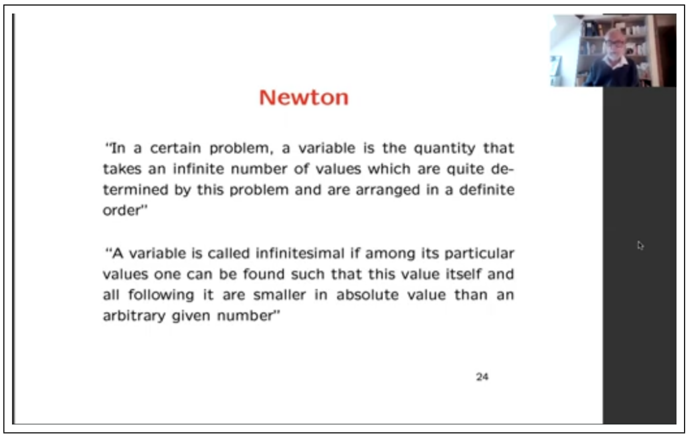
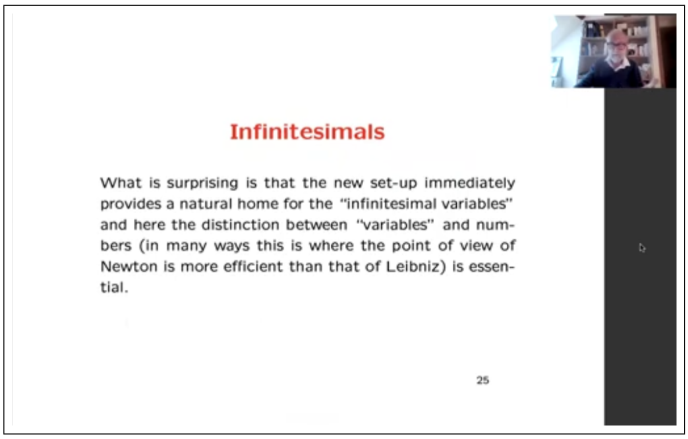


One first thing which is remarkable is that if you read Newton, you'll find that, provided you read what he wrote in the quantum mechanics formalism, it will immediately give you the right answer for what are infinitesimals. So, first of all, Newton was not interested in numbers, he was interested in variables. What he says is that :

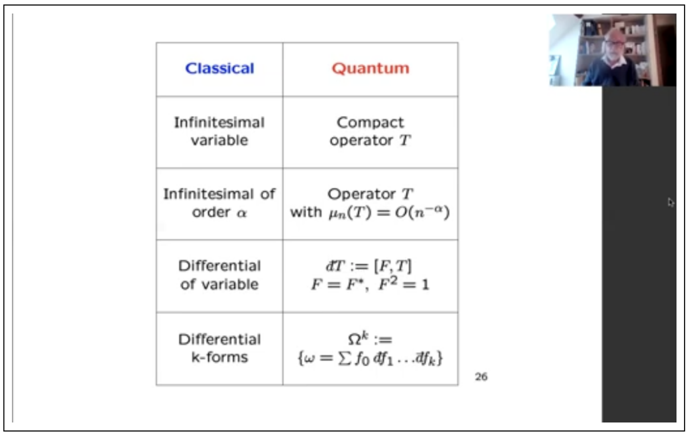
> *"In a certain problem, a variable is the quantity that takes an infinite number of values which are quite determined by this problem and are arranged in a definite order".*

And then he talks about the infinitesimals. And for him, an infinitesimal is a variable. So,

> *"A variable is called infinitesimal if among its particular values one can be found such that this value itself and all following it are smaller in absolute value than an arbitrary given number."*

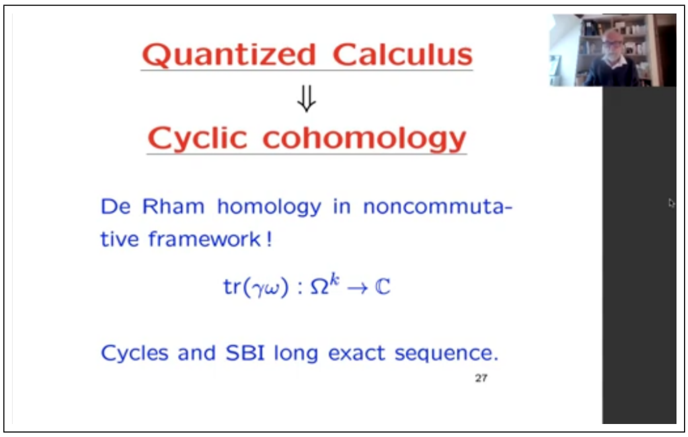


Now, what is amazing is that when you apply this notion of infinitesimal which is essentially there, which is essentially defined in the words of Newton, then you find that they correspond on the nose to a notion which is well-known in operator theory



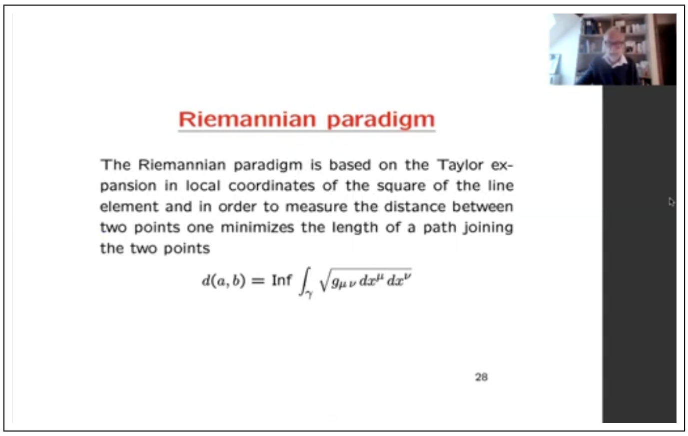
and which is compact operators. Because compact operators, well, they are variables, as we saw, because you know, variables are corresponding to operators, but moreover, they have exactly the property that Newton was saying, namely that if you take their characteristic values, so the characteristic values are eigenvalues for the absolute value of the operator. And these characteristic values have the property that for any ε, there are only finitely many of them which are larger than ε. So they correspond exactly to what Newton had in mind. In this formalism, you have a rule, immediately, for what is an infinitesimal of order α. So for that, you look at the rate of decay of the characteristic values, and for instance, an infinitesimal of order 1 is one such that its characteristic value decay like $1/n$. This is fundamental for later because such things are not tracable because the series $1/n$ is divergent but when you look at their trace, it has a logarithmic divergency, and it is the coefficient of this logarithmic divergency that gives you something local.

There is also the differential of variables. Normally, you try to differentiate the functions, and so on, but here, it's just defined, for a bounded operator, it's just defined as a commutator with the operator F which satisfies two conditions : the condition that it is self-adjoint and the condition that its square is 1. So there is no content in the operator F itself, what is really important is the relation between the operator F and the operator T. Because F^2 is 1, you can easily show that the square of the differential in the graded sense is 0, and then you have the notion of differential k-forms, which are obtained only by taking operators sums of products of (k-forms) 1-forms, if you are defining them in an obvious way. So many properties come out naturally. But the most important was that this quantized calculus

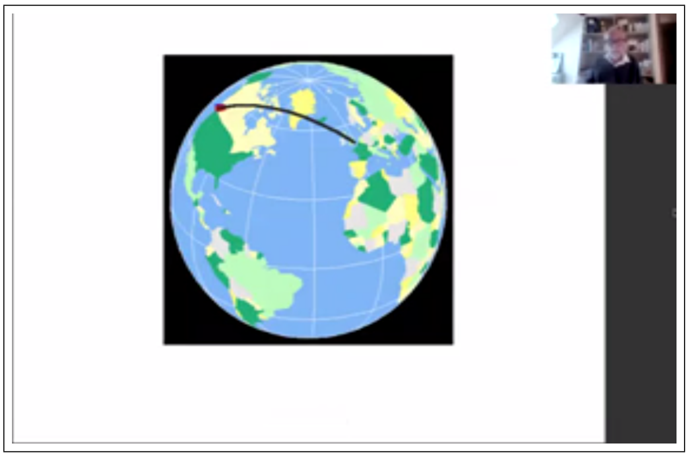


led me in 1980-1981 to the cyclic-cohomology. And cyclic cohomology is really playing a fundamental role in non-commutative geometry as a de Rham cohomology in this non-commutative framework.

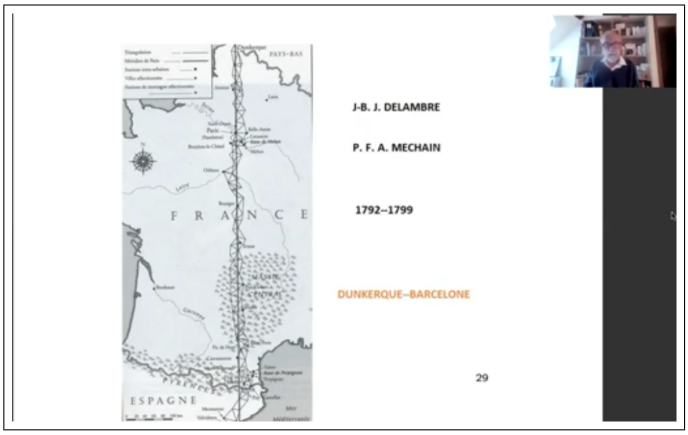
I gave a talk in 1981 in Oberwolfach whose title was “Spectral sequences and cohomology of currents for non-commutative algebras”, where if you want all the basic properties, the fundamental properties of cyclic cohomology follow if one takes seriously this quantized calculus. Because in the quantized calculus, what you get is what is called the cycle, because you can use the trace to integrate the differential forms, and you get that this cycle is closed, and so on. And moreover, if you can integrate form of dimension k, you can also integrate forms of dimension $k + 2$. There is a distinction between even and odd cases, and this gives to the operator S a periodicity, and to the SBI an exact sequence that is the corresponding spectral sequence. So of course, this is just one instance of the use of quantized calculus. There are many other instances. Someone is for instance that you can do differential geometry in the group ring of the free group, using the quantized calculus which is defined by actually the group on a tree. But these are tools, and now, with these tools, we want really to come to the geometry itself, in the metric sense. So for the geometry itself, in the metric sense, we have to make a little discussion, going backwards, to the riemannian paradigm



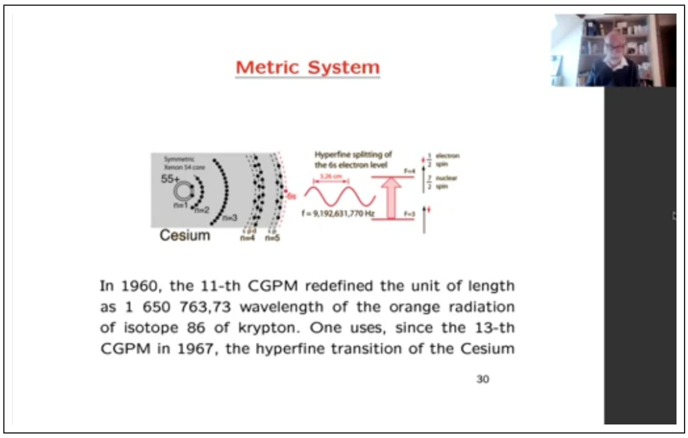
and to the way the metric system evolves. The riemannian paradigm is based on the Taylor expansion... Riemann had this fantastic inaugural talk, to which we shall come back later, in which he had the insight to define the metric locally, by looking at the Taylor expansion of the line element in local coordinates, and in fact, he was looking at the square of the line element. So there is a squareroot involved, and the distance between two points, as you know very well, is computed by minimizing the length of the path, like here,



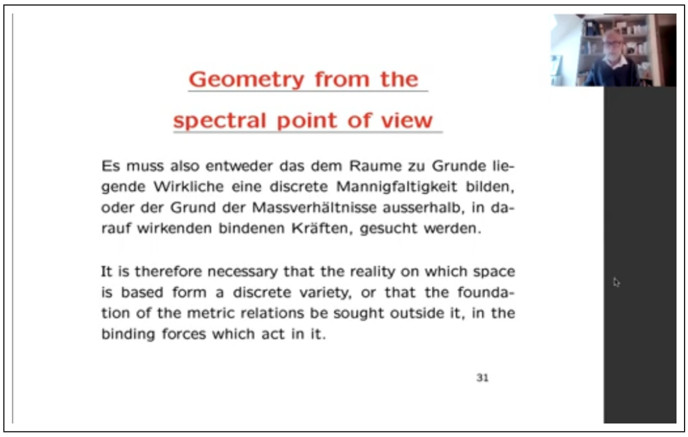
for instance between Seattle and London. And so, it’s a very concrete definition of length and somehow, it fits very well with



how the metric system was developed and typically, what happened... what I am showing you the fact that during the French revolution, they wanted to have a unification of the unit of length, so they defined it as $\frac{1}{40\,000\,000}$ times the circumference of the Earth and of course, we are measuring just an angle, that they knew from the stars, and I mean this angle, this distance was between Dunkerque and Barcelone and two physicists, astronomers, went along and they made a concrete measurement and out of this concrete measurement (they were Delambre and Méchain) was fabricated a platinium bar, which was deposited near Paris in Pavillon de Breteuil in Sèvres, which was supposed to be the unit of length. Now what happened was very interesting. Because what happened around the years 1925 or something like that, physicists discovered that the unit of length which was deposited near Paris and so on and so forth, didn't have a constant length. How did they do that ? Well they compared it with the wavelength of Krypton. There is a certain orange in the wavelength of Krypton which they used to measure this platinium bar, and they found that the length was changing. So I mean, of course, this was not a good definition, and they shifted, I mean physicists shifted, in the Conference on metric system,

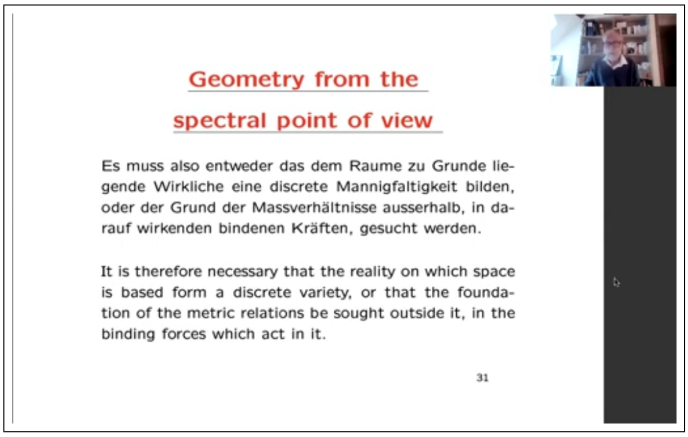


they shifted first to define the unit of length by this orange radiation of Krypton, a certain multiple of this wavelength. But then they found that there was a better way of doing it, which was with Cesium. And in fact, with this definition of Cesium, you can buy, in a store, some apparatus which allows you to make immediate measurements with a precision which is 10 decimal places.



I mean, it's a big step forward, and what happened is that there is an hyperfine transition between two levels, and it corresponds roughly to

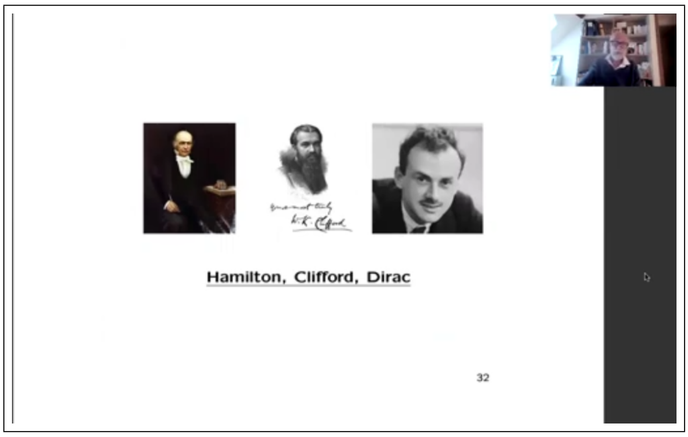
a wavelength of about 3.26 centimeters and then, you redefine the whole thing, so in fact, you define the speed of light, you fix the speed of light at this number (I heard that Grothendieck was furious when he heard that, because he wanted it to be 300 000 000), but the reason why you cannot do that is that there are already measurements which are precise which are done, so you have to accord this. So I mean there is a strange value which is taken. Then you define the second, as a number of periods of this radiation coming from the hyperfine transition, and as a corollary of that, you know the meter, the unit of length, is therefore defined as being this proportion (it's a rational number) times the wavelength of the hyperfine transition. Now what is extremely amazing is that this transition which the physicists have done, long ago, between the platinium bar and the spectral definition, is exactly parallel to the transition between the riemannian paradigm and the spectral paradigm, which I will explain.



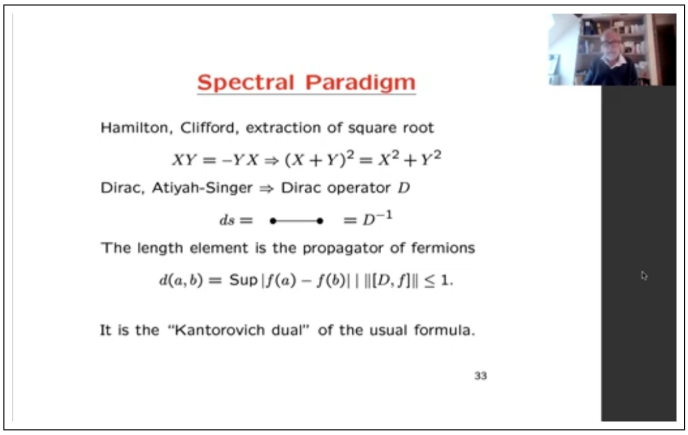
What is also amazing is that Riemann was incredibly careful in his inaugural lecture, to say, I mean, he didn't really believed that his notion of metric would continue to make sense in the very very small. And the reason that he was advocating was that... because he was working with solid bodies and he was using lightrays in his theory so what he was writing was that when you work in the very small, solid bodies no longer make sense, and neither do lightrays and so, what he wrote for instance was that

> *"It is therefore necessary that the reality on which space is based form a discrete variety, or that the foundation of the metric relations be sought outside it, in the binding forces which act in it".*

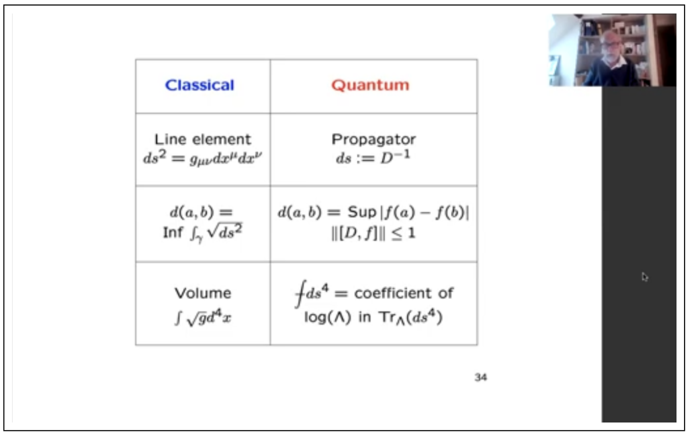
and as we shall see, this is exactly what happens, in the spectral framework.



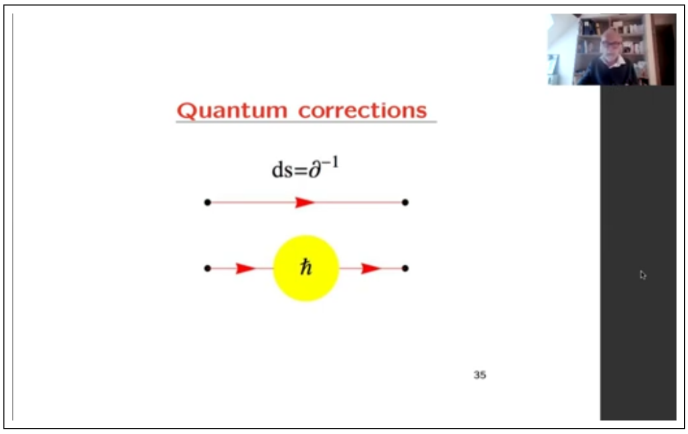
So the possibility to do that, to transfer to the spectral framework these ideas is in fact coming from work of Hamilton, Clifford and Dirac, and essentially what is the way to extract the squareroot in the formula of Riemann ($d(a,b) = \mathrm{Inf}\int_\gamma \mathrm{g}_{\mu\nu}\mathrm{dx}^\mu\mathrm{dx}^\nu$). When there is the squared root of line element, one would like in fact to have not the squared but the line element itself. It's possible to extract this squareroot at the level of the quantum formalism, at the level of operators. And it's possible thanks to Hamilton, Clifford and Dirac. Hamilton was the first one to write really the Dirac operators, because he had the quaternions, and he wrote, you know, i,d by dx plus j, d by dy, plus k, d by dz, which is an example of Dirac operator. And the key to all of this stuff, is that when you have two operators X and Y, which anti-commute,



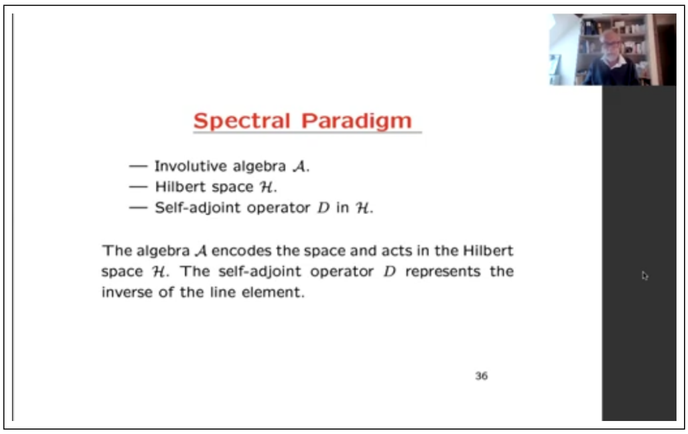
then in fact, you can write $X^2 + Y^2$ as a single square, namely as the square of $X + Y$. So through the work of Dirac and also of Atyiah-Singer, who defined the Dirac operator for arbitrary spin-manifolds, then emerges the Dirac operator D. In the spectral theory, the line element, which is the squareroot of the Riemann's ds^2, is an operator, it is an infinitesimal when the variety is compact, and what is it ? It's simply the inverse of the Dirac operator. Of course, there are minor things to which you have to be careful about, what about the zeros and so on, but I mean, this line element is what is called the fermion propagator, and you have to think of it as physicists write it when they write Feynman diagrams : it's a very very tiny little line, which is joining two points which are very close by. And then, rather ought of this inverse which is the operator D, you can compute a distance between two points, and this distance is no longer computed by the infimum of an arc joining the two points, but it's computed by looking at the maximal waveshift, between the value at a and the value at b, when you subject the waves to the fact that their frequences are bounded. It is what is called, mathematically speaking, the "Kantorovich dual" of the usual formula.



So you have this dictionary now that the line element is ds which is the propagator of fermions, the distance is computable, it's computed not by an infimum on arcs, but by a supremum, and by this way, notice it applies to many more spaces, because there are many spaces in which you cannot join two points by an arc, think about a space which is disconnect, whereas the formula on the right makes perfectly good sense. And the volume, for instance, is defined as the integral of the power of the line element that will be of order 1, and as I said before, when something is of order 1, an infinitesimal of order 1, then it means that its trace is logarithmically divergent. So what you do is that you take the coefficient of the logarithmic divergency and this will give you the volume. Also, what one has to understand is that if this formalism in which geometry is defined is the quantum formalism, this immediately



allows you to understand how to incorporate the quantum corrections. Why ? Because we know very well that the fermion propagator when we do quantum fields theory doesn't stay as it was before. It acquires quantum corrections. They are minute modifications of the geometry, which are given by some kind of power series, but which can be incorporated in the spectral formalism. So the spectral formalism is encoded in

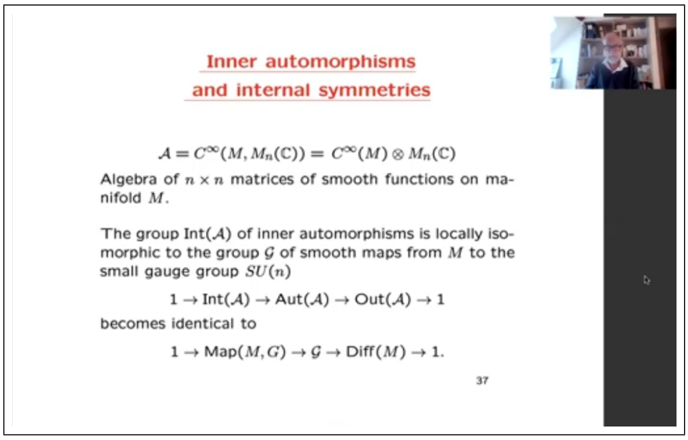


what is called a spectral triple. So such a triple contains three data :

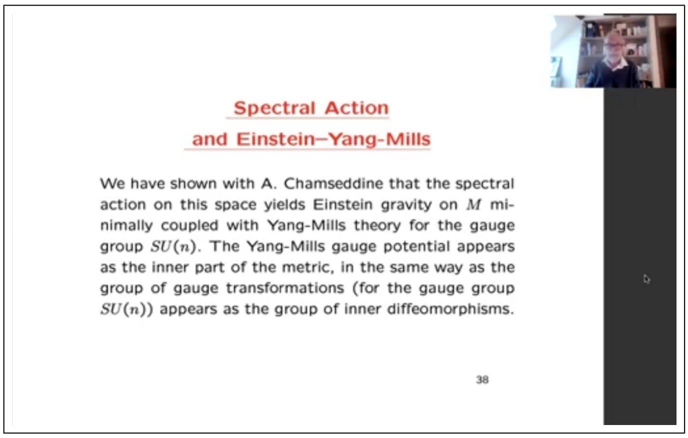
- the data of an involutive algebra, which gives you the space essentially, the coordinates of the space ; this algebra is *acting* in a Hilbert space ;
- the Hilbert space is fixed.
- and moreover you have the selfadjoint element, which is the propagator, which is acting in the Hilbert space $\mathcal{H}$.

In most cases, by the way, what you will find out is that the representation of both $\mathcal{A}$ and D is, when you take them together, irreducible.

So this is the spectral paradigm. And what I want to explain will illustrate the power of this paradigm by a number of cases.

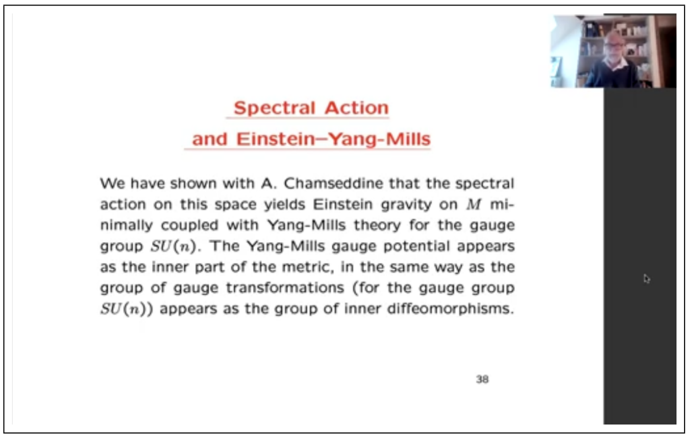


So the first thing which happens is that now, because you can talk about the geometry when the algebra is no longer commutative, now you don't have the $g_{\mu\nu}$ which depends on x and so on and so forth, just because of that, you can look at the most simple example. The simplest example which is not commutative is to replace the algebra of functions on a manifold M by $n \times n$ matrices over this algebra. So, if you do that, you just look at the algebra for a while, what you find out, as I said before, when an algebra is not commutative, it has this non trivial exact sequence, when you have the trivial automorphisms, which are the inner ones, and which form a normal subgroup of automorphisms, and then you go to the quotient which is outer automorphisms. Now, when you apply this sequence, which is general, when you apply it to the algebra of $n \times n$ matrices over manifold, what you obtain is an exact sequence where the inner automorphisms become the maps from the manifold M to the group G which is in this case the group $SU(n)$, if you take $n \times n$ matrices, and then this goes to the group of automorphisms, and it goes to diffeomorphisms. So what you find is that automatically by this very simple non-commutative extension, you have enhanced the group of diffeomorphisms to a group which physicists know very well, because this is the group of invariance of the action functional if they couple, minimally, gravity with Yang-Mills theory, with group $SU(n)$.

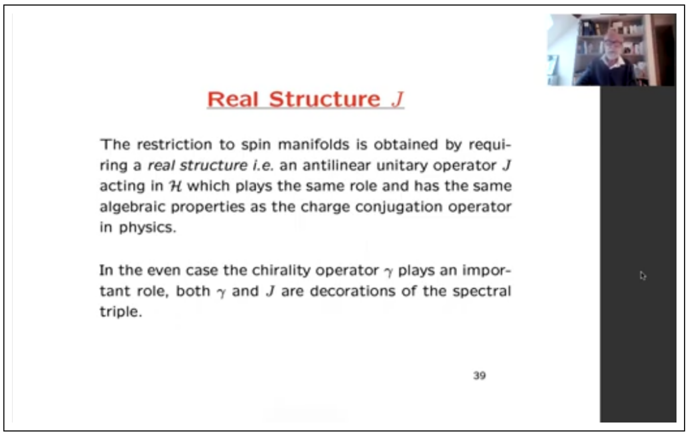


So in our work with Ali Chamseddine, what we found was the action functional. We found that if we take the above very simple case of taking $n \times n$ matrices over a manifold, and if we look at the action that would replace the Einstein action, which is the spectral action, so this spectral action, it can hardly be more invariant, this action only depends on the spectrum of the line element. What you do is that you write the asymptotic expansion and you get the Einstein action. You get the cosmological term to which I will come back much later.

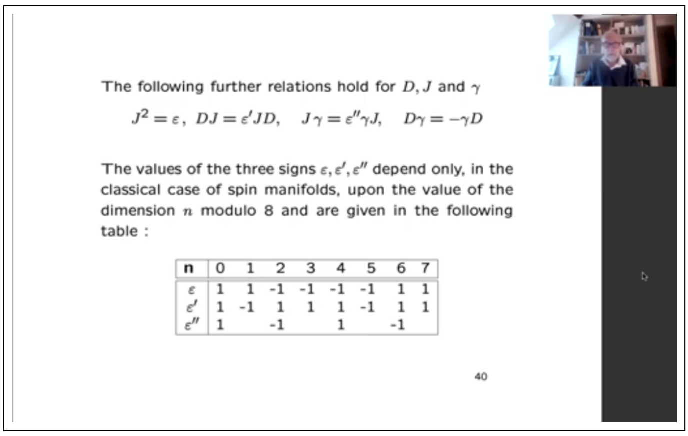
So you get this Einstein gravity but if you take this Einstein gravity minimally coupled with Yang-Mills theory, when you do the calculation. And Yang-Mills gauge potential as they appear, appear as the inner part of the metric. So in exactly the same way that I just said, the group of gauge transformations of second kind, the gauge group $SU(n)$ appear as the group of inner diffeomorphisms. So you have this blending together, which just comes, you know, from having replaced the algebra of functions by matrices over N. So this is a very entire thing and with Ali Chamseddine, we have done a lot of work, then, with Matilde Marcolli, with Walter van Suijlekom, and also with Slava Mukhanov, we have done a very great amount of work in order to go much further than just this simple instance of Einstein Yang-Mills.



In this work, there is an essential role which is played by the real structure. So what happened if you want is that there is a sort of reconstruction that allows you to reconstruct the manifold from the spectral data. And in order to restrict to spin manifolds, you have to think to spin c of things like that, one needs to incorporate a little decoration in the spectral data, which is that of a real structure. So it's an anti-linear unitary operator, and we shall see what it is in the physics language and in the maths language, but essentially, you also have to add another decoration in the case of even dimension which is the chirality operator. So we have these two, and

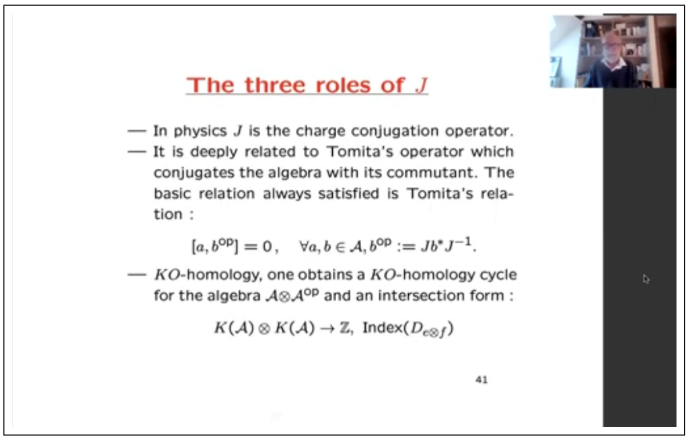


they fulfilled some commutation rules. And these commutation rules in fact, they tell you that you are dealing in fact with eight fold-theories, there are 8 possible theories that in the ordinary manifold case depend on the dimension modulo 8. If you want, the underlying conceptual theory is what is called KO-homology, and the reason why this KO-homology plays a fundamental role is that, if you try to understand at a conceptual level what is a manifold, in the ordinary situation, in the ordinary differential geometry what is a manifold, you will find out, and this is a work which goes back to the 1970's, in particular by Dennis Sullivan, you will find out that what you need to do...

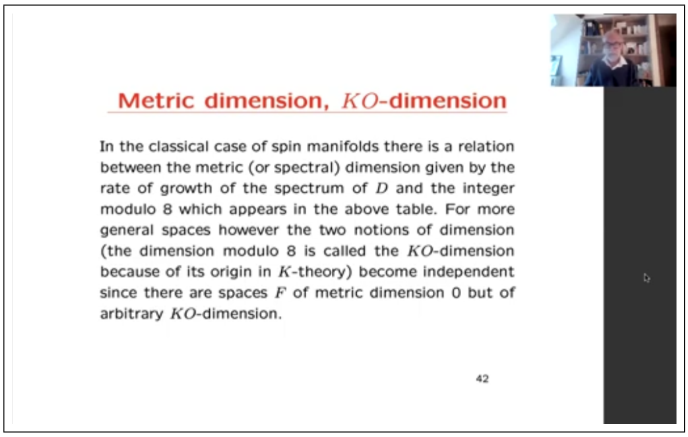


first you assume of course that the manifold has Poincaré duality in ordinary homology, but this is not sufficient at all, it only suffices to put the space in question into euclidean space so that it has a normal micro-bundle. But this micro-bundle is by no means a vector bundle. And the difficulty in order to transform it into a manifold is to elevate the structure of this micro-bundle into the structure of a vector bundle.

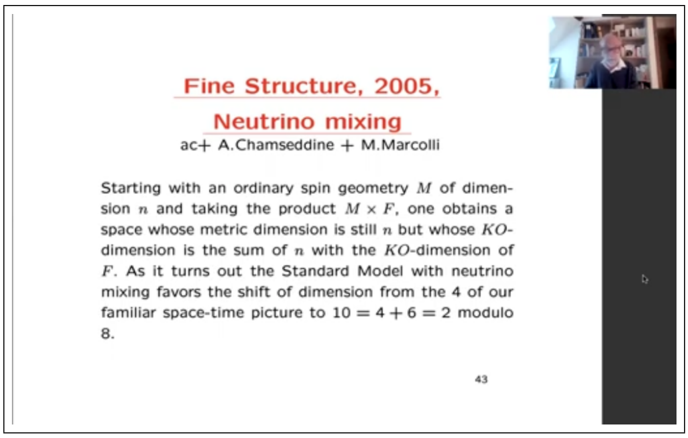
Now to encapsulate things very briefly, in the simply connected case, what you find out is that the obstruction to do that is that you should have also Poincaré duality in the deeper theory which is KO-homology. Now thanks to the work of Atiyah and Singer on the index theorem, they found out that the representative of cycles in KO-homology is in fact exactly given by the data that you need to build a Dirac operator, and that you have 8 possible theories in this KO-homology, and they are corresponding to the various possibilities that I was exhibiting here. So in fact this



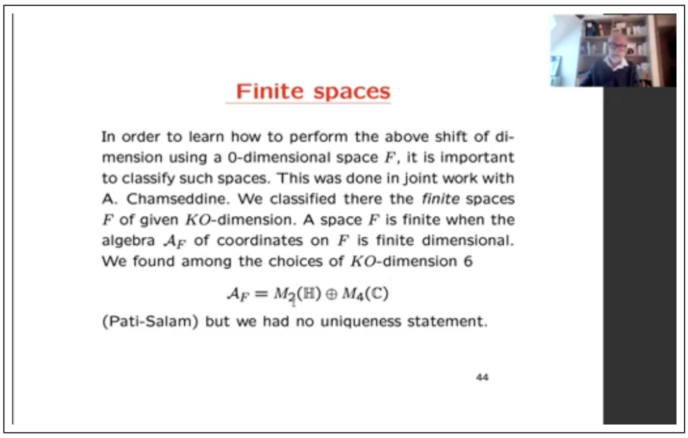
J, this real structure, has three roles. In physics, well, people will recognize it as what is called a charge conjugation operator, we are working in euclidean, in imaginary time, mathematically, this turns out to be very deeply related to Tomita's operator, why ? Because, in the non-commutative case, what you want is that you want to restaurate the commutativity in some way, and how do you do that ? You do that with a sort of trick, you flip, you are able to flip the algebra to its commutant by using this operator J. Tomita's theory allows you to do that in general. I mean, he proved a theorem that is that if you take a factor in Hilbert space, which has a cyclic and separating vector that is always the case in type III, then you can always find such an operator J that flips it to its commutant. And finally, the deepest meaning, if you want, of this J as I said, is to say that you have Poincaré duality in KO-homology, and this gives you the fact that, because of the J, you not only have KO-homology cycle for the algebra $\mathcal{A}$, but also for the algebra tensored by its opposite. And in particular, you have an intersection form, and so on and so forth. Now it turns out that this has played a key role in the development of the understanding of the Standard Model, in the sense that usually, when you work with spin-manifolds, there is a link between the metric dimension and this KO-dimension modulo 8.



What happens is that there is a notion of metric dimension, for a spectral geometry, and this metric dimension just comes from the growth of the eigenvalues of the spectrum of the Dirac operator. But there is also a KO-dimension as I just mentioned and it turns out that normally, the KO-dimension is equal to the metric dimension modulo 8, but when you look at spaces of dimension 0, you find out that this is not necessarily true : you can fabricate spaces of dimension 0, but which are of arbitrary dimension modulo 8. This could look as a curiosity but in fact, it's not at all, and it has played an absolute key role in 2005, in our joint work with Chamseddine and Marcolli,

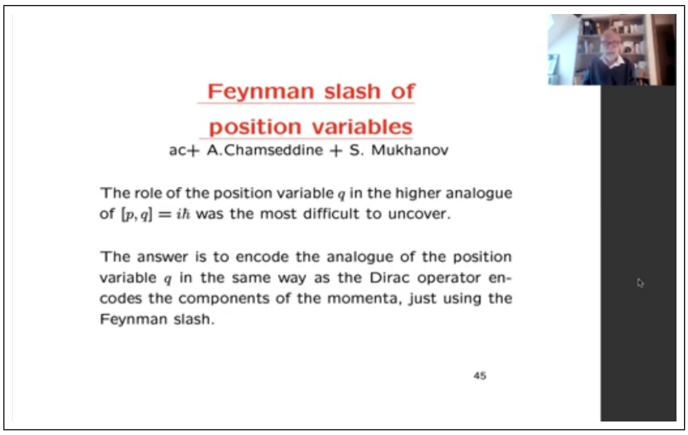


and what we have discovered, you know, with Chamseddine, we had been abandoning our work of understanding the Standard Model in 1998, we had done it in 1996 and we had been abandoning in 1998 because of the discovery of neutrino mixing. And it seemed to be impossible to accomodate neutrino mixing with what we had. But in what we discovered in 2005, the three of us, is that in fact, if you try the various KO-dimension for the finite space, you know, as you put it in the fine structure of space-time, then amazingly, you find that if you take dimension KO-dimension 6, for this finite space that has of course metric dimension 0, then, not only the neutrino mixing comes out absolutely naturally, but also the seesaw mechanism. And I must say I was amazed because I didn't now seesaw mechanism and then I did the calculation of what we had, and I re-discovered the seesaw mechanism. But unlike in physics, it is not put by hand, you find it as a consequence of the calculation.



So what we did later, with Ali[1], we classified the various finite spaces of various KO-dimensions, and of course, we were interested in KO-dimension 6, and among them, we found one which we found extremely interesting, where the algebra that was underlying the finite space (so it's a finite dimensional algebra) was 2×2 matrices over quaternions, + 4×4 matrices over complex numbers. In what we were doing, the breaking to the Standard Model gauge group was done by what we called the order ??, but then by joint work with van Suijlekom, we analyzed the full model, without reduction to Standard Model, and we found a beautiful Patti-Salam model which is in fact much more interesting and symmetric, than the Standard Model itself, in particular because asymptotic freedom.

Okay so in fact, at this point, we had found, and we would, you know, be extremely interested in some kind of other way of finding the same algebra. But this algebra is strange in the sense that the real dimension is different from the two sides. You have 32 and 16, so it looks like a very difficult thing to obtain this in a natural manner.

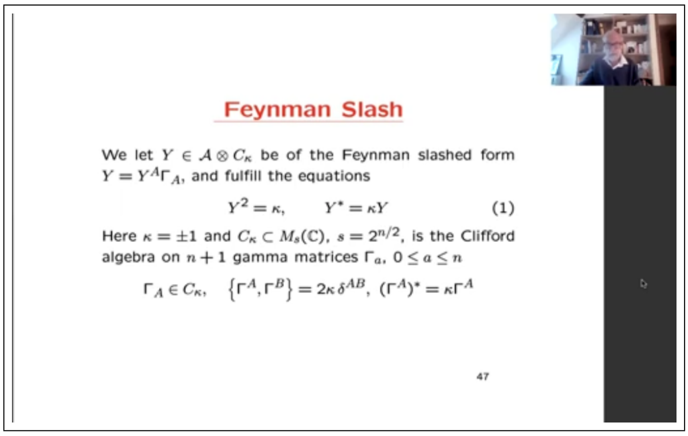


But this is what we did in our work with Mukhanov. The new idea that came up there is that now, we are not only going to encode, if you want, the full momenta together, assemble all the momenta

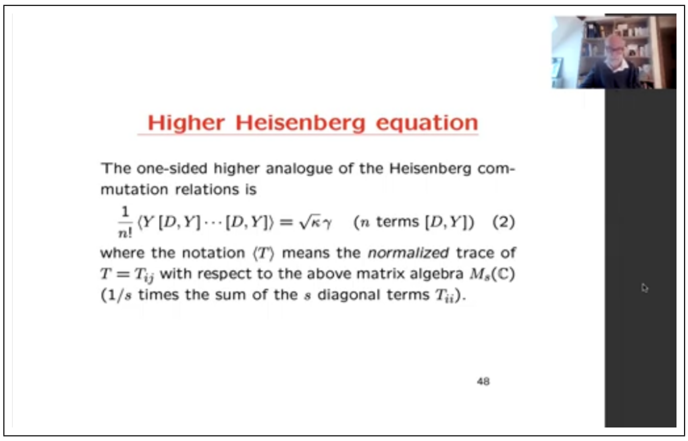
[1] Chamseddine

together, as Dirac did, you know, using the Dirac operator, which was blending together all the components of the momentum to a single entity, but we investigated what would happen if we did the same thing with the coordinates. And what we obtained is an higher analogue of the Heisenberg commutation relations.

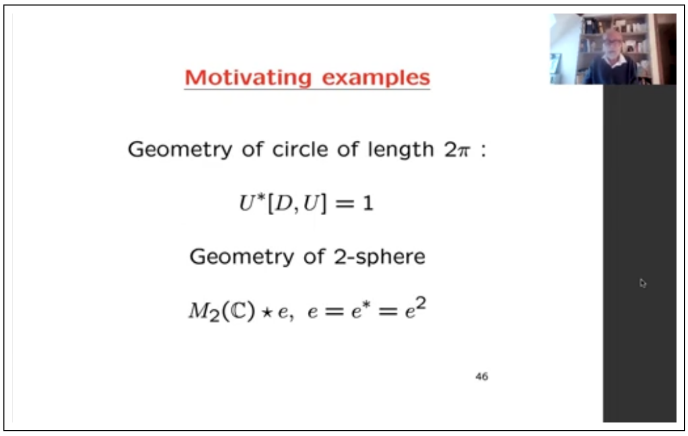
What we first investigated, as I will explain, we wanted to blend together the coordinates to a single operator, so we started of course with the



Feynman slash, okay ? We wanted to assemble them into a single entity, and what we found very quickly is that the right condition to ones assemble, because of the Clifford matrices and the relations they were fulfilling, was to have a self-adjoint Y, or ?-adjoint depending on the value of κ that is plus or minus one, that satisfies $Y^2 = \kappa$ (either is self-adjoint of ???-adjoint). This is based on gamma matrices, and at first it is very reminiscent of the sphere because then, the components Y ? A has to satisfy that the sum of their squares is equal to 1.



So at first, we wrote a sort of Heisenberg higher type of equation, which is like a commutation relation : if you want to understand what is behind this, I have to come back to a much simpler example,



which is the geometry of the circle of length 2π. It's an easy exercize to show that if you look at the geometry of the circle of length 2π, it is uniquely specified by an equation, an operator theoretic equation, which is $U^*[D,U] = 1$. U is a unitary operator, D is a self-adjoint operator, and there is a unique, essentially up to a parameter that plays no role in the metric, irreducible representation of these relations into Hilbert space operators. And when you compute, you find that the spectrum of U has to be the circle and the operator D defines the metric and you find that the corresponding circle has length 2π. And of course, you find this geometry in the right way. Similarly we had started in fact many years ago with Gianni Landi to do the geometry of the 2-sphere in a similar manner by combining 2×2 matrices with projections.