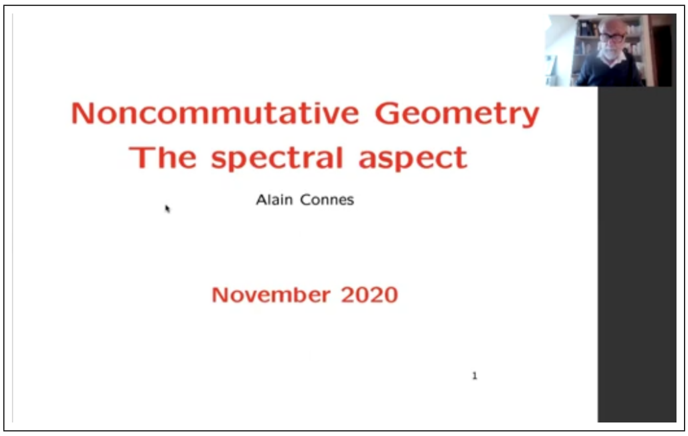
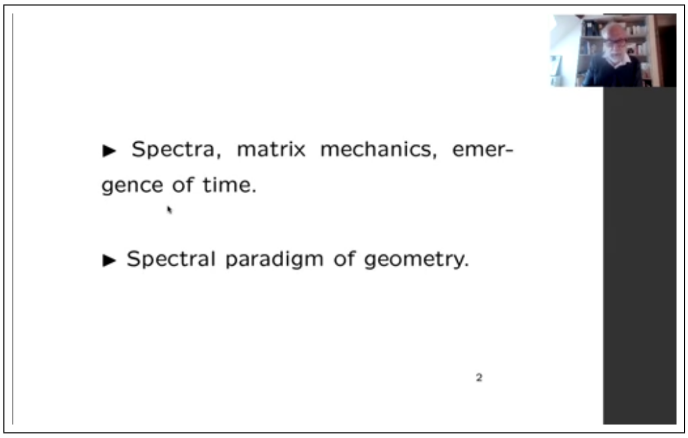
Noncommutative geometry, the spectral aspect
Alain CONNES
23.11.2020



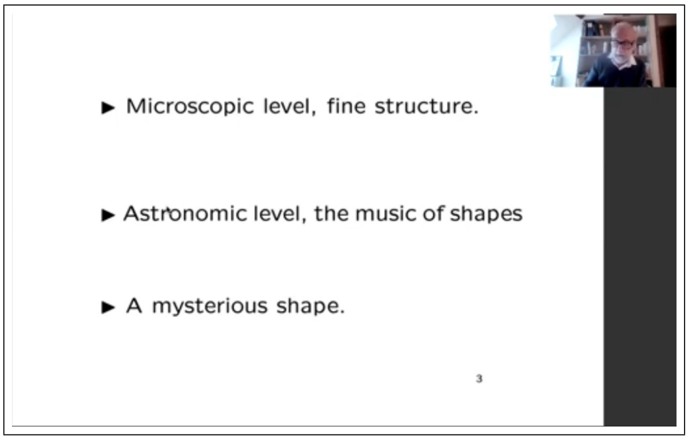
OK, so let me start by saying that I am really grateful for this occasion to talk about noncommutative geometry. And I will concentrate on the spectral aspect of the subject. So somehow, I will start by explaining the origin.



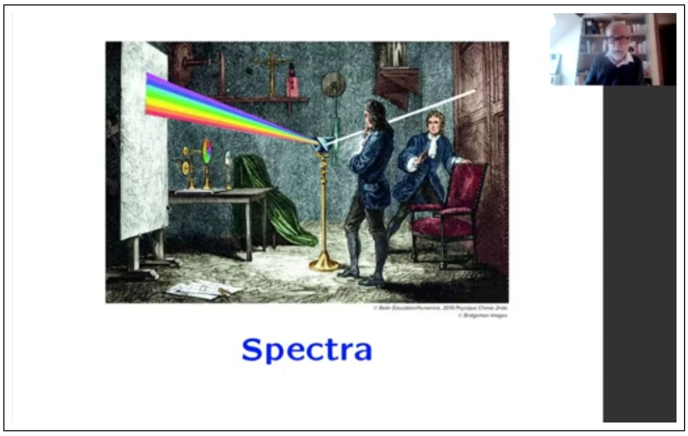
They are spectra, how it leds Heisenberg to matrix mechanics, and emergence of time, as I will explain, which is related to the ideas of von Neumann. Now, the next point will be the spectral paradigm, the new paradigm that comes from dealing with noncommutative spaces, which is spectral. And this will be analysed and explained at two levels.

First at the microscopic level, it will give the fine structure of space-time at the euclidean level. And at the astronomic level, it will reveal the music of shapes. And I will end by exhibiting a mysterious shape which is related to recent work with Katia Consani.

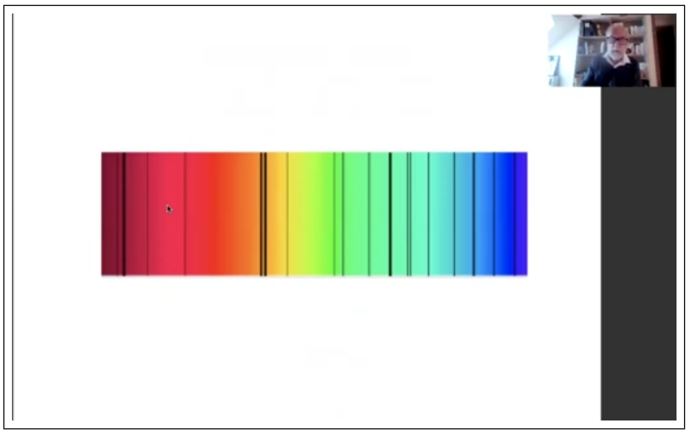
Conférence donnée à distance dans le cadre du cycle de conférences de l'Université de Harvard "Lecture series Mathematical Science Literature",
Vidéo visionnable ici https://youtu.be/AwVRss0F6zI.
Transcription Denise Vella-Chemla, novembre 2020.



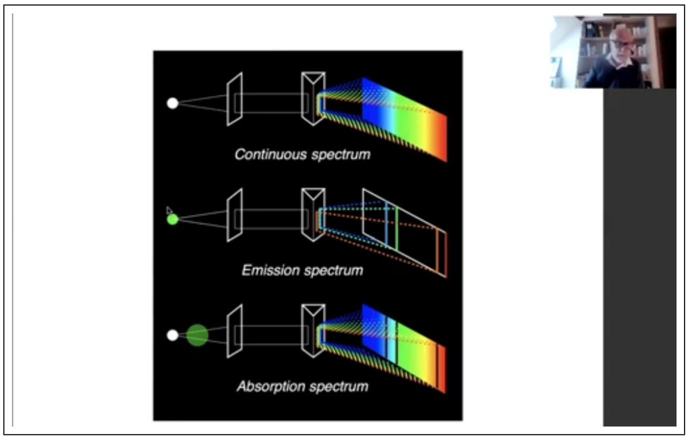
So let me start with the old times. This picture represents what happened for instance when Newton was decomposing a straight ray of light coming from the sun by letting it go through a prism.



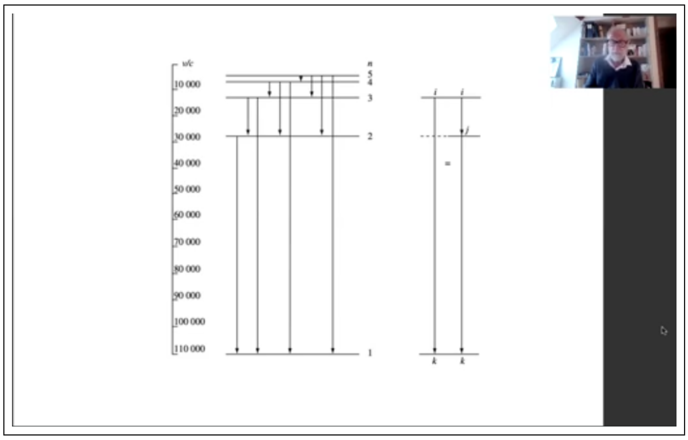
And one obtains the rainbow. What is really interesting about this rainbow is that when you look at it very carefully, you find out that there are some missing lines,



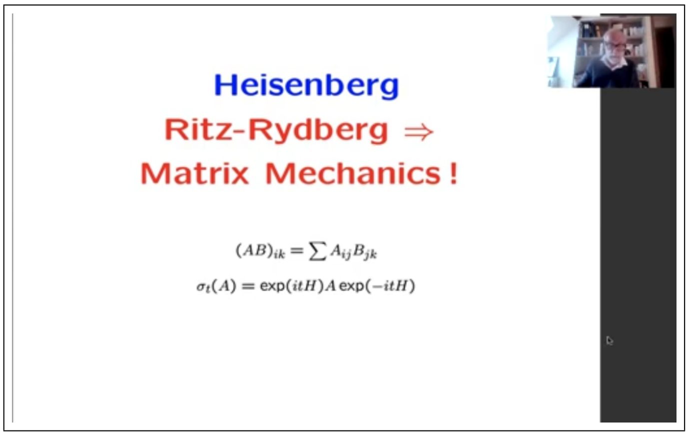
there are some dark lines. At first, one was discovered for sodium. The real discovery was made by Fraunhofer at the beginning of the 19^{th} century. He exhibited about five hundred of these dark lines which are understood now as the absorption lines, in the sense of what happens is that when the light goes through some chemical like in the neighborhood of the sun, then the presence of these chemicals has a consequence which is that the sort of signature of the chemicals appears in negative, through these dark lines. Somehow, few years later



around 1860, what was discovered by Bunsen and Kirchhoff was that in fact one could obtain the same lines but now as bright lines over a dark background, it is if you want the negative of the previous and after that, they were able to identify many many of the lines which had been identified as absorption lines by Fraunhofer, they were able to identify most of them as coming from chemicals. So this means that each chemical has a sort of bar-code that is its own signature. And what they found also is that there were few of these lines that actually would not pertain to any chemical body that was known on Earth, so they invented a new chemical body which they called Helium, in the honour of the Sun, of course, and what is amazing is that at the beginning of the twentieth century, there was an eruption of the Vesuvio and people did spectral analysis of the lava coming out from the volcano and amazingly, they found that the corresponding emission spectrum was exactly corresponding missing lines found before and it was Helium. And of course, you know, now, Helium is used on Earth. This is just clearly a featuring of the fact that chemicals have their own bar-code.

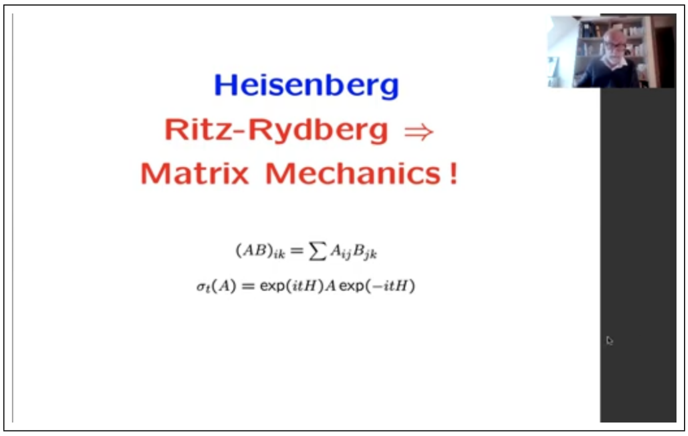


Now these bar-codes were studied by physicists and what happens is that they have a quite remarkable compatibility property that is that some of these lines, when you express them in terms of frequency, you have to be very careful that you should express them in terms of frequency and not wavelength, some of them actually add up. And in order to understand how they add up, it's Ritz-Rydberg who found what is called the Ritz-Rydberg principle, and the idea is that these lines would be indexed not by one index but by two indices, it could be greek letters whatever you want, and the point is that Ritz-Rydberg principle tells you that the line with indices $\alpha\beta$ will combine with the line with indices $\beta\gamma$, so I mean the second index of the first line has to be the first index of the second line, and then they combine and they give you the line corresponding to $\alpha\gamma$. Now, this Ritz-Rydberg combination principle had one amazing consequence in the hands of Heisenberg



and what Heisenberg found out is that thanks to this principle, he was doing calculations when he was alone

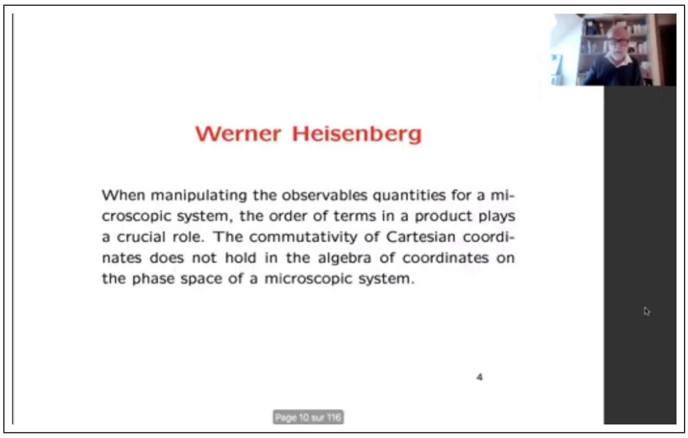
in Helgoland where he had been sent because he had allergy, he had been sent by his university, because there was no cure except to send people in a place where there was no source of pollen. So he was there and he had all the time he wanted to work and at some point, during a night, I think it was near four o'clock in the morning, he had proven that the energy is conserved[1],



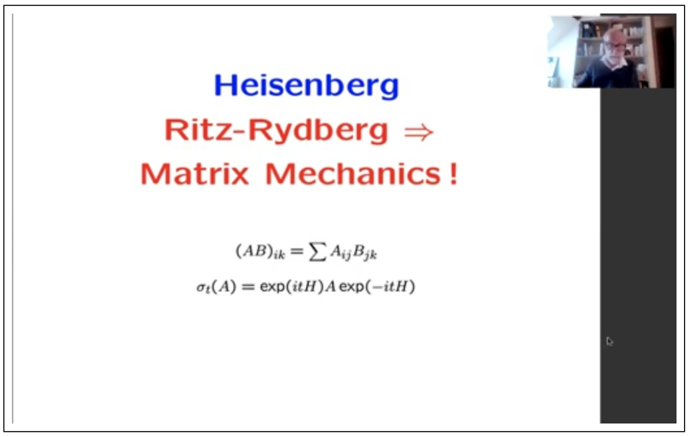
because if you take H to be A, then, there is a commutation between these two terms and you will get that H is preserved by time evolution.

[1]surrounding last line.

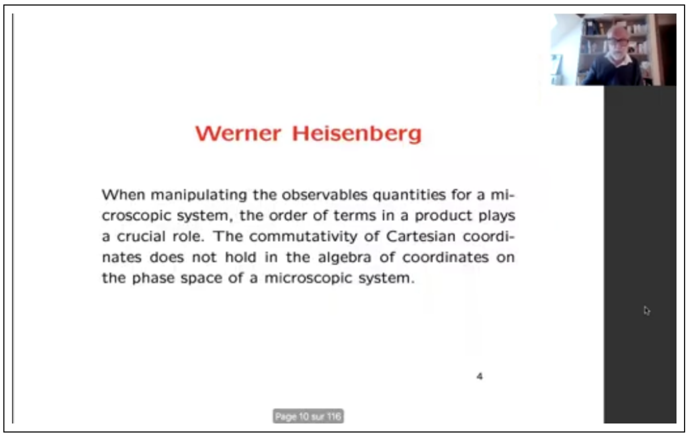
Then, instead of going to bed, what he did was to climb over one of the peaks which was along the coast and he waited for the sunrise on the top of this peak. And he explains that he was seeing, of course in his mind in his discovery, an incredible landscape. What he had discovered had one peculiar consequence, and that consequence was that, because matrices don't commute,



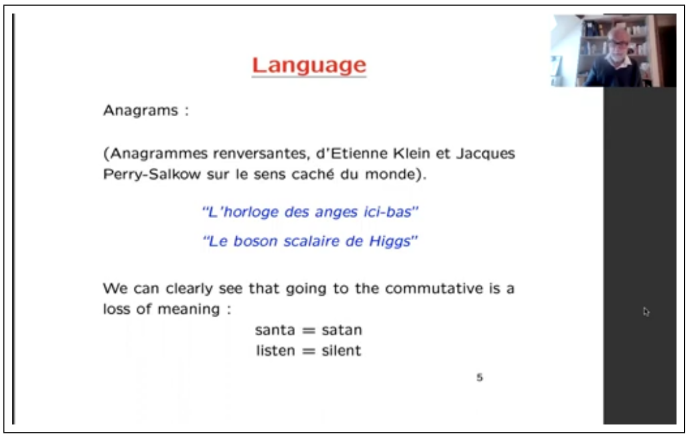
when you work with observable quantities for a microscopic system, yu have to pay attention to the order of terms in a product. In fact, the order of terms in a product plays a crucial role. And in fact, if you come back to the evolution equation of Heisenberg,



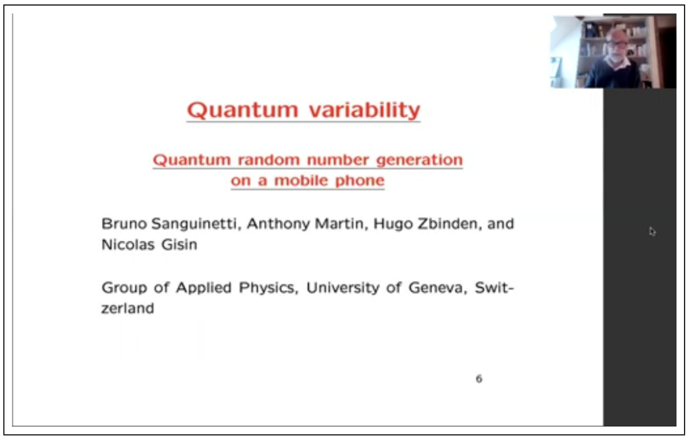
you find of course that if everything would commute, this evolution should be the identity. In fact, as we shall see much later, the commutative world is static, whereas quantum world is dynamical, and this is the first instance.



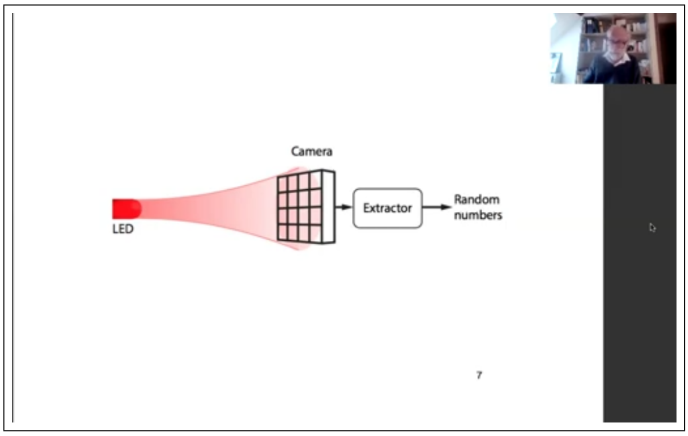
Now, in particular, what it means is that the commutativity of cartesian coordinates does not hold in the algebra of coordinates on the phase space. And this is one fundamental instance of appearance of such a noncommutative space.



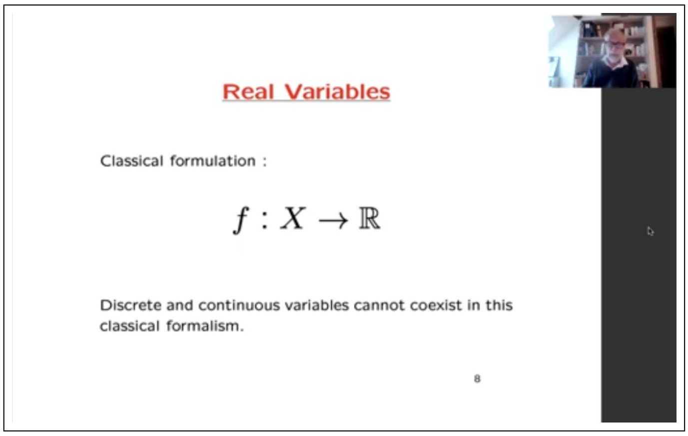
Now, as a corollary of this, you might think that this is very strange, and that, you know, dealing with this care with the order is something we are not used to, but this is wrong. We are perfectly used to that, in the language. I mean, when we use words, we need, of course, to pay attention to the order of the letters, and the order of the words, otherwise, you get anagrams. What I have shown here is a french anagram which is quite amazing but somehow, one can clearly see that when you go to the commutative, you lose meaning. For instance, I have written here for example, Santa and Satan are the same in the commutative world (there are two a, one s, one n, one t). Listen is the same thing as silent, and so on. So in fact, what you find is that this quantum way, this way of being forced to pay attention to the order of the letters, is a way to keep meaning. So in ordinary algebraic geometry, one forgets completely about these nuances.



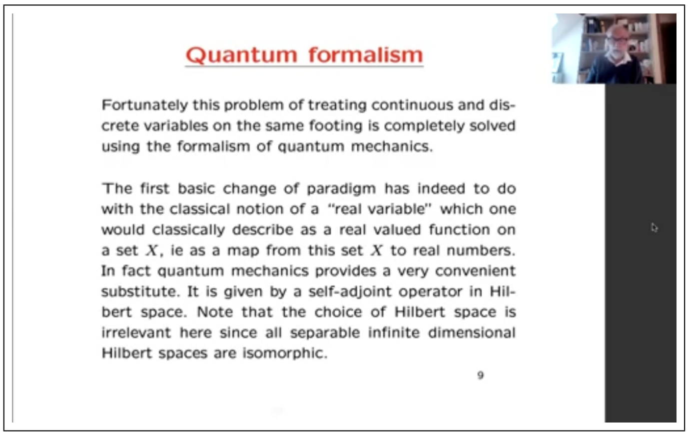
Now one corollary of the non-commutativity, of the Heisenberg uncertainty principle, is quantum variability. And to understand this quantum variability, one needs for instance to give an example.



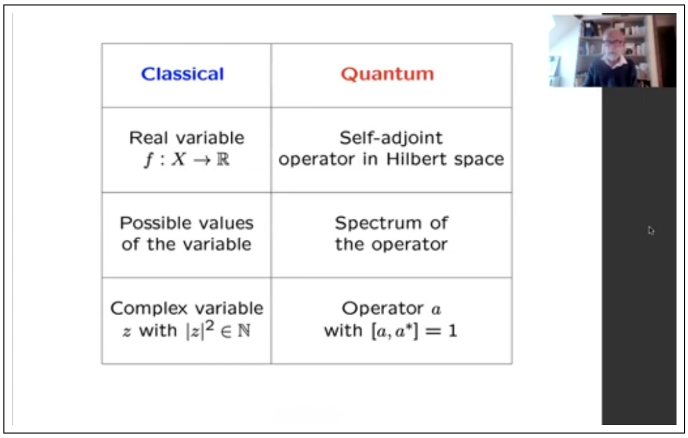
Several swiss engineers have manufactured a small device which you can use in a mobile phone and which will generate random numbers. But the way they will generate those random numbers is simply by letting a photon go through a small slit and land somewhere on a photo cell, and which one of these cells it will land on is something which is totally unpredictible, by the uncertainty principle of Heisenberg. And so, what this gives for you is a way to generate random number, and this way of generating a random number cannot be attacked. For security reasons, it's a way which contrary what you would obtain if you would generate quantum numbers from a computer. It's totally different not only by experiment, but also by the theory : you know that they are not reproducible. So there is this fundamental variability,



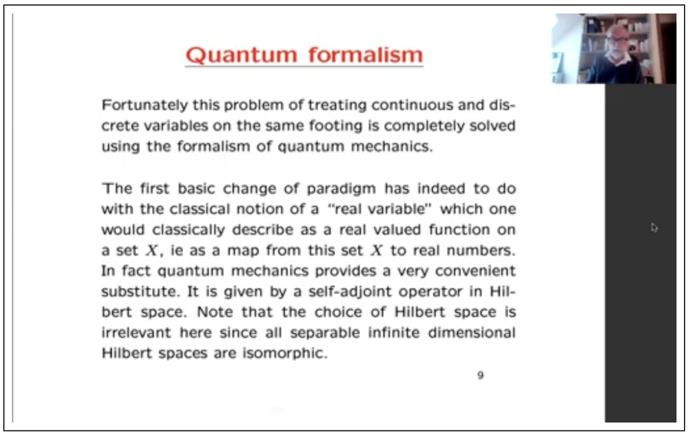
which is in quantum mechanics and when you think about it, you will find out that quantum mechanics is in fact a much more better formalism of variability than ordinary classical mathematics. For instance, if you ask a mathematician what is a real variable, very often you will get as an answer the fact that it's just a map f from some set X to the real line. Now it turns out that this formalism is in fact rather poor because you cannot have coexistence of discrete and continuous variables, in this classical formalism. The reason is very simple. The reason is that if you have a continuous variable, in the given X, then this given X has to be uncountable. And then, any variable meant to be discrete will in fact take some value an infinite number of time, and in fact more than an infinite countable number of times. So they don't coexist.



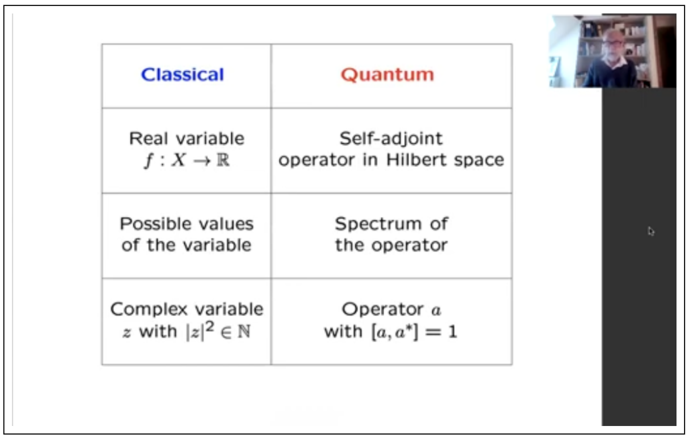
And amazingly, they coexist in the quantum formalism. So if you want, the continuous and the discrete coexist in the quantum formalism because in this formalism, a real variable becomes a self-adjoint operator



in Hilbert space. And in the same Hilbert space, you can have self-adjoint operator which is for instance a multiplication by x in the Hilbert space which is L^2 functions over $[0, 1]$, but this Hilbert space of L^2 functions over $[0, 1]$ is isomorphic to the Hilbert space which is the Hilbert space of ℓ^2 sequences on the integers in which you also have another variable, if you want, which is the multiplication by n, which is self-adjoint, and which is obviously discrete.

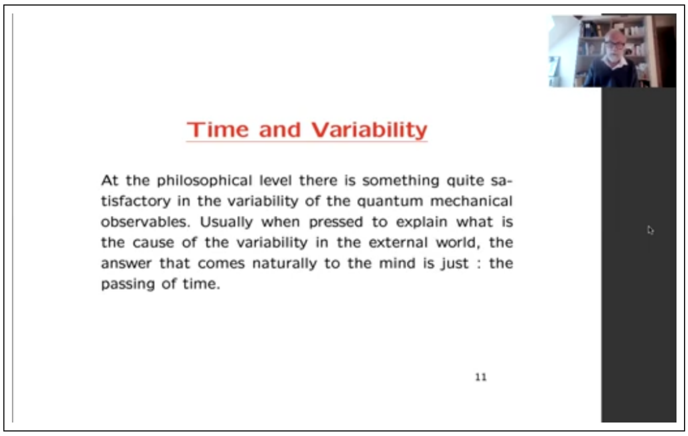


So if you want, because there is only one Hilbert space, namely infinite dimensional with countable basis, what you find out is that there is coexistence of the discrete variables, with the continuous variables with the only proviso that they cannot commute. There is this nuance, and this nuance will play a fundamental role later as we shall see.

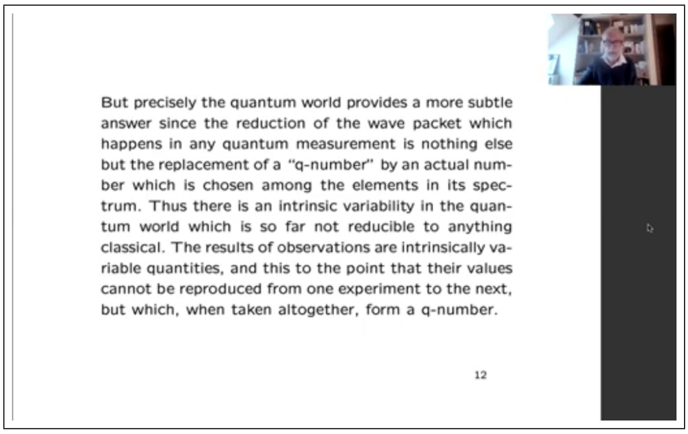


Classical	Quantum
Real variable $f : X \to \mathbb{R}$	Self-adjoint operator in Hilbert space
Possible values of the variable	Spectrum of the operator
Complex variable z with $\|z\|^2 \in \mathbb{N}$	Operator a with $[a, a^*] = 1$

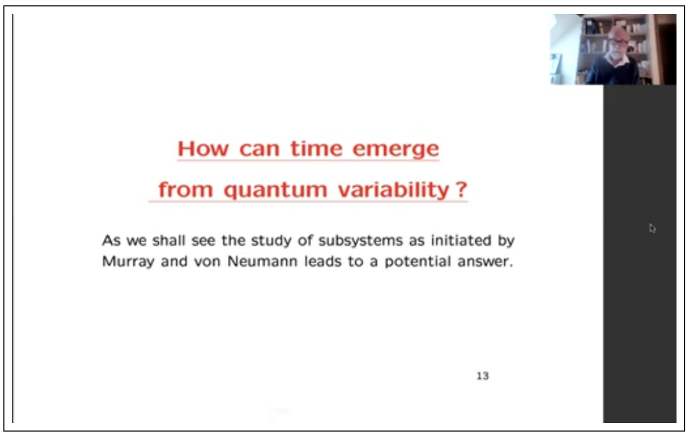
So we have this dictionary, which is coming from the quantum. And of course, the values of a real variable is just a spectrum of the self-adjoint operator, but physicists have been very very early own capable of applying this notion to complex variables. In fact, they applied it to a very peculiar situation where you would like to have a complex variable z which is such that $|z|^2$ is an integer. This is related to the Planck discovery in 1900 and to what Einstein wrote in 1906 which is that the energy of an oscillator should only take integral multiples of $h\nu$. The oscillator was first understood in a paper of Born, Heisenberg and Jordan, I think in 1925, and then Dirac was able to use this very same ansatz in which you replace variable z, that was supposed to be a complex variable, you replace it by an operator a, and the only condition on that operator a is that its commutator with its adjoint is equal to 1. That suffices to ensure that the spectrum will be formed of positive integers, it's a little exercize. And in the hands of Dirac, this allowed him to actually prove what Einstein had guessed when he had guessed the constants A and B of emission and absorption of an atom. So this is a very successful and a very amazing formalism, which replaces the classical formalism.



And there is something in fact which is quite striking if you want at the level of the variability, which is that normally, when we are addressed by people to explain what is really, the essence of variability, what is the cause of variability in the external world, the usual answer that comes, I remember giving this answer when I was in highschool, the natural answer that comes to mind is just the passing of time. This is the only sort of reasonable answer we are able to give. But now, because of this intrinsic and sort of fundamental variability which there is in the quantum, comes a very natural question and

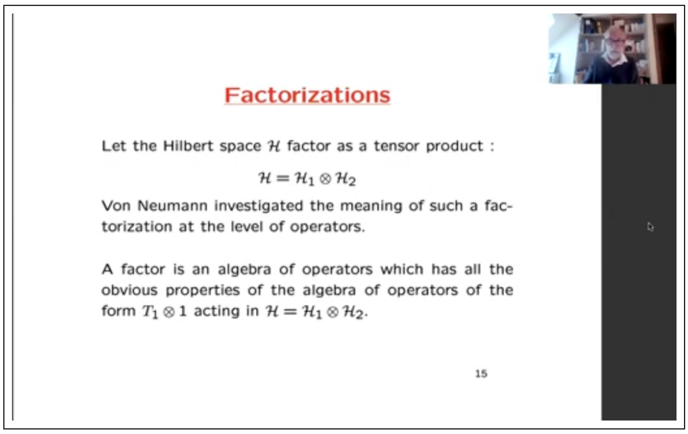


this question is... you know, of course, we have not been able in the formalism of quantum mechanics to reduce this variability because of the reduction of the wave packet which is something which is outside the time evolution, so if you want, this intrinsic variability in the quantum world sort of poses a very natural question and this natural question is

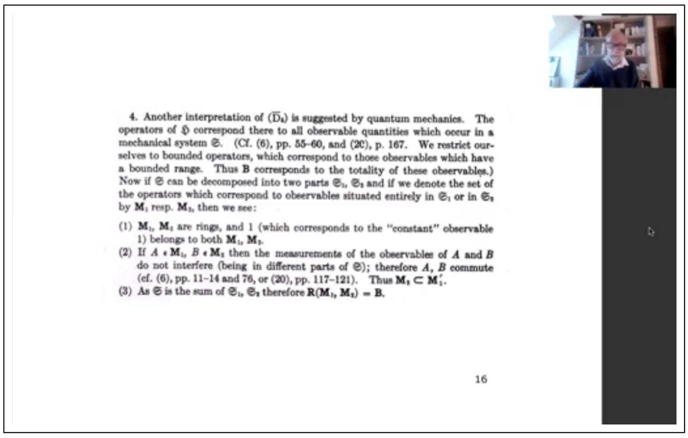


would it be more primitive that the passing of time ? Namely, how could time emerge from this quantum variability ? And what I want to explain briefly is that the study of sub-systems which was initiated by Murray and von Neumann, in the 1930-1940, leads in fact to a potential answer to this question. What did they do ? This is just a picture

just to make sure not to forget that von Neumann is also very well known for inventing computers. But what did they do ?



They studied, they started by studying space factorizations. And in that respect, they were motivated by quantum mechanics. So they wanted to understand that if you happen to have a Hilbert space $\mathcal{H}$ which is a tensor product, which splits as a tensor product, then you can consider in this Hilbert space the operators which are of the form $T_1 \otimes 1$ where T_1 is acting in $\mathcal{H}_1$ and 1 is the identity in $\mathcal{H}_2$. Somehow you want to understand algebraically what are the algebras which appear in this way. So they motivated their work by quantum mechanics,



4. Another interpretation of $(\overline{D}_4)$ is suggested by quantum mechanics. The operators of $\mathfrak{H}$ correspond there to all observable quantities which occur in a mechanical system $\mathfrak{S}$. (Cf. (6), pp. 55–60, and (2C), p. 167. We restrict ourselves to bounded operators, which correspond to those observables which have a bounded range. Thus $\mathbf{B}$ corresponds to the totality of these observables.) Now if $\mathfrak{S}$ can be decomposed into two parts $\mathfrak{S}_1$, $\mathfrak{S}_2$ and if we denote the set of the operators which correspond to observables situated entirely in $\mathfrak{S}_1$ or in $\mathfrak{S}_2$ by $\mathbf{M}_1$ resp. $\mathbf{M}_2$, then we see:

(1) $\mathbf{M}_1$, $\mathbf{M}_2$ are rings, and 1 (which corresponds to the "constant" observable 1) belongs to both $\mathbf{M}_1$, $\mathbf{M}_2$.
(2) If $A \in \mathbf{M}_1$, $B \in \mathbf{M}_2$ then the measurements of the observables of A and B do not interfere (being in different parts of $\mathfrak{S}$); therefore A, B commute (cf. (6), pp. 11–14 and 76, or (20), pp. 117–121). Thus $\mathbf{M}_2 \subset \mathbf{M}_1'$.
(3) As $\mathfrak{S}$ is the sum of $\mathfrak{S}_1$, $\mathfrak{S}_2$ therefore $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$.

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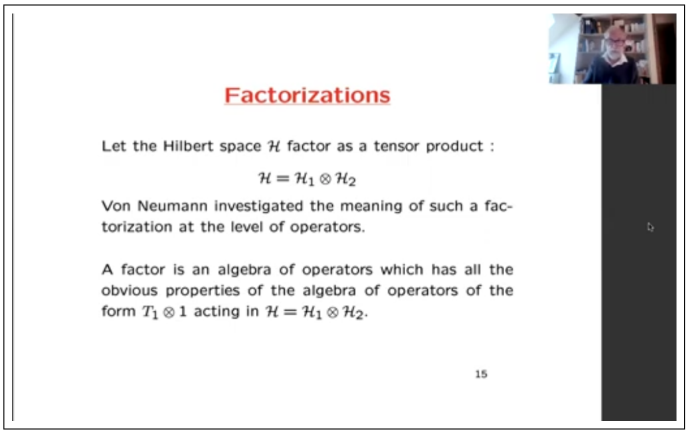
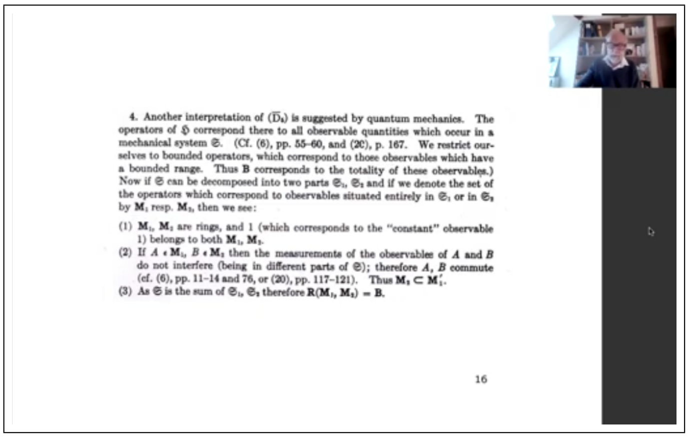
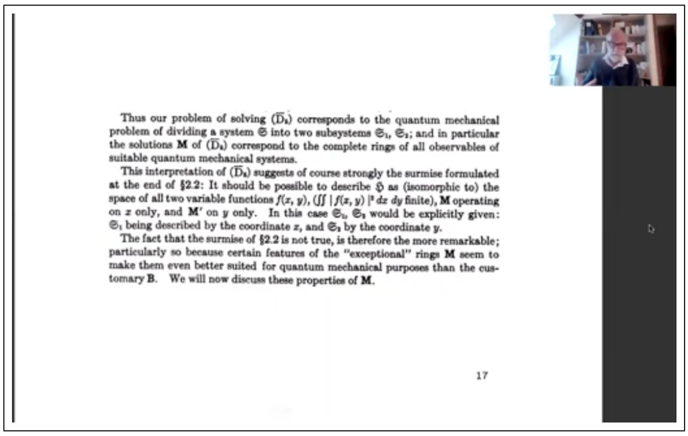
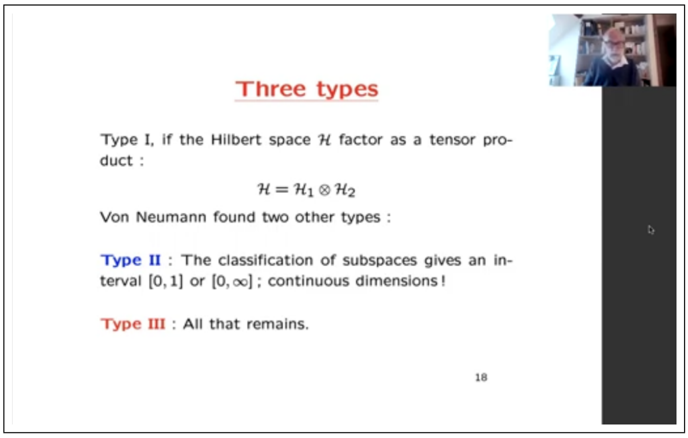
by of course saying that you want to consider observable quantities which occur in a sub-system, then of course you are dealing with rings of operators, with algebras of operators, and you have a commutation, between what happens in one system and in the complementary system, and so on. So they studied these factorizations and the term factor comes from designing algebras that you will have, that would imitate this situation of a tensor product.



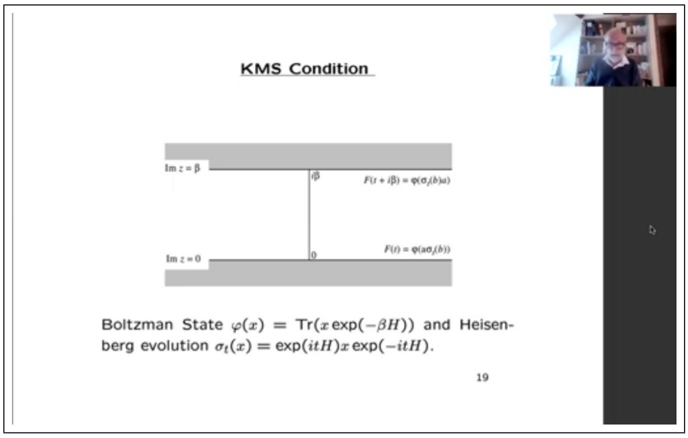




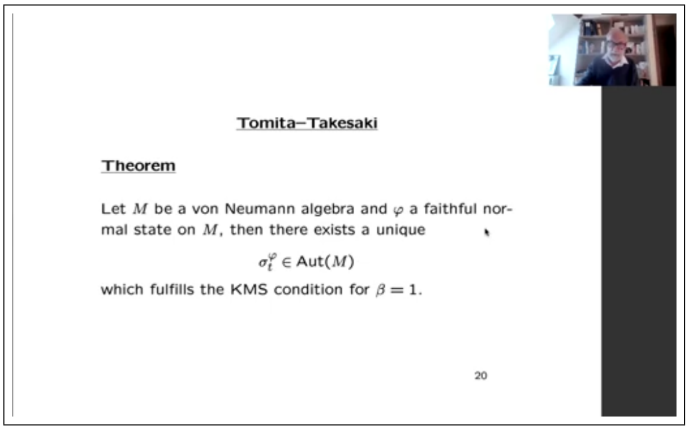
But amazingly, what Murray and von Neumann found is that, besides the factorization, which occurs from factoring the underlying Hilbert space, it turns out that there are factorizations which do not come from there. And so the factorizations that come from factorizing the Hilbert space are called of type I, they are the simplest by far.



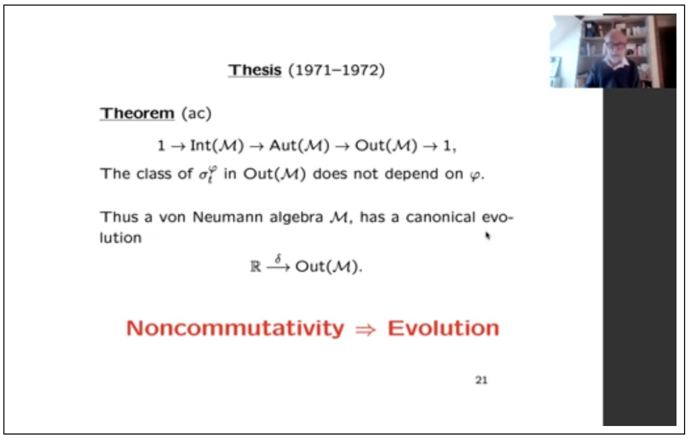
But they found two other types. They found what are called type II, and type II, what does it mean, in which way, if you want, the type II factorizations are different, are distinct from the type I factorizations, well, they are very distinct, because when you consider a type I factorization, after all, the algebra will just be the algebra of operators in a given Hilbert space. So, if you want, what would correspond to the subspaces are classified by the integers, by the dimension of the subspace, it could be infinite of course. Now, in the case of the type II, what happens is that what correspond to the subspaces are no longer classified by an integer but they are classified depending on type II_1 or type II_∞, either by the interval $[0, 1]$ or $[0, \infty]$. And I mean, this is the first appearance of continuous dimensions which... I remember reading a paper of von Neumann when I was in École Normale, and this, really, intrigued me a lot, the fact that there are those continuous dimensions that appear. And then, what do you have, you have the type III and the type III is all that remains.



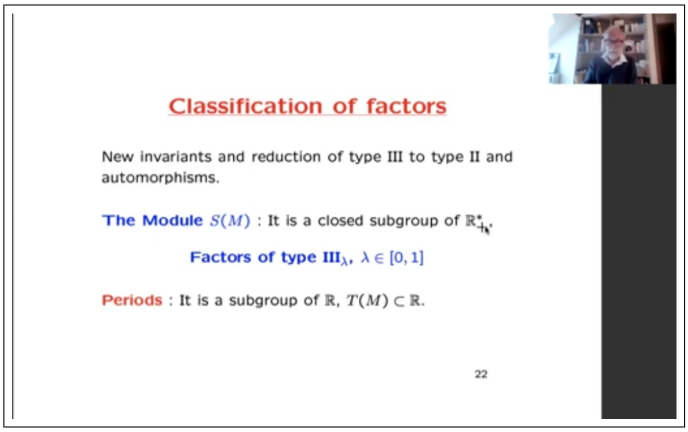
I mean in fact, came as an important tool the fact that the link between the Boltzman state which is given when you consider all operators in the Hilbert space by the trace of x multiplied by the exponential of $-\beta H$ where H is the Hamiltonian and β is the inverse temperature. So this is related to the Heisenberg time evolution which I showed you before, namely $\sigma_t(x) = exp(itH)xexp(-itH)$. They are related together by something which can be formulated purely algebraically in terms of the state itself and the time evolution. And this is the Kubo-Martin-Schwinger (KMS) condition, which is a condition that can be formulated in terms of holomorphic functions.



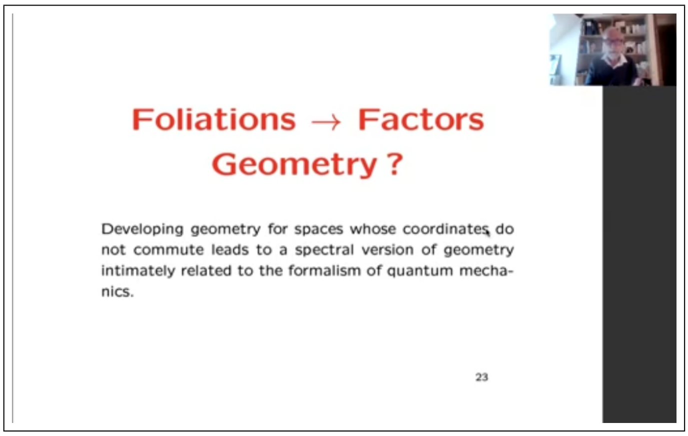
And a very important step was done by Tomita and Takesaki around 1970 when they proved that this association between a state and a one-parameter group of automorphisms actually holds for any von Neumann algebra. So if you take a von Neumann algebra, and take any faithful normal state on it, then there exist a unique one-parameter group of automorphisms that actually fulfills this KMS condition of the association for $\beta = 1$. I started my thesis and in my thesis, what I proved



in 1971-1972, in april 1972 is that in fact, this one-parameter group of automorphisms is unique, when you look at it in the quotient of the group of automorphisms of $\mathcal{M}$ divided by inner-automorphisms. You see, when an algebra is not commutative, it admits trivial automorphisms, namely automorphisms that are obtained by conjugating an element by a unitary element in the algebra, so by x goes to UxU^*. And because these automorphisms are completely trivial in a certain way, they form a normal subgroup of the group of automorphisms and the interesting automorphisms are forming a quotient group which is the group $Out(\mathcal{M})$. So what I proved in my thesis which was under Jacques Dixmier, I proved that in fact, there is a unique, independent of the choice of the state, homomorphism from the real line to the group $Out(\mathcal{M})$ of automorphism classes of $\mathcal{M}$. This is an amazing fact in the sense that what it tells you that this algebra, just from its non-commutativity, acquires an evolution. This of course gave me



the classification of factors. So I could define new invariants, and I could also reduce type III to type II and automorphisms. In fact, I left one case open which was later done by Takesaki. But I had defined two fundamental invariants, the module $S(M)$ which is a closed subgroup of $\mathbb{R}_+^*$ and which allowed to classify, if you want, the factors of type III into type III_λ where λ belongs to $[0,1]$ and the reduction from type III to type II, I did in the case where λ was different from 1. The III_0 case was particularly interesting. And I also defined the group of periods, which is a subgroup of the real line, but this time, it's not a closed subgroup, it can be quite wild. And it's a remarkable subgroup in the sense that what additive it is, there are certain times, from the subgroup of the line, which are periods of the factors namely which the factor doesn't move.



Once I have done this work, I arrived in IHES in Bures, and I found out that, of course I was a specialist of a specific topic, but the people preoccupations were rather far from mine and I had the luck to meet Dennis Sullivan, and to discuss with him a lot, and after these discussions, I found that there was a completely canonical way to associate a von Neumann algebra that in the most case is a factor to foliations. So foliations are very familiar objects in differential geometry, essentially what they are are decompositions of the product but given locally only and what is interesting is not their local properties which are trivial but their global properties. And what was amazing is that this association I had found from foliations to factors allowed me to exhibit the most exotic factors in the simplest case of foliations. For instance if you take the Kronecker foliation of the torus this gives you type II_∞ hyperfinite, if you take for instance the ... of foliations of the sphere bundle of a Riemann surface, this gives you the unique type III_1 hyperfinite factor which is extremely exotic. On the other hand you know what happens is that this association from foliations to factors in von Neumann algebras was only taking into account the major theory of foliations. But foliations are much richer in a way. They belong to functional geometry. So they have differential structure. They have a topology and so on and so forth. And this led to develop geometry for spaces whose coordinates do not commute, because when you deal with algebra of foliations, of course the factors you get are not commutative. This non commutativity comes from the fact that you are allowed to slide along the leaves. So this led me to a spectral version of geometry, which I want to present, and this is closely related to the formalism of quantum mechanics. And as a warm-up, one has to understand what is sort of miraculous in this formalism of quantum mechanics and why it can be so pertinent and so useful, for doing geometry.