you look at its dimension, I mean, involving the previous ideas that you have to use the Hermann Weyl theorem and so on, you find that it has dimension 0. Dimension 0, it means that it's sort of hopeless to find it among the shapes that we know. But amazingly, it does exist in the non-commutative world, and it is the quantum sphere,

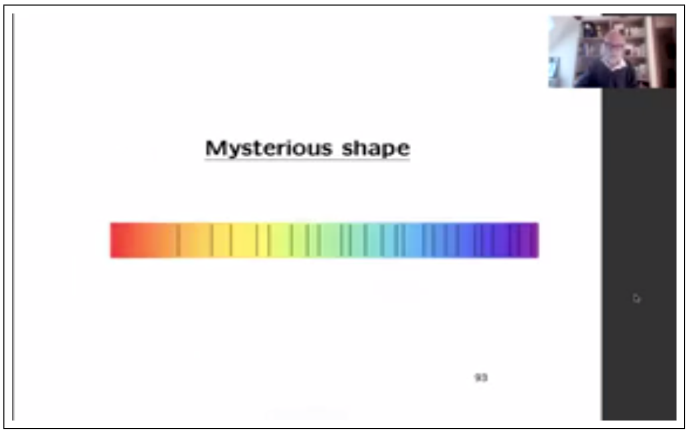


which is a deformation of the sphere, and whose beauty is due to the fact that not only, it has the spectrum that we would like to have, but also, it has the symmetries we would like to have, namely, like a sphere, which has the full group of symmetries which are acting transitively, the quantum sphere has a quantum group of symmetries, which is acting transitively in the suitable sense.

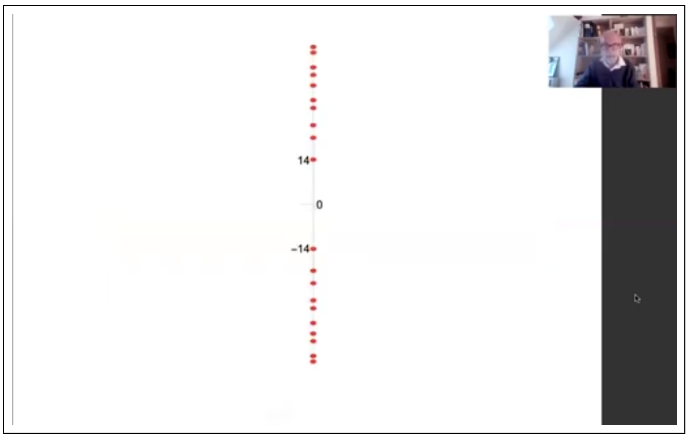
Let me now come to a very important topic which will be like ending of my talk.



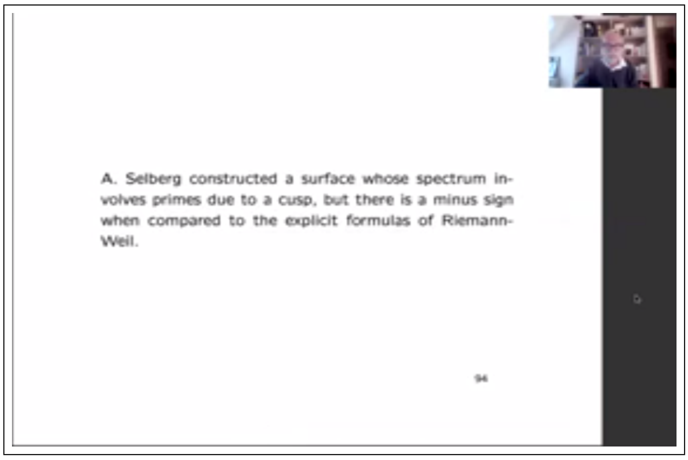
There is a book which I have written with my wife and with Jacques Dixmier, which is a kind of Prelude to this last topic.



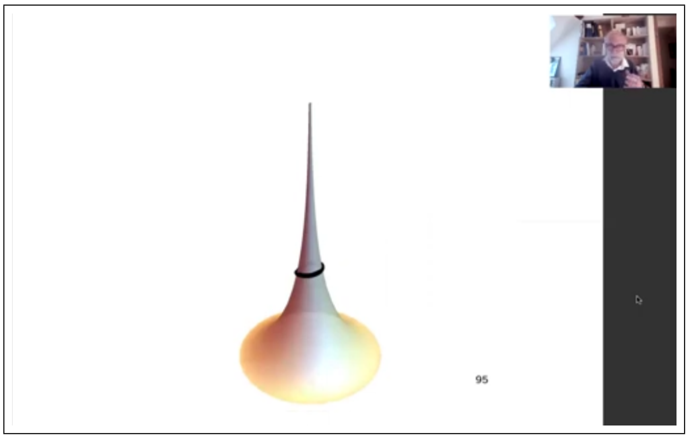
So this last topic is a mysterious shape, there is a fairly mysterious shape, which I am showing you, as far as this spectrum is concerned, and this is how it appears,



when you first look at it, and what is it ? Well, people who know a little bit of number theory will have recognized the zeroes of the Riemann zeta function. Now this is a very mysterious shape, and you have to admit that it looks a bit like a kind of spectrum of some Dirac operator, this remark was made to me by Atyiah, and you know

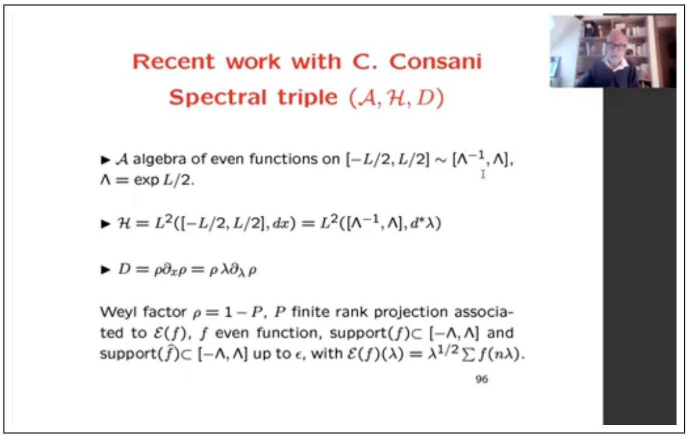


Selberg tried to find, to construct a surface whose spectrum could be quite related to this. He constructed a surface, which, because of a cusp, was related to primes, but when you compute with this, you find that there is a minus sign which appears, and when you compare to the explicit formulas of Riemann-Weil.

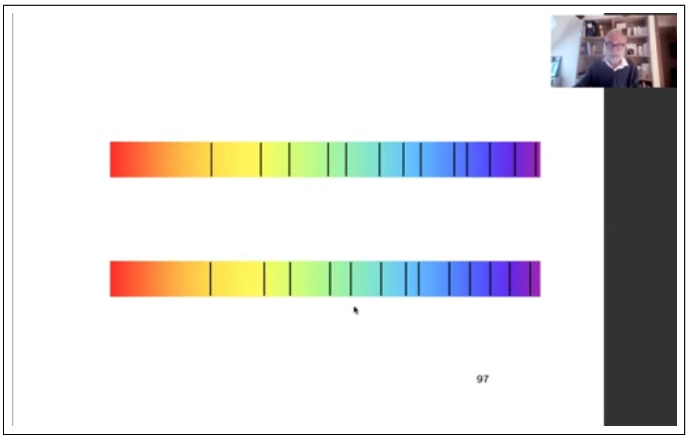


So this sort of didn't quite work, and this is the type of cusp that Selberg was getting and which was coming rather close.

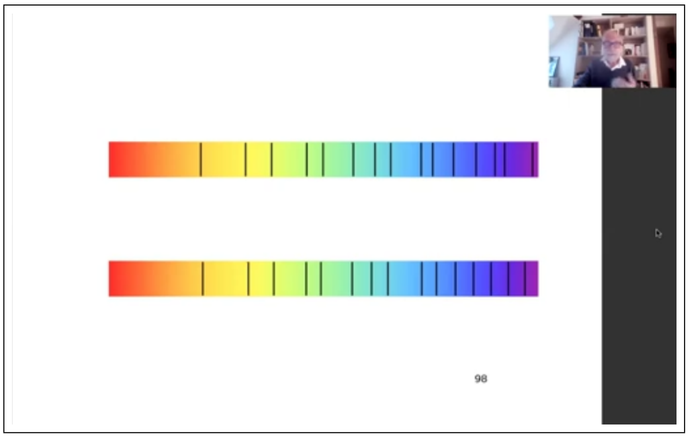
Now the reason why I mention that is that a very recent work which was done in the last month with Katia Consani, what we have found, is a non-commutative geometry, but this non-commutative geometry is of course given by a spectral triple, but the algebra is commutative.



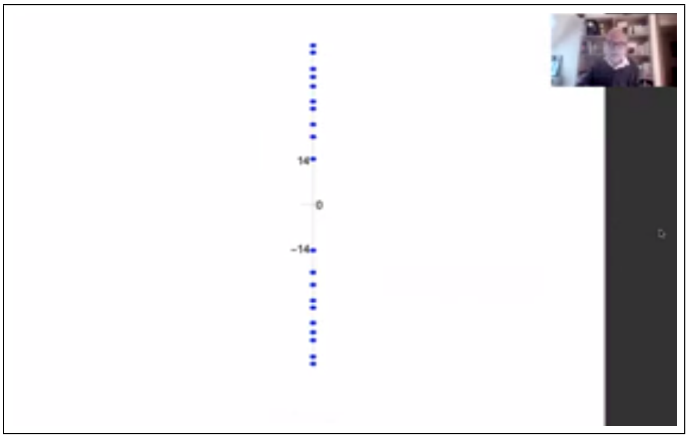
so the algebra is just the algebra of ordinary even functions on the interval $[-L/2, L/2]$, I think you better think of it multiplicatively $([\Lambda^{-2}, \Lambda]$, this is a bit better. So you think of it as functions from lambda length to lambda by using the exponential. Okay. The Hilbert space is equally simple, it's the Hilbert space of all L^2-functions, on this interval, to dx or to Haar measure multiplicative $d^*\lambda$ when you think multiplicatively, and what about the Dirac operator ? The Dirac operator is a very tiny perturbation of the ordinary Dirac. And the ordinary Dirac is given by D by ∂x or by λD by ∂_λ when you work with the exponential, and this tiny perturbation is by a Weyl factor : it turns out in general that you can write the Dirac operator from a given metric to the Dirac operator from the new metric obtained by just introducing a Weyl factor by the formula $\rho\partial\rho$. So this is the formula I am using here, and the Weyl factor couldn't be simpler, it's of the form $1 - P$ where P is the finite rank projection, which is associated to even functions whose support is between $-\Lambda$ and Λ, now I am in the multiplicative framework, and whose support of the Fourier transform is also in $[-\Lambda, \Lambda]$. So one has to be very careful, I mean, this is impossible, but it's possible up to ε and this leads to prolate functions. Okay. So what we did with Katia, we defined this spectral triple, it depends on the length of the interval, and we were able to compute the corresponding spectrum of Dirac operator, only in very simple cases because the formulas for the prolate functions are quite complicated. So we did compute it for small values of L and to our amazement, you know,



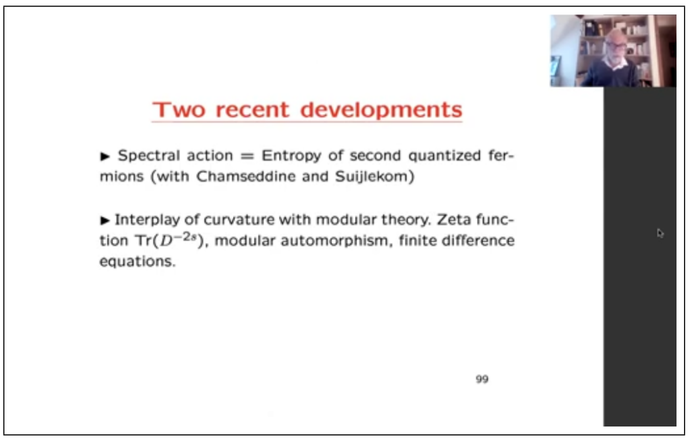
I mean, this thing is the spectrum of the zeroes of zeta, and this is the spectrum that we found, in the first example, and in the second example, we had an even better coincidence between the two.



So we are now exploring this coincidence, trying to understand in which sense, in the limits the two spectra coincide, and that is just the tip of an iceberg in a huge program that we are pursuing with Katia Consani on the Riemann zeta function and which of course, I mean, has many connections to non-commutative geometry but finally, if you want, it has a connection with the spectral point of view. It has many connections with other sides of non-commutative geometry, in fact, to the singular spaces, like the adele class spaces, and also to topos theory of Grothendieck, and so on and so forth.



So this is the spectrum we obtained which is so similar, the spectrum of our Dirac operator, so similar to the zeroes of zeta,



and I want to end my talk by mentioning two recent very active developments. There is one development which I like very much which is that when we developed the spectral action, with

Chamseddine, the spectral action is depending on a function, it depends very little of this function because you just give the asymptotic expansion but we had no way to choose the function. Now it turns out that with Ali Chamseddine and Walter van Suijlekom, we showed that in fact, the spectral action is equal to the entropy of the second quantized fermions, but for a very very specific test function, which is related to the Riemann zeta function. And the other development, which I didn't have time to mention, is the interplay of the curvature, which is typically a Riemannian curvature, an extension of the Riemannian curvature to the non-commutative case, there is a fantastic interplay between this curvature and the modular theory, the modular operators, that I mentioned with respect to time evolution. So this theory is amazing in the sense that one has to compute asymptotic expansion, and as far as I am concerned, you know, I started to work on that with Paula Tretkoff, Paula Cohen at the end of the 1980's, and this was revised more recently to prove the Gauss-Bonnet theorem in the non-commutative setup, this Gauss-Bonnet theorem was proved in a particular case, but then was proved by Masoud Khalkhali and his collaborators in the general case, and this work, as far as I am concerned, really acquired incredible substance in my collaboration with Henri Moscovici. And what we found in particular is that the formulas were fulfilling certain finite differential equations that were allowing you to compute the functions of several variables which were occurring in the interplay of curvature with modular theory and these things can be put on computer and can be checked, we were absolutaly amazed that the theory was predicting some very complicated relations which were actually fulfilled and an higher case of this was done in my collaboration with Farzad Fathizadeh, when we computed the a_4 terms in the asymptotic expansion.

I will end with a diagram which shows the relations between non-commutative geometry and other branches of mathematics. So I mean non-commutative geometry fits itself tremendously on its relation with physics, high energy physics as I explained, its relation with number theory, with the space underlying the adele classes and so on, with operator algebras of course, from the very start, with K-theory, Index theory and fantastic KK-theory of Kasparov which is one of the key tools, with of course algebraic topology, geometric group theory and so on and so forth, and also with differential geometry, because in all these cases, there is a feedback, for instance in differential geometry what Skandalis and his collaborators have shown is how much it is relevant, not only to study manifolds but to study smooth groupoids, and to study smooth groupoids, you need non-commutative geometry. So okay, I think I will end here, and I will thank you for your patience, thanks a lot.