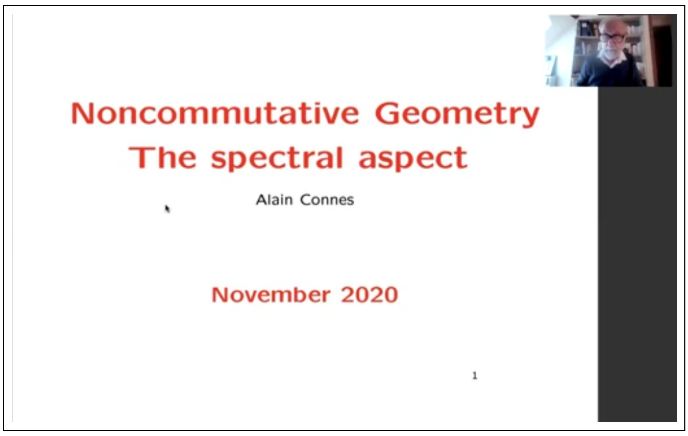
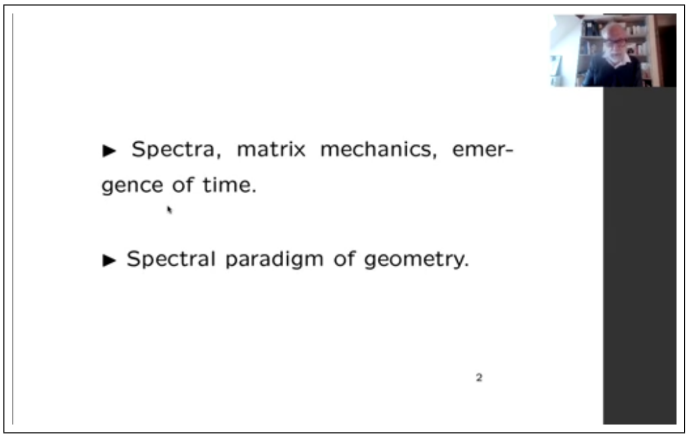
Noncommutative geometry, the spectral aspect
Alain CONNES
23.11.2020



OK, so let me start by saying that I am really grateful for this occasion to talk about noncommutative geometry. And I will concentrate on the spectral aspect of the subject. So somehow, I will start by explaining the origin.



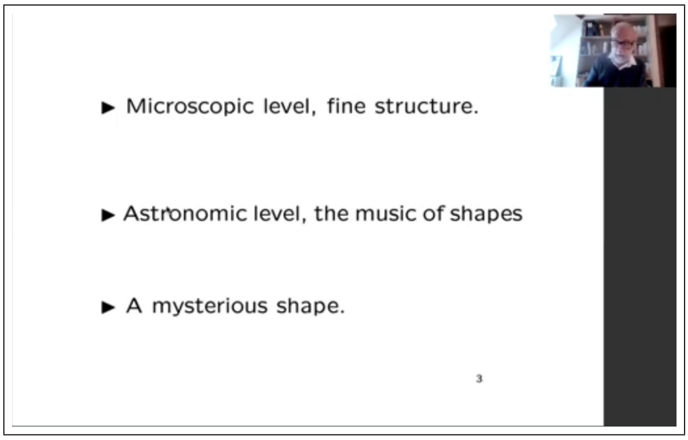
They are spectra, how it leds Heisenberg to matrix mechanics, and emergence of time, as I will explain, which is related to the ideas of von Neumann. Now, the next point will be the spectral paradigm, the new paradigm that comes from dealing with noncommutative spaces, which is spectral. And this will be analysed and explained at two levels.

First at the microscopic level, it will give the fine structure of space-time at the euclidean level. And at the astronomic level, it will reveal the music of shapes. And I will end by exhibiting a mysterious shape which is related to recent work with Katia Consani.

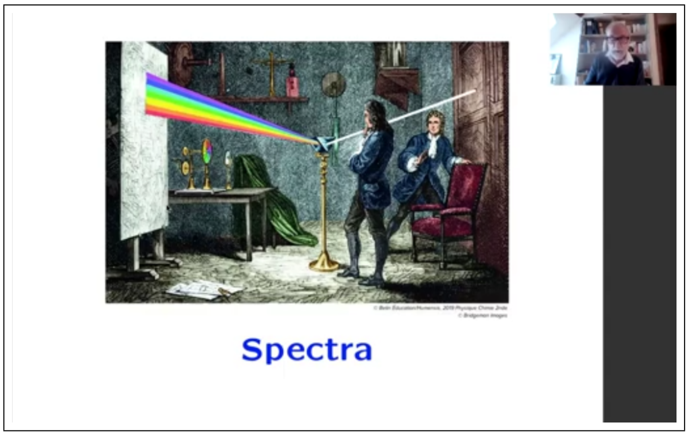
Conférence donnée à distance dans le cadre du cycle de conférences de l'Université de Harvard "Lecture series Mathematical Science Literature",

Vidéo visionnable ici https://youtu.be/AwVRss0F6zI.

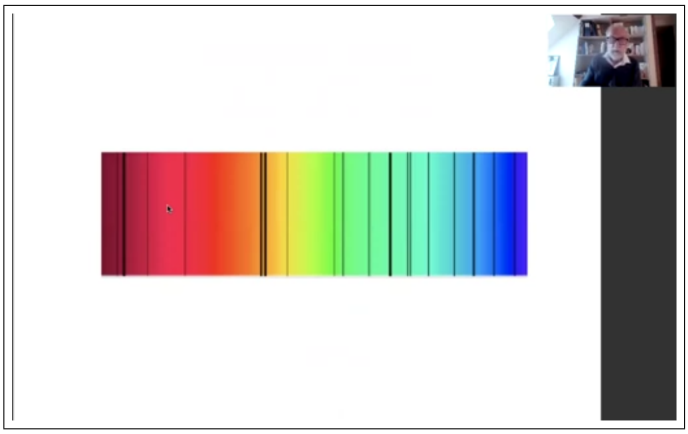
Transcription Denise Vella-Chemla, novembre 2020.



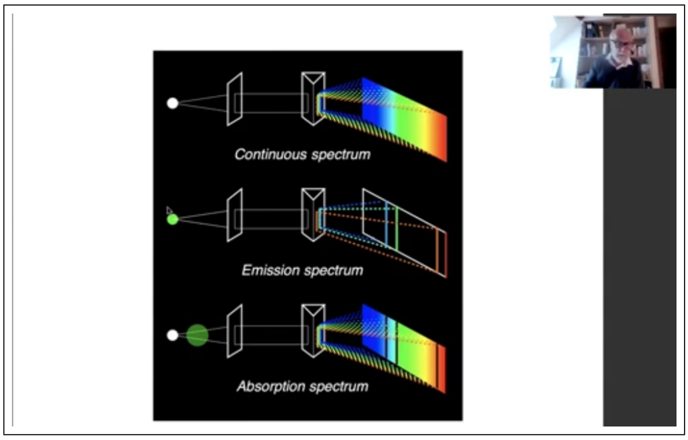
So let me start with the old times. This picture represents what happened for instance when Newton was decomposing a straight ray of light coming from the sun by letting it go through a prism.



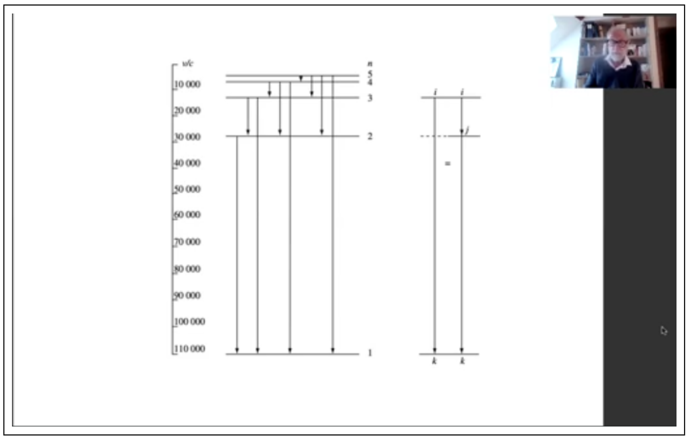
And one obtains the rainbow. What is really interesting about this rainbow is that when you look at it very carefully, you find out that there are some missing lines,



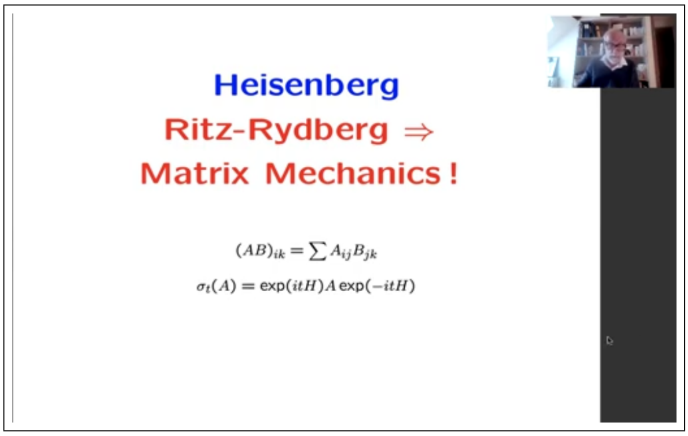
there are some dark lines. At first, one was discovered for sodium. The real discovery was made by Fraunhofer at the beginning of the 19$^{\text{th}}$ century. He exhibited about five hundred of these dark lines which are understood now as the absorption lines, in the sense of what happens is that when the light goes through some chemical like in the neighborhood of the sun, then the presence of these chemicals has a consequence which is that the sort of signature of the chemicals appears in negative, through these dark lines. Somehow, few years later



around 1860, what was discovered by Bunsen and Kirchhoff was that in fact one could obtain the same lines but now as bright lines over a dark background, it is if you want the negative of the previous and after that, they were able to identify many many of the lines which had been identified as absorption lines by Fraunhofer, they were able to identify most of them as coming from chemicals. So this means that each chemical has a sort of bar-code that is its own signature. And what they found also is that there were few of these lines that actually would not pertain to any chemical body that was known on Earth, so they invented a new chemical body which they called Helium, in the honour of the Sun, of course, and what is amazing is that at the beginning of the twentieth century, there was an eruption of the Vesuvio and people did spectral analysis of the lava coming out from the volcano and amazingly, they found that the corresponding emission spectrum was exactly corresponding missing lines found before and it was Helium. And of course, you know, now, Helium is used on Earth. This is just clearly a featuring of the fact that chemicals have their own bar-code.

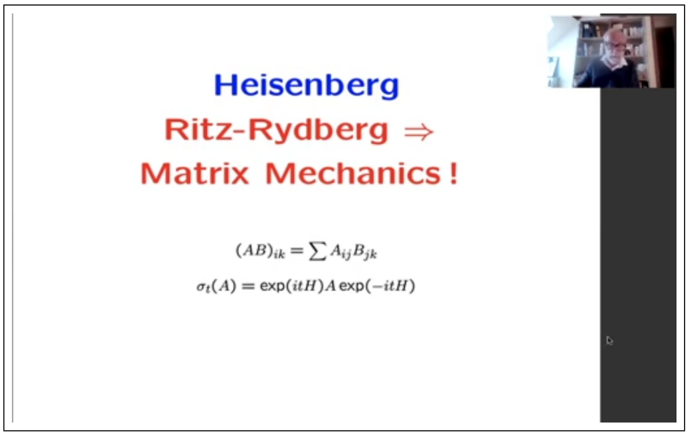


Now these bar-codes were studied by physicists and what happens is that they have a quite remarkable compatibility property that is that some of these lines, when you express them in terms of frequency, you have to be very careful that you should express them in terms of frequency and not wavelength, some of them actually add up. And in order to understand how they add up, it's Ritz-Rydberg who found what is called the Ritz-Rydberg principle, and the idea is that these lines would be indexed not by one index but by two indices, it could be greek letters whatever you want, and the point is that Ritz-Rydberg principle tells you that the line with indices $\alpha\beta$ will combine with the line with indices $\beta\gamma$, so I mean the second index of the first line has to be the first index of the second line, and then they combine and they give you the line corresponding to $\alpha\gamma$. Now, this Ritz-Rydberg combination principle had one amazing consequence in the hands of Heisenberg



and what Heisenberg found out is that thanks to this principle, he was doing calculations when he was alone

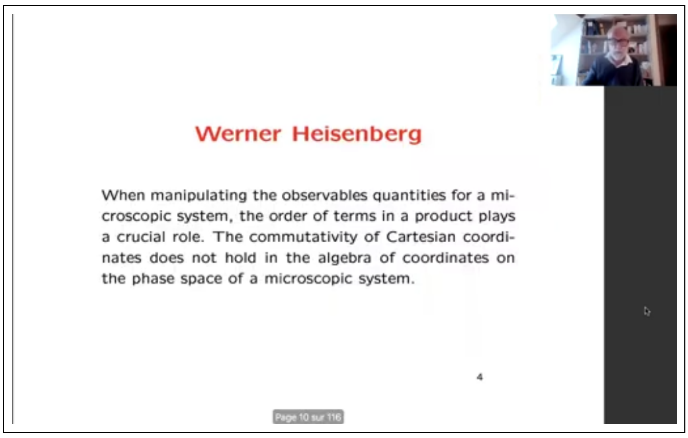
in Helgoland where he had been sent because he had allergy, he had been sent by his university, because there was no cure except to send people in a place where there was no source of pollen. So he was there and he had all the time he wanted to work and at some point, during a night, I think it was near four o'clock in the morning, he had proven that the energy is conserved[1],



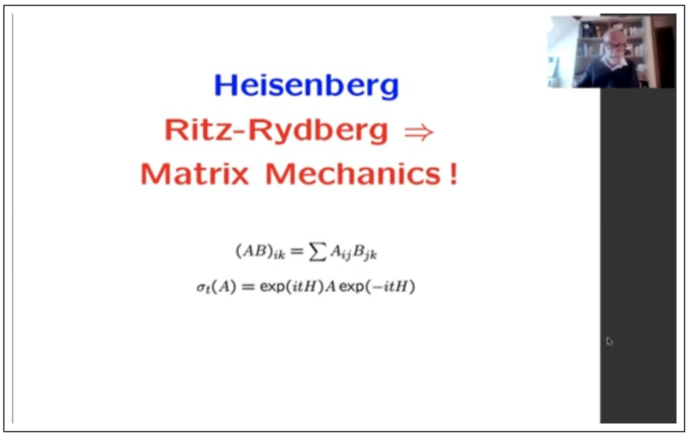
because if you take H to be A, then, there is a commutation between these two terms and you will get that H is preserved by time evolution.

[1] surrounding last line.

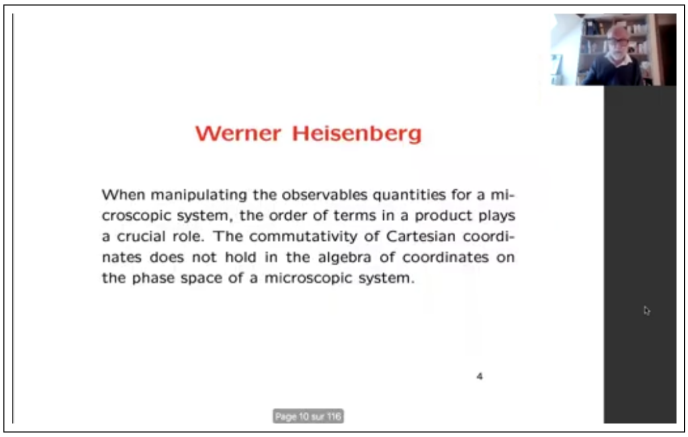
Then, instead of going to bed, what he did was to climb over one of the peaks which was along the coast and he waited for the sunrise on the top of this peak. And he explains that he was seeing, of course in his mind in his discovery, an incredible landscape. What he had discovered had one peculiar consequence, and that consequence was that, because matrices don't commute,



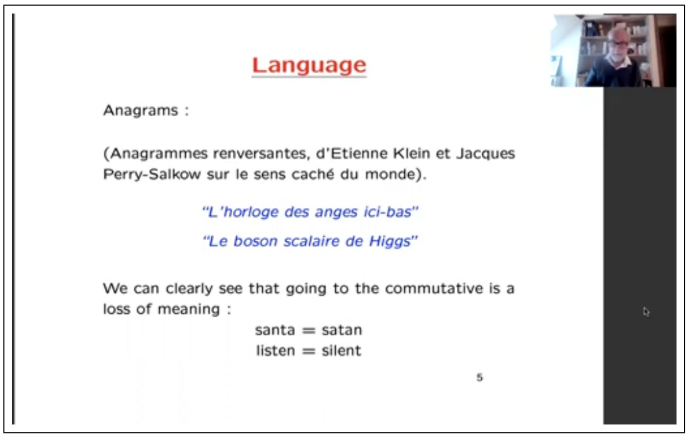
when you work with observable quantities for a microscopic system, yu have to pay attention to the order of terms in a product. In fact, the order of terms in a product plays a crucial role. And in fact, if you come back to the evolution equation of Heisenberg,



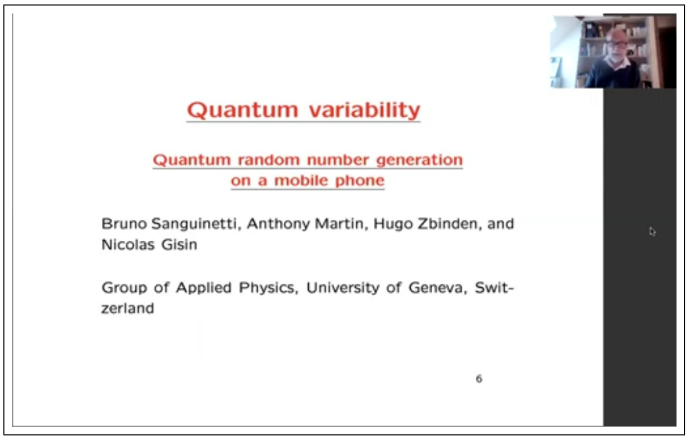
you find of course that if everything would commute, this evolution should be the identity. In fact, as we shall see much later, the commutative world is static, whereas quantum world is dynamical, and this is the first instance.



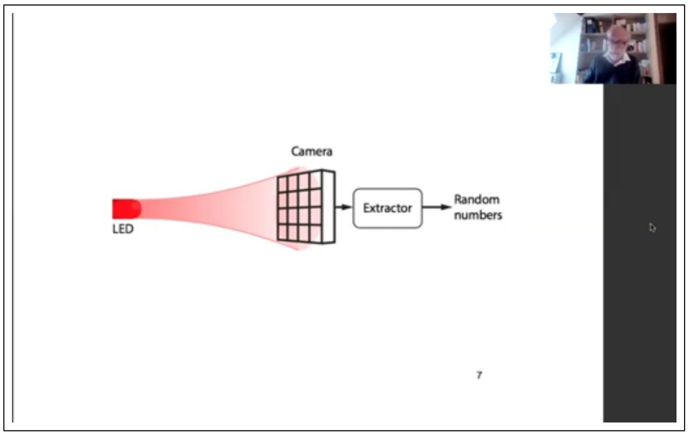
Now, in particular, what it means is that the commutativity of cartesian coordinates does not hold in the algebra of coordinates on the phase space. And this is one fundamental instance of appearance of such a noncommutative space.



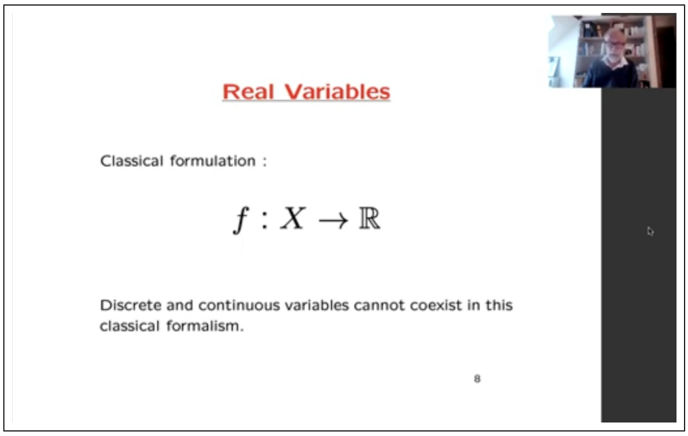
Now, as a corollary of this, you might think that this is very strange, and that, you know, dealing with this care with the order is something we are not used to, but this is wrong. We are perfectly used to that, in the language. I mean, when we use words, we need, of course, to pay attention to the order of the letters, and the order of the words, otherwise, you get anagrams. What I have shown here is a french anagram which is quite amazing but somehow, one can clearly see that when you go to the commutative, you lose meaning. For instance, I have written here for example, Santa and Satan are the same in the commutative world (there are two a, one s, one n, one t). Listen is the same thing as silent, and so on. So in fact, what you find is that this quantum way, this way of being forced to pay attention to the order of the letters, is a way to keep meaning. So in ordinary algebraic geometry, one forgets completely about these nuances.



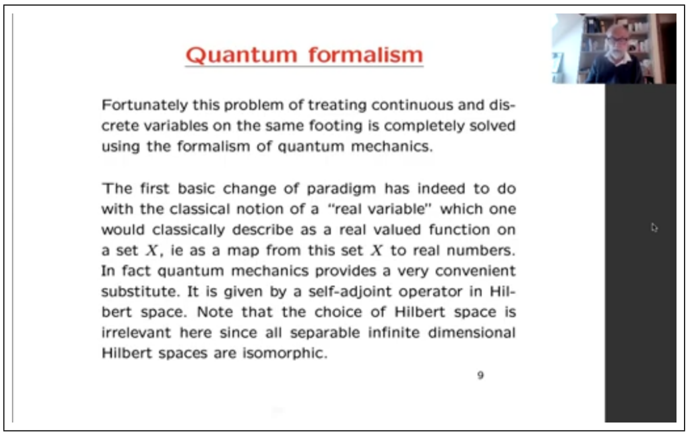
Now one corollary of the non-commutativity, of the Heisenberg uncertainty principle, is quantum variability. And to understand this quantum variability, one needs for instance to give an example.



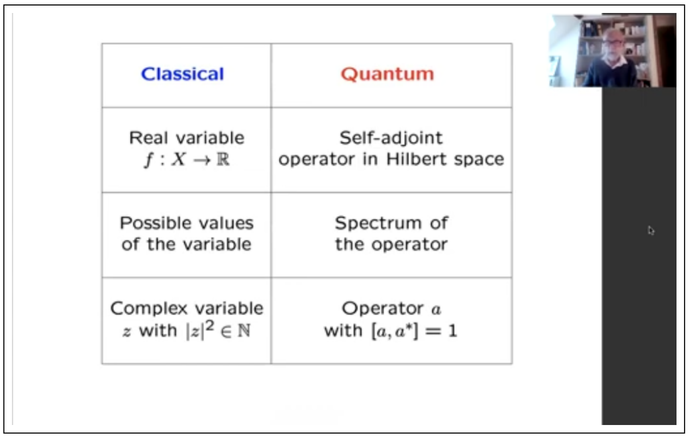
Several swiss engineers have manufactured a small device which you can use in a mobile phone and which will generate random numbers. But the way they will generate those random numbers is simply by letting a photon go through a small slit and land somewhere on a photo cell, and which one of these cells it will land on is something which is totally unpredictible, by the uncertainty principle of Heisenberg. And so, what this gives for you is a way to generate random number, and this way of generating a random number cannot be attacked. For security reasons, it's a way which contrary what you would obtain if you would generate quantum numbers from a computer. It's totally different not only by experiment, but also by the theory : you know that they are not reproducible. So there is this fundamental variability,



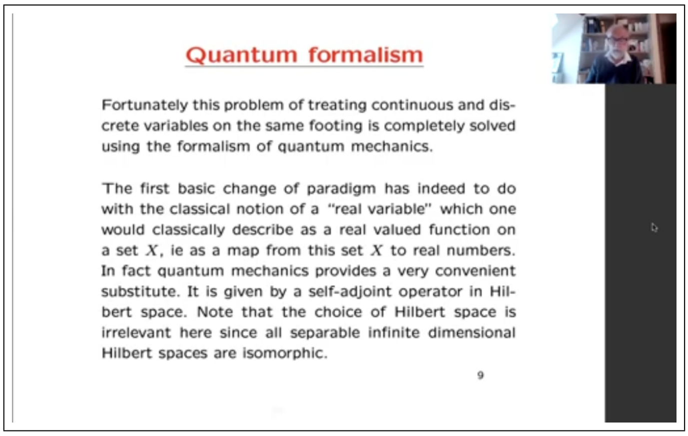
which is in quantum mechanics and when you think about it, you will find out that quantum mechanics is in fact a much more better formalism of variability than ordinary classical mathematics. For instance, if you ask a mathematician what is a real variable, very often you will get as an answer the fact that it's just a map f from some set X to the real line. Now it turns out that this formalism is in fact rather poor because you cannot have coexistence of discrete and continuous variables, in this classical formalism. The reason is very simple. The reason is that if you have a continuous variable, in the given X, then this given X has to be uncountable. And then, any variable meant to be discrete will in fact take some value an infinite number of time, and in fact more than an infinite countable number of times. So they don't coexist.



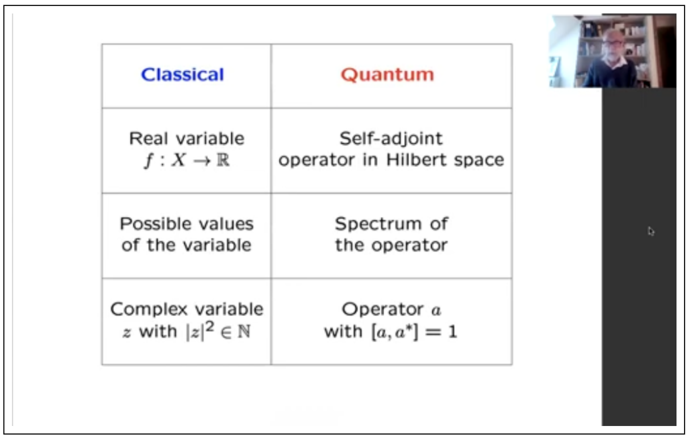
And amazingly, they coexist in the quantum formalism. So if you want, the continuous and the discrete coexist in the quantum formalism because in this formalism, a real variable becomes a self-adjoint operator



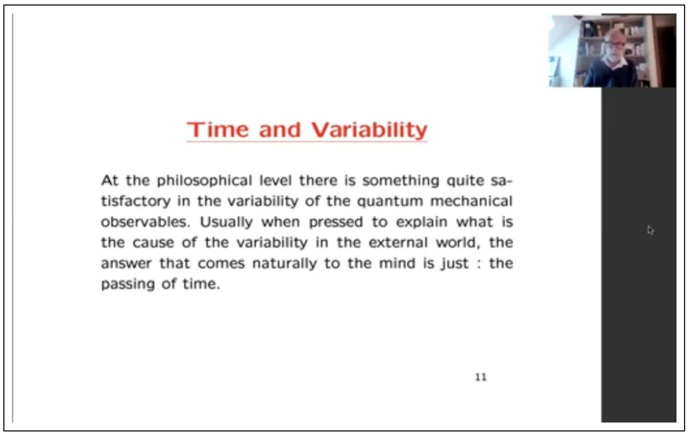
in Hilbert space. And in the same Hilbert space, you can have self-adjoint operator which is for instance a multiplication by x in the Hilbert space which is L^2 functions over $[0, 1]$, but this Hilbert space of L^2 functions over $[0, 1]$ is isomorphic to the Hilbert space which is the Hilbert space of ℓ^2 sequences on the integers in which you also have another variable, if you want, which is the multiplication by n, which is self-adjoint, and which is obviously discrete.



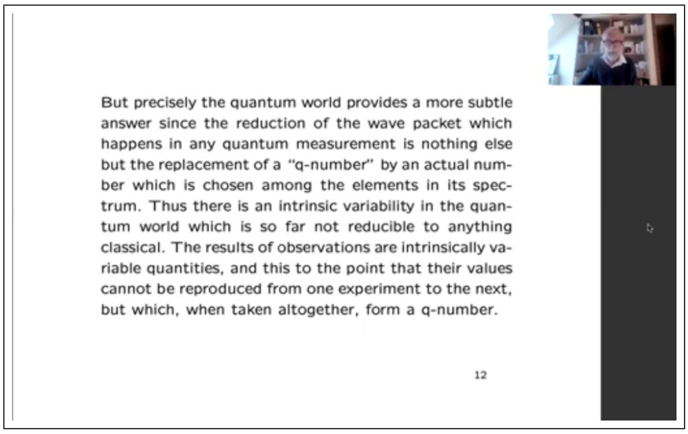
So if you want, because there is only one Hilbert space, namely infinite dimensional with countable basis, what you find out is that there is coexistence of the discrete variables, with the continuous variables with the only proviso that they cannot commute. There is this nuance, and this nuance will play a fundamental role later as we shall see.



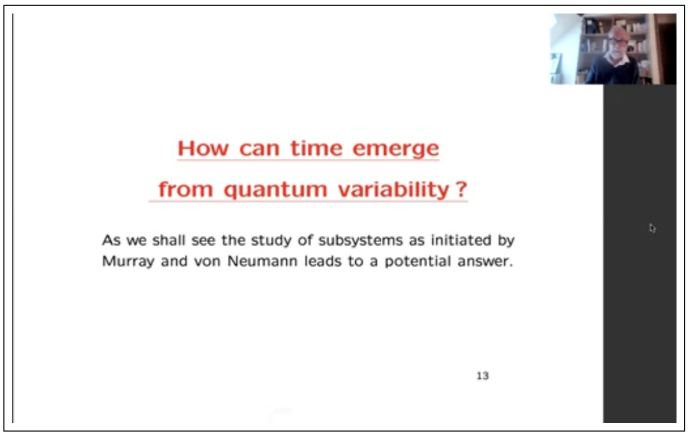
So we have this dictionary, which is coming from the quantum. And of course, the values of a real variable is just a spectrum of the self-adjoint operator, but physicists have been very very early own capable of applying this notion to complex variables. In fact, they applied it to a very peculiar situation where you would like to have a complex variable z which is such that $|z|^2$ is an integer. This is related to the Planck discovery in 1900 and to what Einstein wrote in 1906 which is that the energy of an oscillator should only take integral multiples of $h\nu$. The oscillator was first understood in a paper of Born, Heisenberg and Jordan, I think in 1925, and then Dirac was able to use this very same ansatz in which you replace variable z, that was supposed to be a complex variable, you replace it by an operator a, and the only condition on that operator a is that its commutator with its adjoint is equal to 1. That suffices to ensure that the spectrum will be formed of positive integers, it's a little exercize. And in the hands of Dirac, this allowed him to actually prove what Einstein had guessed when he had guessed the constants A and B of emission and absorption of an atom. So this is a very successful and a very amazing formalism, which replaces the classical formalism.



And there is something in fact which is quite striking if you want at the level of the variability, which is that normally, when we are addressed by people to explain what is really, the essence of variability, what is the cause of variability in the external world, the usual answer that comes, I remember giving this answer when I was in highschool, the natural answer that comes to mind is just the passing of time. This is the only sort of reasonable answer we are able to give. But now, because of this intrinsic and sort of fundamental variability which there is in the quantum, comes a very natural question and

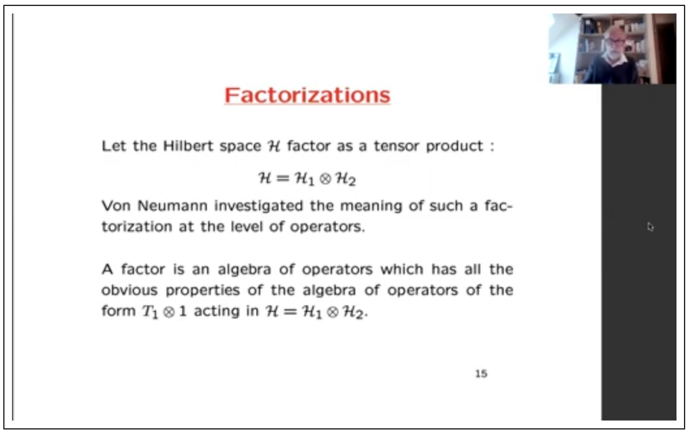


this question is... you know, of course, we have not been able in the formalism of quantum mechanics to reduce this variability because of the reduction of the wave packet which is something which is outside the time evolution, so if you want, this intrinsic variability in the quantum world sort of poses a very natural question and this natural question is

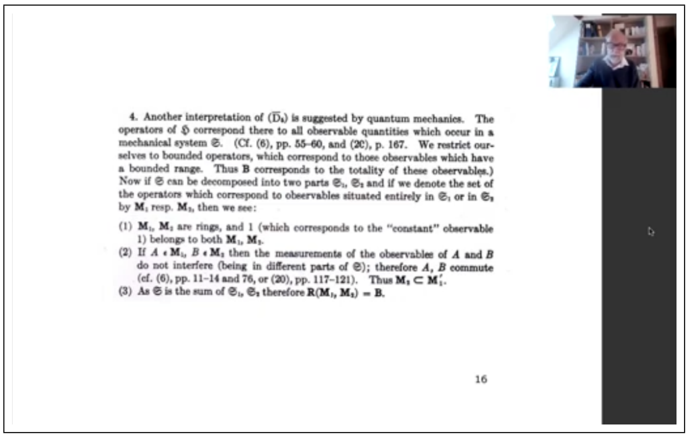


would it be more primitive that the passing of time ? Namely, how could time emerge from this quantum variability ? And what I want to explain briefly is that the study of sub-systems which was initiated by Murray and von Neumann, in the 1930-1940, leads in fact to a potential answer to this question. What did they do ? This is just a picture

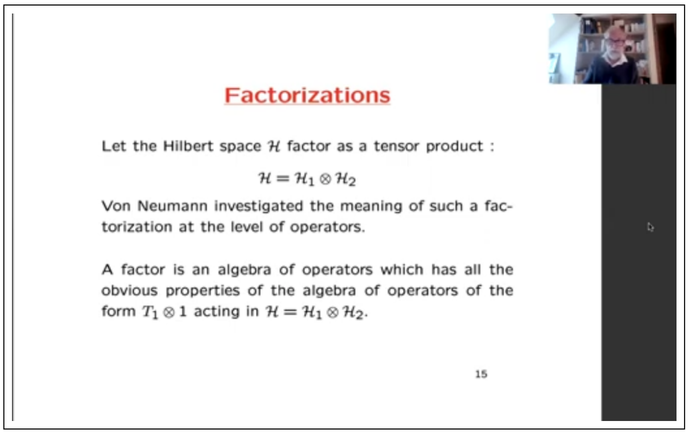
just to make sure not to forget that von Neumann is also very well known for inventing computers. But what did they do ?

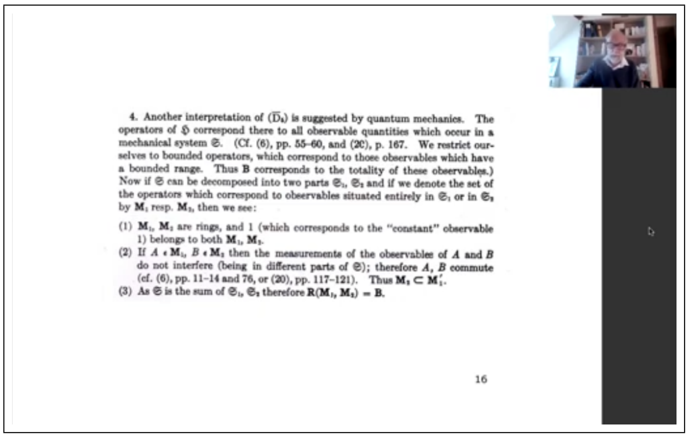


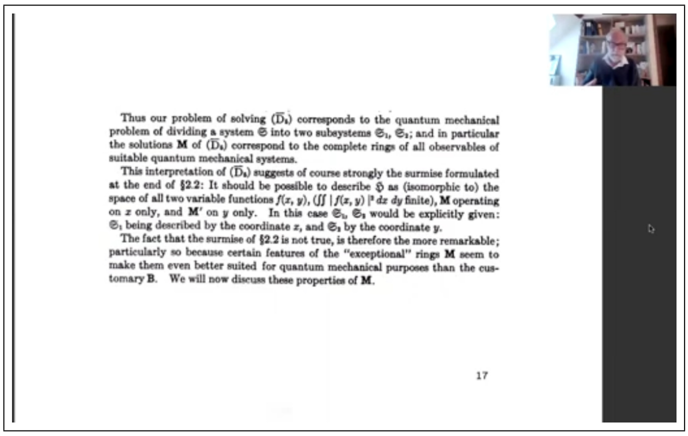
They studied, they started by studying space factorizations. And in that respect, they were motivated by quantum mechanics. So they wanted to understand that if you happen to have a Hilbert space $\mathcal{H}$ which is a tensor product, which splits as a tensor product, then you can consider in this Hilbert space the operators which are of the form $T_1 \otimes 1$ where T_1 is acting in $\mathcal{H}_1$ and 1 is the identity in $\mathcal{H}_2$. Somehow you want to understand algebraically what are the algebras which appear in this way. So they motivated their work by quantum mechanics,



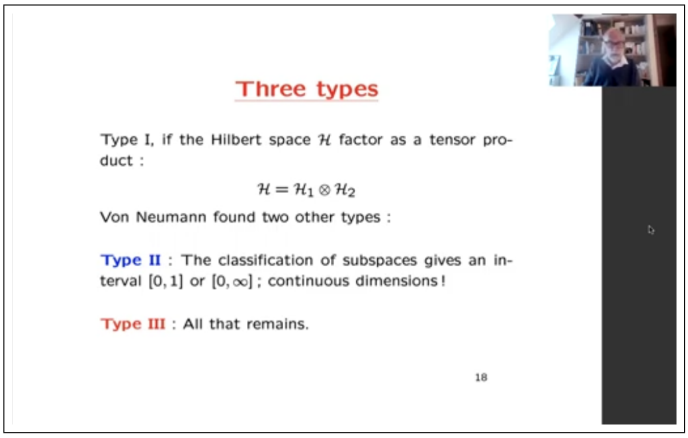
by of course saying that you want to consider observable quantities which occur in a sub-system, then of course you are dealing with rings of operators, with algebras of operators, and you have a commutation, between what happens in one system and in the complementary system, and so on. So they studied these factorizations and the term factor comes from designing algebras that you will have, that would imitate this situation of a tensor product.



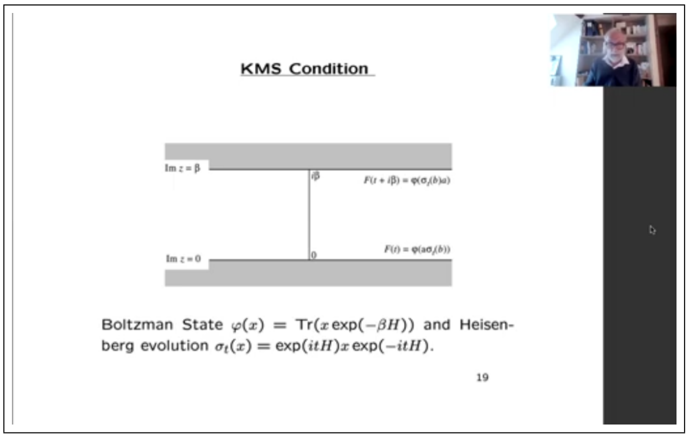




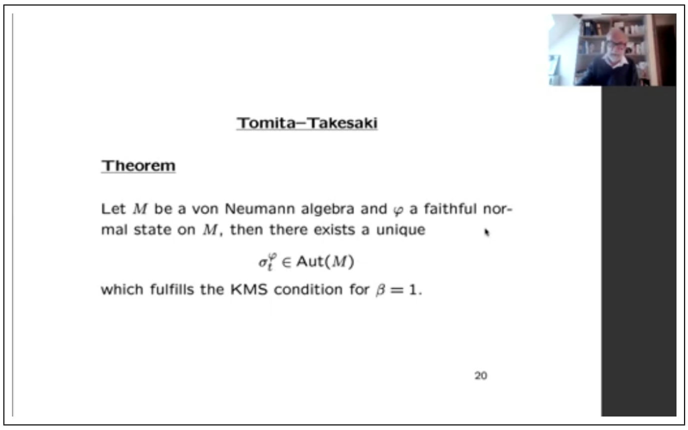
But amazingly, what Murray and von Neumann found is that, besides the factorization, which occurs from factoring the underlying Hilbert space, it turns out that there are factorizations which do not come from there. And so the factorizations that come from factorizing the Hilbert space are called of type I, they are the simplest by far.



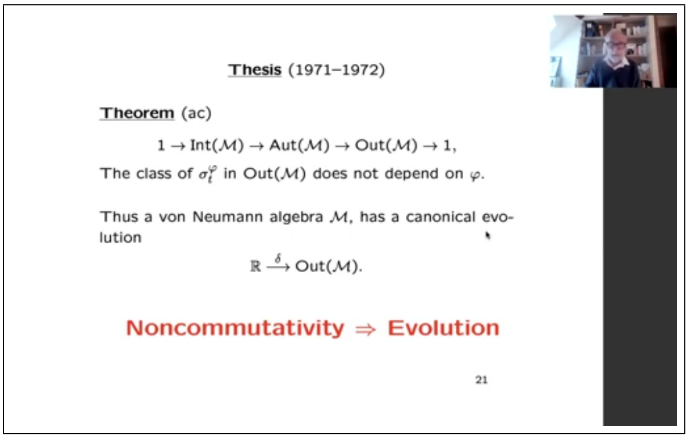
But they found two other types. They found what are called type II, and type II, what does it mean, in which way, if you want, the type II factorizations are different, are distinct from the type I factorizations, well, they are very distinct, because when you consider a type I factorization, after all, the algebra will just be the algebra of operators in a given Hilbert space. So, if you want, what would correspond to the subspaces are classified by the integers, by the dimension of the subspace, it could be infinite of course. Now, in the case of the type II, what happens is that what correspond to the subspaces are no longer classified by an integer but they are classified depending on type II_1 or type II_∞, either by the interval $[0, 1]$ or $[0, \infty]$. And I mean, this is the first appearance of continuous dimensions which... I remember reading a paper of von Neumann when I was in École Normale, and this, really, intrigued me a lot, the fact that there are those continuous dimensions that appear. And then, what do you have, you have the type III and the type III is all that remains.



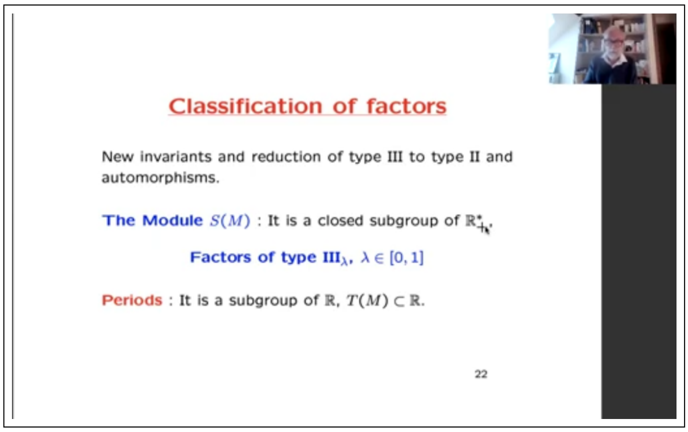
I mean in fact, came as an important tool the fact that the link between the Boltzman state which is given when you consider all operators in the Hilbert space by the trace of x multiplied by the exponential of $-\beta H$ where H is the Hamiltonian and β is the inverse temperature. So this is related to the Heisenberg time evolution which I showed you before, namely $\sigma_t(x) = exp(itH)xexp(-itH)$. They are related together by something which can be formulated purely algebraically in terms of the state itself and the time evolution. And this is the Kubo-Martin-Schwinger (KMS) condition, which is a condition that can be formulated in terms of holomorphic functions.



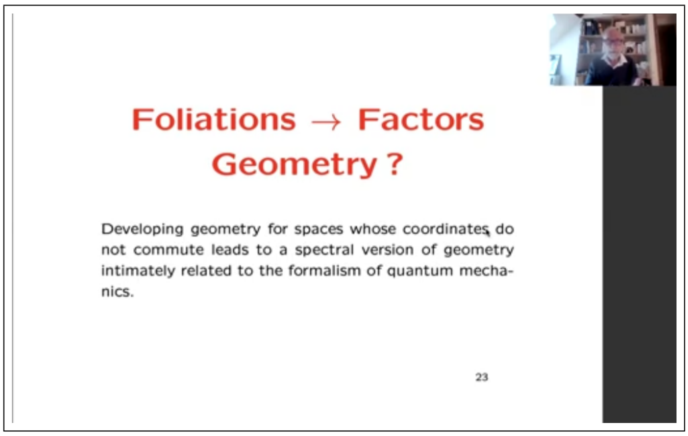
And a very important step was done by Tomita and Takesaki around 1970 when they proved that this association between a state and a one-parameter group of automorphisms actually holds for any von Neumann algebra. So if you take a von Neumann algebra, and take any faithful normal state on it, then there exist a unique one-parameter group of automorphisms that actually fulfills this KMS condition of the association for $\beta = 1$. I started my thesis and in my thesis, what I proved



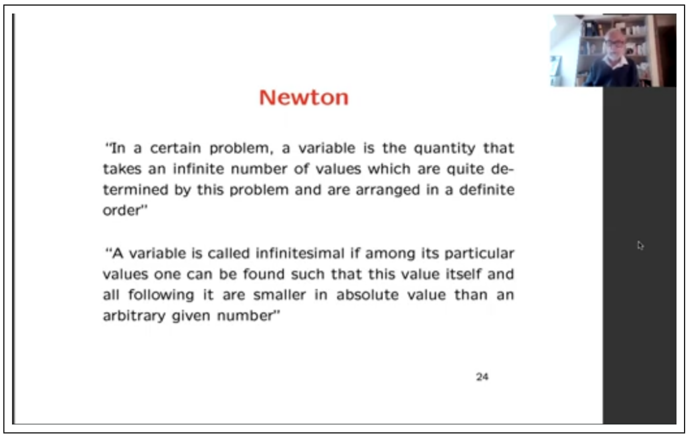
in 1971-1972, in april 1972 is that in fact, this one-parameter group of automorphisms is unique, when you look at it in the quotient of the group of automorphisms of $\mathcal{M}$ divided by inner-automorphisms. You see, when an algebra is not commutative, it admits trivial automorphisms, namely automorphisms that are obtained by conjugating an element by a unitary element in the algebra, so by x goes to UxU^*. And because these automorphisms are completely trivial in a certain way, they form a normal subgroup of the group of automorphisms and the interesting automorphisms are forming a quotient group which is the group $Out(\mathcal{M})$. So what I proved in my thesis which was under Jacques Dixmier, I proved that in fact, there is a unique, independent of the choice of the state, homomorphism from the real line to the group $Out(\mathcal{M})$ of automorphism classes of $\mathcal{M}$. This is an amazing fact in the sense that what it tells you that this algebra, just from its non-commutativity, acquires an evolution. This of course gave me



the classification of factors. So I could define new invariants, and I could also reduce type III to type II and automorphisms. In fact, I left one case open which was later done by Takesaki. But I had defined two fundamental invariants, the module $S(M)$ which is a closed subgroup of $\mathbb{R}_+^*$ and which allowed to classify, if you want, the factors of type III into type III_λ where λ belongs to $[0,1]$ and the reduction from type III to type II, I did in the case where λ was different from 1. The III_0 case was particularly interesting. And I also defined the group of periods, which is a subgroup of the real line, but this time, it's not a closed subgroup, it can be quite wild. And it's a remarkable subgroup in the sense that what additive it is, there are certain times, from the subgroup of the line, which are periods of the factors namely which the factor doesn't move.



Once I have done this work, I arrived in IHES in Bures, and I found out that, of course I was a specialist of a specific topic, but the people preoccupations were rather far from mine and I had the luck to meet Dennis Sullivan, and to discuss with him a lot, and after these discussions, I found that there was a completely canonical way to associate a von Neumann algebra that in the most case is a factor to foliations. So foliations are very familiar objects in differential geometry, essentially what they are are decompositions of the product but given locally only and what is interesting is not their local properties which are trivial but their global properties. And what was amazing is that this association I had found from foliations to factors allowed me to exhibit the most exotic factors in the simplest case of foliations. For instance if you take the Kronecker foliation of the torus this gives you type II_∞ hyperfinite, if you take for instance the ... of foliations of the sphere bundle of a Riemann surface, this gives you the unique type III_1 hyperfinite factor which is extremely exotic. On the other hand you know what happens is that this association from foliations to factors in von Neumann algebras was only taking into account the major theory of foliations. But foliations are much richer in a way. They belong to functional geometry. So they have differential structure. They have a topology and so on and so forth. And this led to develop geometry for spaces whose coordinates do not commute, because when you deal with algebra of foliations, of course the factors you get are not commutative. This non commutativity comes from the fact that you are allowed to slide along the leaves. So this led me to a spectral version of geometry, which I want to present, and this is closely related to the formalism of quantum mechanics. And as a warm-up, one has to understand what is sort of miraculous in this formalism of quantum mechanics and why it can be so pertinent and so useful, for doing geometry.

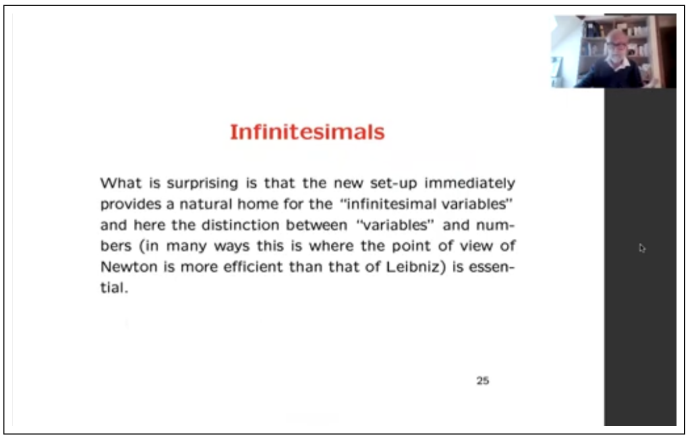


One first thing which is remarkable is that if you read Newton, you'll find that, provided you read what he wrote in the quantum mechanics formalism, it will immediately give you the right answer for what are infinitesimals. So, first of all, Newton was not interested in numbers, he was interested in variables. What he says is that :

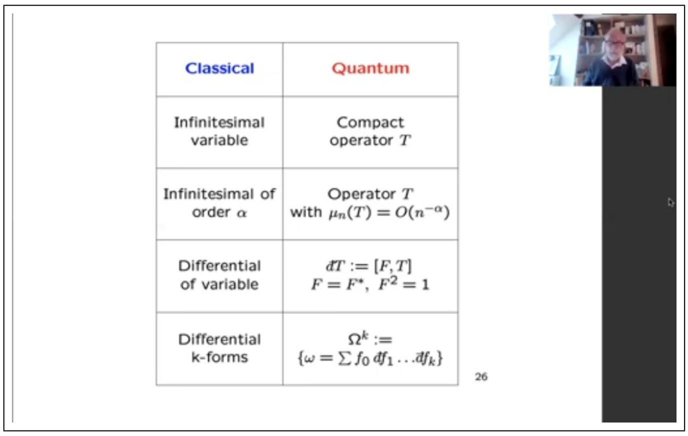
> *"In a certain problem, a variable is the quantity that takes an infinite number of values which are quite determined by this problem and are arranged in a definite order".*

And then he talks about the infinitesimals. And for him, an infinitesimal is a variable. So,

> *"A variable is called infinitesimal if among its particular values one can be found such that this value itself and all following it are smaller in absolute value than an arbitrary given number."*

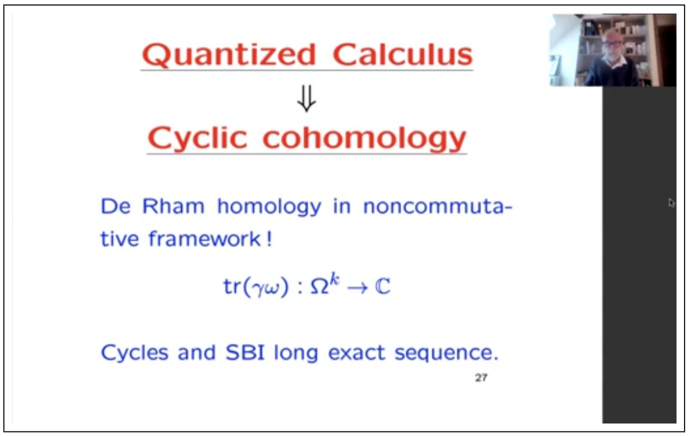


Now, what is amazing is that when you apply this notion of infinitesimal which is essentially there, which is essentially defined in the words of Newton, then you find that they correspond on the nose to a notion which is well-known in operator theory

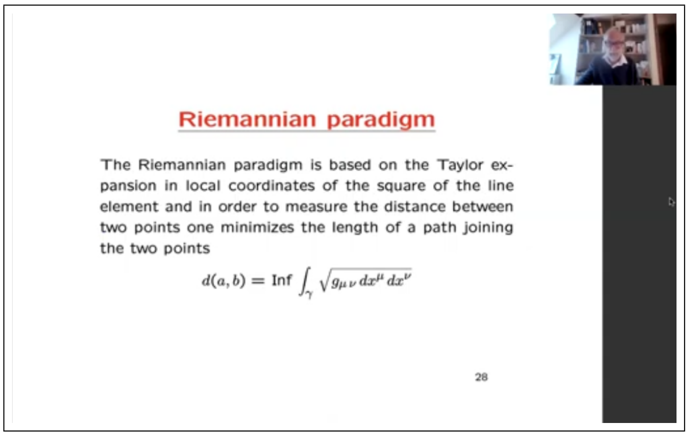


and which is compact operators. Because compact operators, well, they are variables, as we saw, because you know, variables are corresponding to operators, but moreover, they have exactly the property that Newton was saying, namely that if you take their characteristic values, so the characteristic values are eigenvalues for the absolute value of the operator. And these characteristic values have the property that for any ε, there are only finitely many of them which are larger than ε. So they correspond exactly to what Newton had in mind. In this formalism, you have a rule, immediately, for what is an infinitesimal of order α. So for that, you look at the rate of decay of the characteristic values, and for instance, an infinitesimal of order 1 is one such that its characteristic value decay like $1/n$. This is fundamental for later because such things are not tracable because the series $1/n$ is divergent but when you look at their trace, it has a logarithmic divergency, and it is the coefficient of this logarithmic divergency that gives you something local.

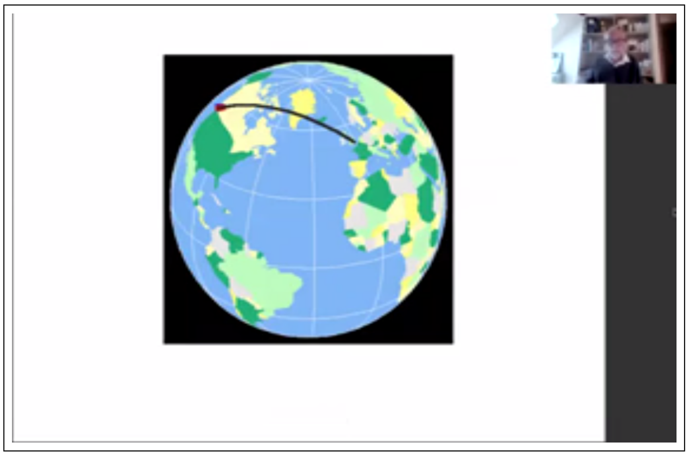
There is also the differential of variables. Normally, you try to differentiate the functions, and so on, but here, it's just defined, for a bounded operator, it's just defined as a commutator with the operator F which satisfies two conditions : the condition that it is self-adjoint and the condition that its square is 1. So there is no content in the operator F itself, what is really important is the relation between the operator F and the operator T. Because F^2 is 1, you can easily show that the square of the differential in the graded sense is 0, and then you have the notion of differential k-forms, which are obtained only by taking operators sums of products of (k-forms) 1-forms, if you are defining them in an obvious way. So many properties come out naturally. But the most important was that this quantized calculus



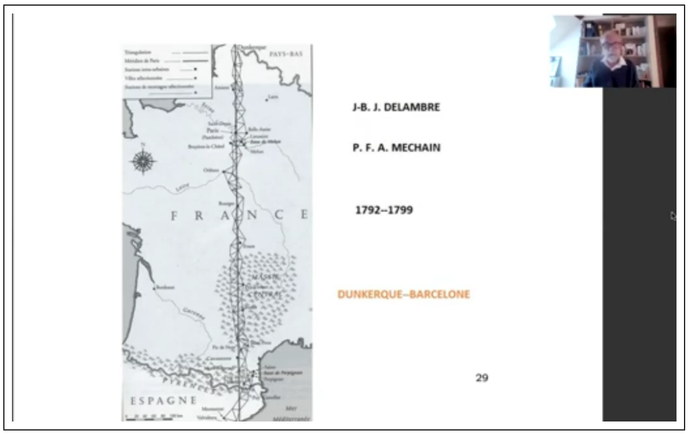
led me in 1980-1981 to the cyclic-cohomology. And cyclic cohomology is really playing a fundamental role in non-commutative geometry as a de Rham cohomology in this non-commutative framework. I gave a talk in 1981 in Oberwolfach whose title was "Spectral sequences and cohomology of currents for non-commutative algebras", where if you want all the basic properties, the fundamental properties of cyclic cohomology follow if one takes seriously this quantized calculus. Because in the quantized calculus, what you get is what is called the cycle, because you can use the trace to integrate the differential forms, and you get that this cycle is closed, and so on. And moreover, if you can integrate form of dimension k, you can also integrate forms of dimension $k+2$. There is a distinction between even and odd cases, and this gives to the operator S a periodicity, and to the SBI an exact sequence that is the corresponding spectral sequence. So of course, this is just one instance of the use of quantized calculus. There are many other instances. Someone is for instance that you can do differential geometry in the group ring of the free group, using the quantized calculus which is defined by actually the group on a tree. But these are tools, and now, with these tools, we want really to come to the geometry itself, in the metric sense. So for the geometry itself, in the metric sense, we have to make a little discussion, going backwards, to the riemannian paradigm



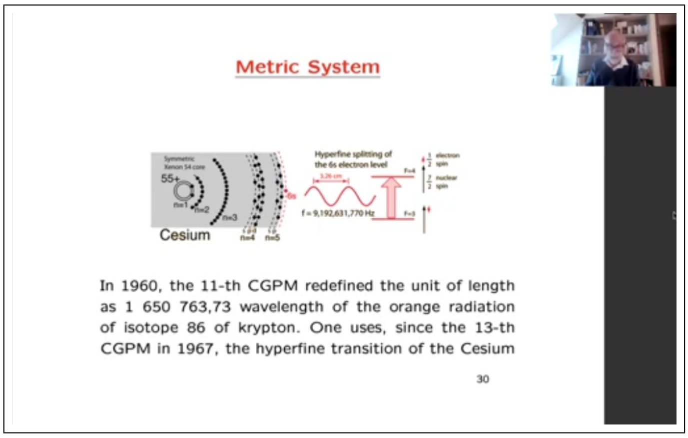
and to the way the metric system evolves. The riemannian paradigm is based on the Taylor expansion... Riemann had this fantastic inaugural talk, to which we shall come back later, in which he had the insight to define the metric locally, by looking at the Taylor expansion of the line element in local coordinates, and in fact, he was looking at the square of the line element. So there is a squareroot involved, and the distance between two points, as you know very well, is computed by minimizing the length of the path, like here,



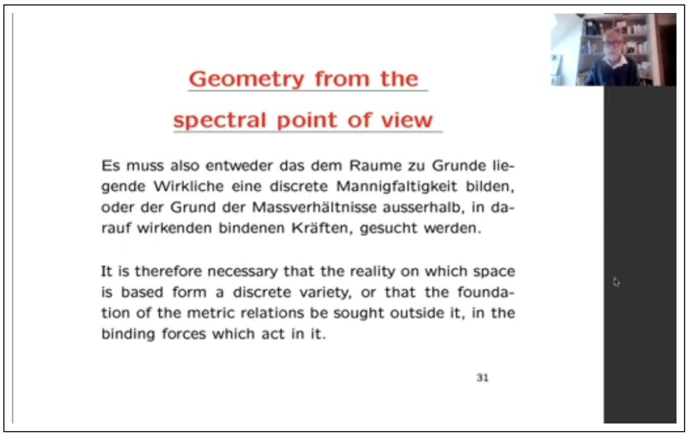
for instance between Seattle and London. And so, it's a very concrete definition of length and somehow, it fits very well with



how the metric system was developed and typically, what happened... what I am showing you the fact that during the French revolution, they wanted to have a unification of the unit of length, so they defined it as $\frac{1}{40\ 000\ 000}$ times the circumference of the Earth and of course, we are measuring just an angle, that they knew from the stars, and I mean this angle, this distance was between Dunkerque and Barcelone and two physicists, astronomers, went along and they made a concrete measurement and out of this concrete measurement (they were Delambre and Méchain) was fabricated a platinium bar, which was deposited near Paris in Pavillon de Breteuil in Sèvres, which was supposed to be the unit of length. Now what happened was very interesting. Because what happened around the years 1925 or something like that, physicists discovered that the unit of length which was deposited near Paris and so on and so forth, didn't have a constant length. How did they do that ? Well they compared it with the wavelength of Krypton. There is a certain orange in the wavelength of Krypton which they used to measure this platinium bar, and they found that the length was changing. So I mean, of course, this was not a good definition, and they shifted, I mean physicists shifted, in the Conference on metric system,

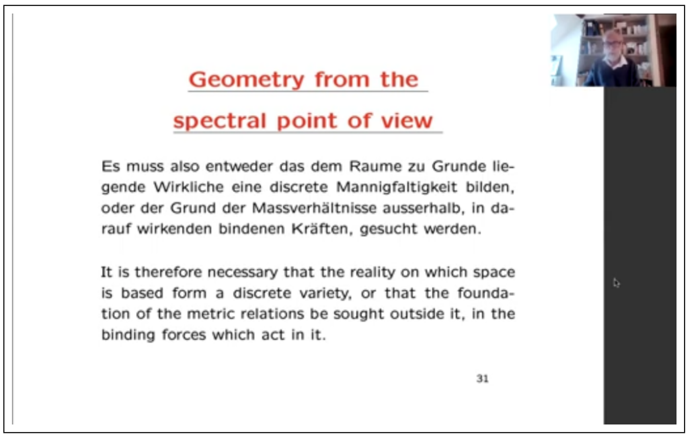


they shifted first to define the unit of length by this orange radiation of Krypton, a certain multiple of this wavelength. But then they found that there was a better way of doing it, which was with Cesium. And in fact, with this definition of Cesium, you can buy, in a store, some apparatus which allows you to make immediate measurements with a precision which is 10 decimal places.



I mean, it's a big step forward, and what happened is that there is an hyperfine transition between two levels, and it corresponds roughly to

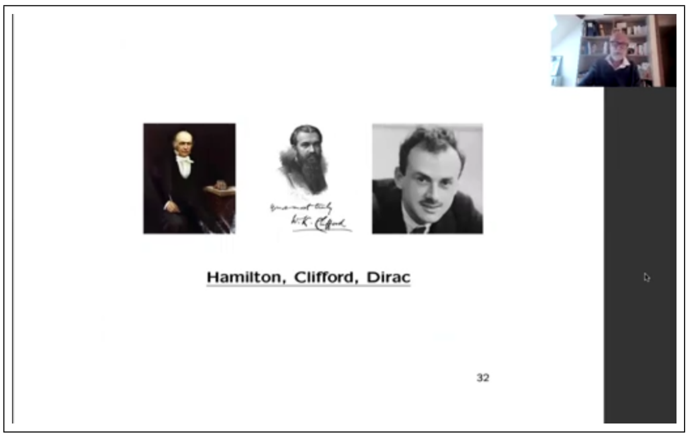
a wavelength of about 3.26 centimeters and then, you redefine the whole thing, so in fact, you define the speed of light, you fix the speed of light at this number (I heard that Grothendieck was furious when he heard that, because he wanted it to be 300 000 000), but the reason why you cannot do that is that there are already measurements which are precise which are done, so you have to accord this. So I mean there is a strange value which is taken. Then you define the second, as a number of periods of this radiation coming from the hyperfine transition, and as a corollary of that, you know the meter, the unit of length, is therefore defined as being this proportion (it's a rational number) times the wavelength of the hyperfine transition. Now what is extremely amazing is that this transition which the physicists have done, long ago, between the platinium bar and the spectral definition, is exactly parallel to the transition between the riemannian paradigm and the spectral paradigm, which I will explain.



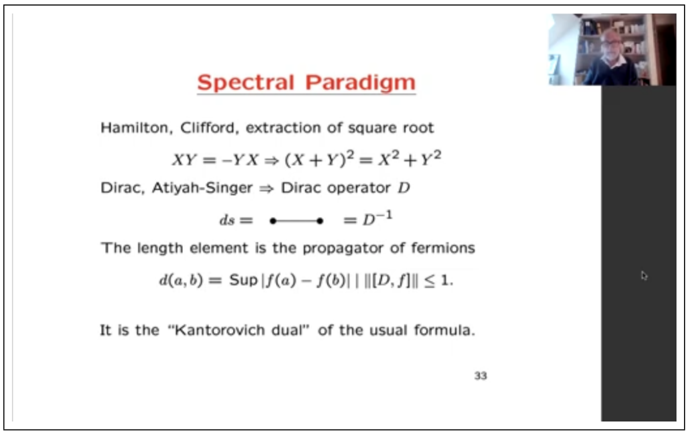
What is also amazing is that Riemann was incredibly careful in his inaugural lecture, to say, I mean, he didn't really believed that his notion of metric would continue to make sense in the very very small. And the reason that he was advocating was that... because he was working with solid bodies and he was using lightrays in his theory so what he was writing was that when you work in the very small, solid bodies no longer make sense, and neither do lightrays and so, what he wrote for instance was that

> *"It is therefore necessary that the reality on which space is based form a discrete variety, or that the foundation of the metric relations be sought outside it, in the binding forces which act in it".*

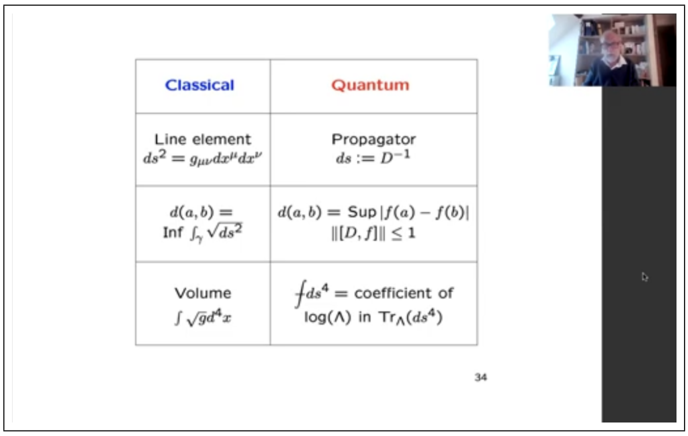
and as we shall see, this is exactly what happens, in the spectral framework.



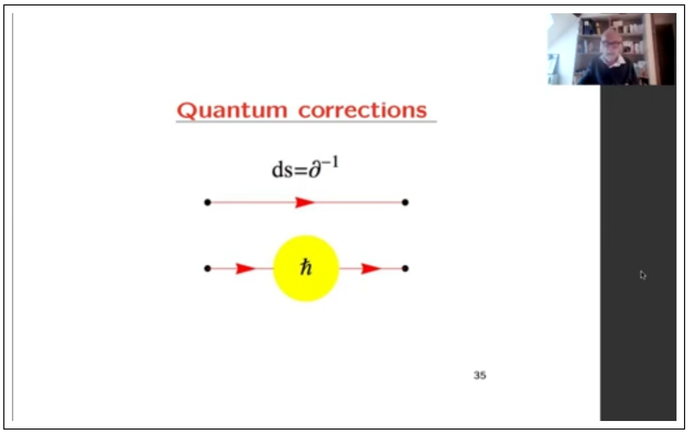
So the possibility to do that, to transfer to the spectral framework these ideas is in fact coming from work of Hamilton, Clifford and Dirac, and essentially what is the way to extract the squareroot in the formula of Riemann ($d(a,b) = \mathrm{Inf} \int_\gamma \mathrm{g}_{\mu\nu}\mathrm{dx}^\mu \mathrm{dx}^\nu$). When there is the squared root of line element, one would like in fact to have not the squared but the line element itself. It's possible to extract this squareroot at the level of the quantum formalism, at the level of operators. And it's possible thanks to Hamilton, Clifford and Dirac. Hamilton was the first one to write really the Dirac operators, because he had the quaternions, and he wrote, you know, i,d by dx plus j, d by dy, plus k, d by dz, which is an example of Dirac operator. And the key to all of this stuff, is that when you have two operators X and Y, which anti-commute,



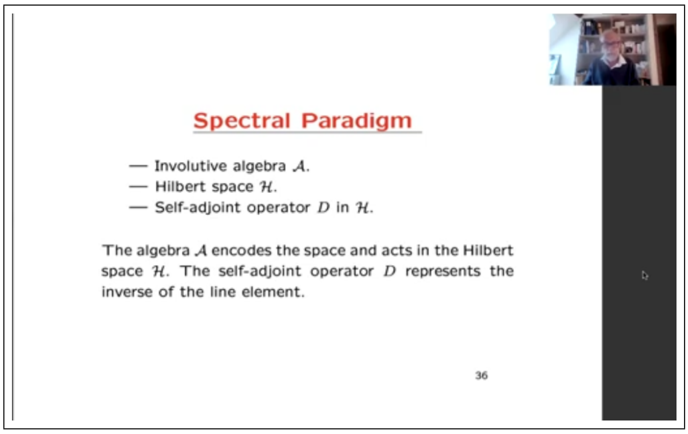
then in fact, you can write $X^2 + Y^2$ as a single square, namely as the square of $X + Y$. So through the work of Dirac and also of Atyiah-Singer, who defined the Dirac operator for arbitrary spin-manifolds, then emerges the Dirac operator D. In the spectral theory, the line element, which is the squareroot of the Riemann's ds^2, is an operator, it is an infinitesimal when the variety is compact, and what is it ? It's simply the inverse of the Dirac operator. Of course, there are minor things to which you have to be careful about, what about the zeros and so on, but I mean, this line element is what is called the fermion propagator, and you have to think of it as physicists write it when they write Feynman diagrams : it's a very very tiny little line, which is joining two points which are very close by. And then, rather ought of this inverse which is the operator D, you can compute a distance between two points, and this distance is no longer computed by the infimum of an arc joining the two points, but it's computed by looking at the maximal waveshift, between the value at a and the value at b, when you subject the waves to the fact that their frequences are bounded. It is what is called, mathematically speaking, the "Kantorovich dual" of the usual formula.



So you have this dictionary now that the line element is ds which is the propagator of fermions, the distance is computable, it's computed not by an infimum on arcs, but by a supremum, and by this way, notice it applies to many more spaces, because there are many spaces in which you cannot join two points by an arc, think about a space which is disconnect, whereas the formula on the right makes perfectly good sense. And the volume, for instance, is defined as the integral of the power of the line element that will be of order 1, and as I said before, when something is of order 1, an infinitesimal of order 1, then it means that its trace is logarithmically divergent. So what you do is that you take the coefficient of the logarithmic divergency and this will give you the volume. Also, what one has to understand is that if this formalism in which geometry is defined is the quantum formalism, this immediately



allows you to understand how to incorporate the quantum corrections. Why ? Because we know very well that the fermion propagator when we do quantum fields theory doesn't stay as it was before. It acquires quantum corrections. They are minute modifications of the geometry, which are given by some kind of power series, but which can be incorporated in the spectral formalism. So the spectral formalism is encoded in

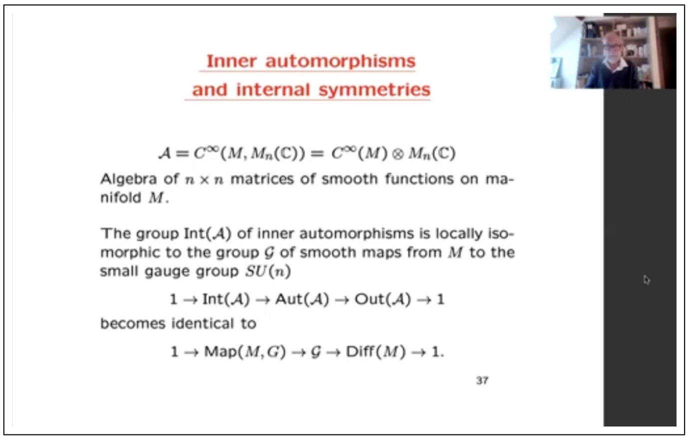


what is called a spectral triple. So such a triple contains three data :

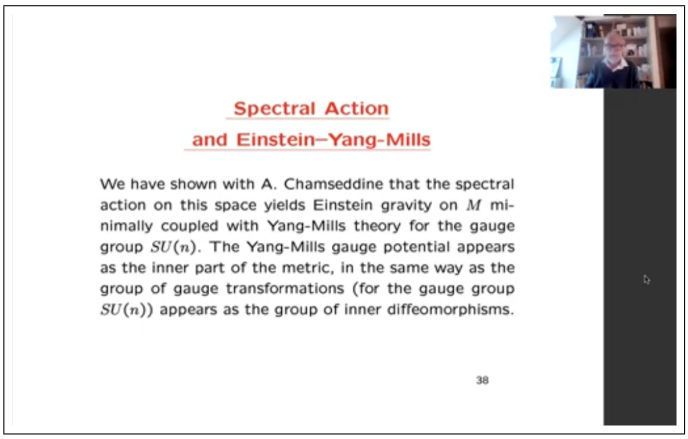
- the data of an involutive algebra, which gives you the space essentially, the coordinates of the space ; this algebra is *acting* in a Hilbert space ;
- the Hilbert space is fixed.
- and moreover you have the selfadjoint element, which is the propagator, which is acting in the Hilbert space $\mathcal{H}$.

In most cases, by the way, what you will find out is that the representation of both $\mathcal{A}$ and D is, when you take them together, irreducible.

So this is the spectral paradigm. And what I want to explain will illustrate the power of this paradigm by a number of cases.

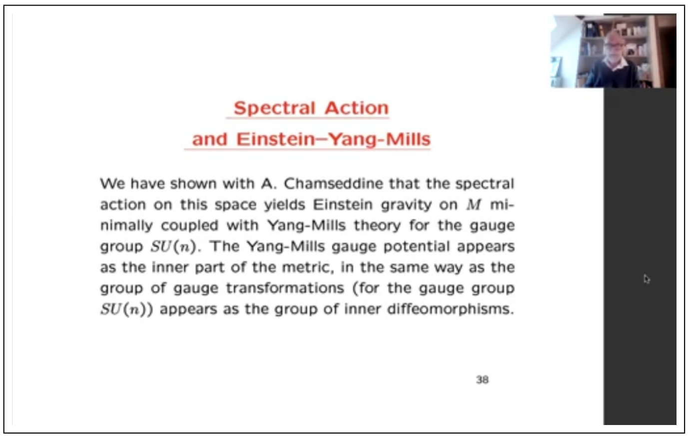


So the first thing which happens is that now, because you can talk about the geometry when the algebra is no longer commutative, now you don't have the $g_{\mu\nu}$ which depends on x and so on and so forth, just because of that, you can look at the most simple example. The simplest example which is not commutative is to replace the algebra of functions on a manifold M by $n \times n$ matrices over this algebra. So, if you do that, you just look at the algebra for a while, what you find out, as I said before, when an algebra is not commutative, it has this non trivial exact sequence, when you have the trivial automorphisms, which are the inner ones, and which form a normal subgroup of automorphisms, and then you go to the quotient which is outer automorphisms. Now, when you apply this sequence, which is general, when you apply it to the algebra of $n \times n$ matrices over manifold, what you obtain is an exact sequence where the inner automorphisms become the maps from the manifold M to the group G which is in this case the group $SU(n)$, if you take $n \times n$ matrices, and then this goes to the group of automorphisms, and it goes to diffeomorphisms. So what you find is that automatically by this very simple non-commutative extension, you have enhanced the group of diffeomorphisms to a group which physicists know very well, because this is the group of invariance of the action functional if they couple, minimally, gravity with Yang-Mills theory, with group $SU(n)$.

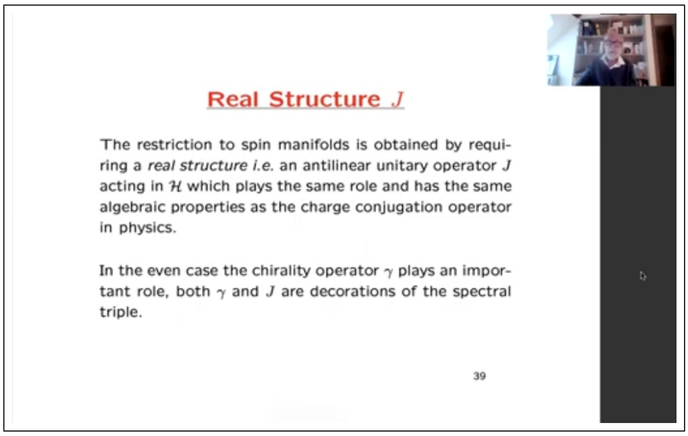


So in our work with Ali Chamseddine, what we found was the action functional. We found that if we take the above very simple case of taking $n \times n$ matrices over a manifold, and if we look at the action that would replace the Einstein action, which is the spectral action, so this spectral action, it can hardly be more invariant, this action only depends on the spectrum of the line element. What you do is that you write the asymptotic expansion and you get the Einstein action. You get the cosmological term to which I will come back much later.

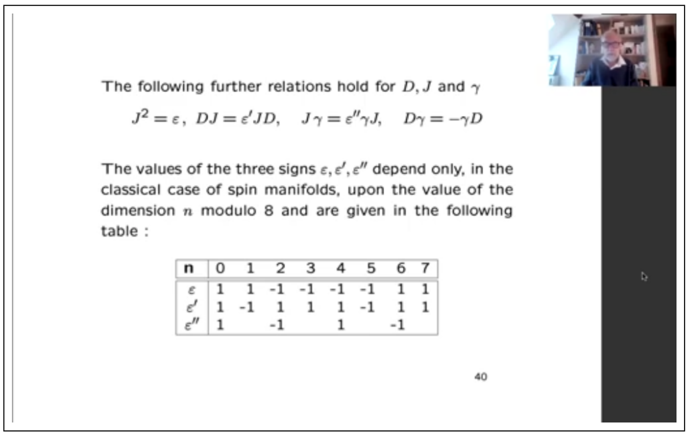
So you get this Einstein gravity but if you take this Einstein gravity minimally coupled with Yang-Mills theory, when you do the calculation. And Yang-Mills gauge potential as they appear, appear as the inner part of the metric. So in exactly the same way that I just said, the group of gauge transformations of second kind, the gauge group $SU(n)$ appear as the group of inner diffeomorphisms. So you have this blending together, which just comes, you know, from having replaced the algebra of functions by matrices over N. So this is a very entire thing and with Ali Chamseddine, we have done a lot of work, then, with Matilde Marcolli, with Walter van Suijlekom, and also with Slava Mukhanov, we have done a very great amount of work in order to go much further than just this simple instance of Einstein Yang-Mills.



In this work, there is an essential role which is played by the real structure. So what happened if you want is that there is a sort of reconstruction that allows you to reconstruct the manifold from the spectral data. And in order to restrict to spin manifolds, you have to think to spin c of things like that, one needs to incorporate a little decoration in the spectral data, which is that of a real structure. So it's an anti-linear unitary operator, and we shall see what it is in the physics language and in the maths language, but essentially, you also have to add another decoration in the case of even dimension which is the chirality operator. So we have these two, and

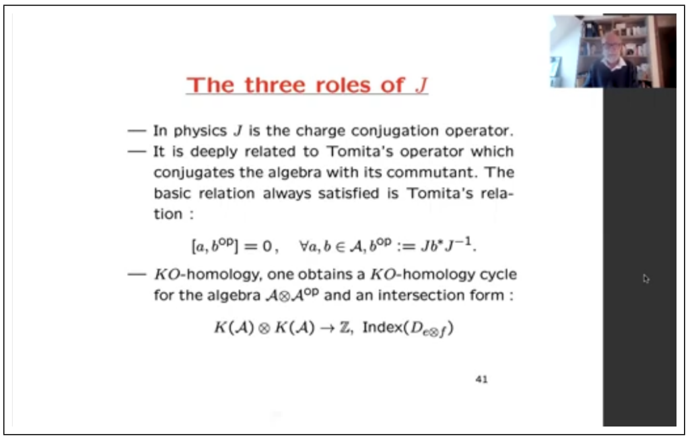


they fulfilled some commutation rules. And these commutation rules in fact, they tell you that you are dealing in fact with eight fold-theories, there are 8 possible theories that in the ordinary manifold case depend on the dimension modulo 8. If you want, the underlying conceptual theory is what is called KO-homology, and the reason why this KO-homology plays a fundamental role is that, if you try to understand at a conceptual level what is a manifold, in the ordinary situation, in the ordinary differential geometry what is a manifold, you will find out, and this is a work which goes back to the 1970's, in particular by Dennis Sullivan, you will find out that what you need to do...

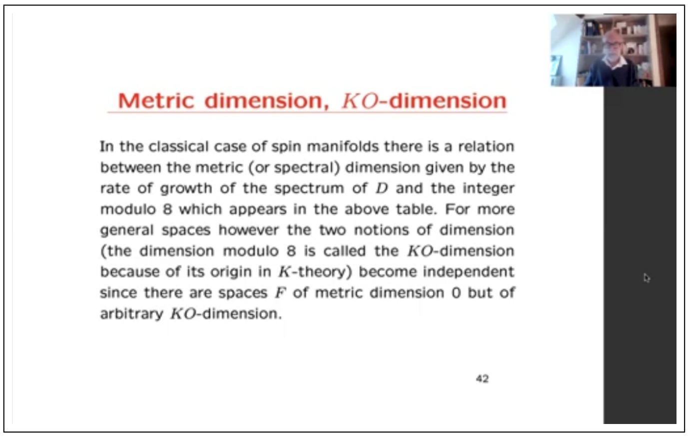


first you assume of course that the manifold has Poincaré duality in ordinary homology, but this is not sufficient at all, it only suffices to put the space in question into euclidean space so that it has a normal micro-bundle. But this micro-bundle is by no means a vector bundle. And the difficulty in order to transform it into a manifold is to elevate the structure of this micro-bundle into the structure of a vector bundle.

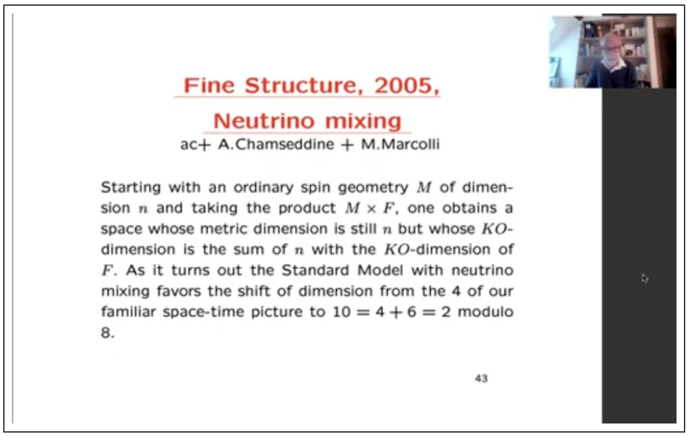
Now to encapsulate things very briefly, in the simply connected case, what you find out is that the obstruction to do that is that you should have also Poincaré duality in the deeper theory which is KO-homology. Now thanks to the work of Atiyah and Singer on the index theorem, they found out that the representative of cycles in KO-homology is in fact exactly given by the data that you need to build a Dirac operator, and that you have 8 possible theories in this KO-homology, and they are corresponding to the various possibilities that I was exhibiting here. So in fact this



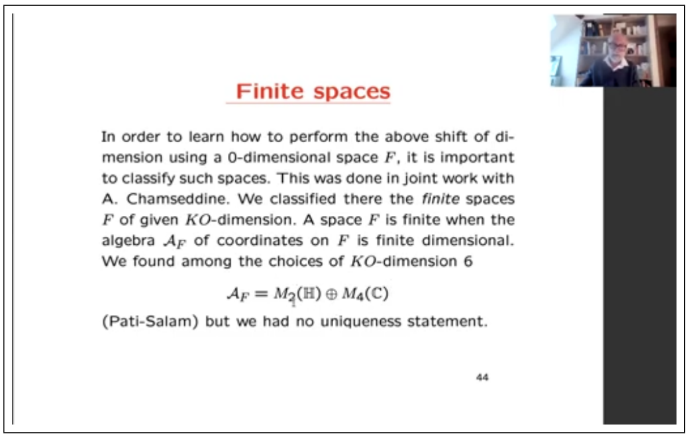
J, this real structure, has three roles. In physics, well, people will recognize it as what is called a charge conjugation operator, we are working in euclidean, in imaginary time, mathematically, this turns out to be very deeply related to Tomita's operator, why ? Because, in the non-commutative case, what you want is that you want to restaurate the commutativity in some way, and how do you do that ? You do that with a sort of trick, you flip, you are able to flip the algebra to its commutant by using this operator J. Tomita's theory allows you to do that in general. I mean, he proved a theorem that is that if you take a factor in Hilbert space, which has a cyclic and separating vector that is always the case in type III, then you can always find such an operator J that flips it to its commutant. And finally, the deepest meaning, if you want, of this J as I said, is to say that you have Poincaré duality in KO-homology, and this gives you the fact that, because of the J, you not only have KO-homology cycle for the algebra $\mathcal{A}$, but also for the algebra tensored by its opposite. And in particular, you have an intersection form, and so on and so forth. Now it turns out that this has played a key role in the development of the understanding of the Standard Model, in the sense that usually, when you work with spin-manifolds, there is a link between the metric dimension and this KO-dimension modulo 8.



What happens is that there is a notion of metric dimension, for a spectral geometry, and this metric dimension just comes from the growth of the eigenvalues of the spectrum of the Dirac operator. But there is also a KO-dimension as I just mentioned and it turns out that normally, the KO-dimension is equal to the metric dimension modulo 8, but when you look at spaces of dimension 0, you find out that this is not necessarily true : you can fabricate spaces of dimension 0, but which are of arbitrary dimension modulo 8. This could look as a curiosity but in fact, it's not at all, and it has played an absolute key role in 2005, in our joint work with Chamseddine and Marcolli,

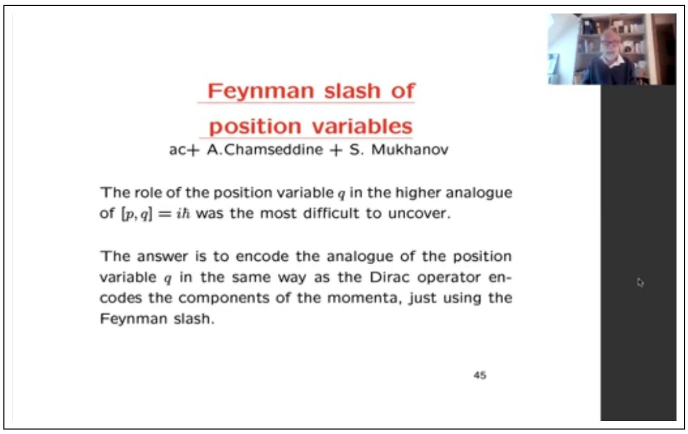


and what we have discovered, you know, with Chamseddine, we had been abandoning our work of understanding the Standard Model in 1998, we had done it in 1996 and we had been abandoning in 1998 because of the discovery of neutrino mixing. And it seemed to be impossible to accomodate neutrino mixing with what we had. But in what we discovered in 2005, the three of us, is that in fact, if you try the various KO-dimension for the finite space, you know, as you put it in the fine structure of space-time, then amazingly, you find that if you take dimension KO-dimension 6, for this finite space that has of course metric dimension 0, then, not only the neutrino mixing comes out absolutely naturally, but also the seesaw mechanism. And I must say I was amazed because I didn't now seesaw mechanism and then I did the calculation of what we had, and I re-discovered the seesaw mechanism. But unlike in physics, it is not put by hand, you find it as a consequence of the calculation.



So what we did later, with Ali[2], we classified the various finite spaces of various KO-dimensions, and of course, we were interested in KO-dimension 6, and among them, we found one which we found extremely interesting, where the algebra that was underlying the finite space (so it's a finite dimensional algebra) was 2×2 matrices over quaternions, + 4×4 matrices over complex numbers. In what we were doing, the breaking to the Standard Model gauge group was done by what we called the order ??, but then by joint work with van Suijlekom, we analyzed the full model, without reduction to Standard Model, and we found a beautiful Patti-Salam model which is in fact much more interesting and symmetric, than the Standard Model itself, in particular because asymptotic freedom.

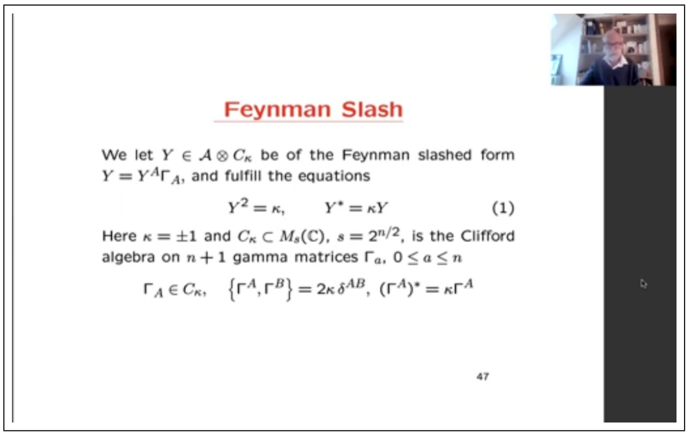
Okay so in fact, at this point, we had found, and we would, you know, be extremely interested in some kind of other way of finding the same algebra. But this algebra is strange in the sense that the real dimension is different from the two sides. You have 32 and 16, so it looks like a very difficult thing to obtain this in a natural manner.



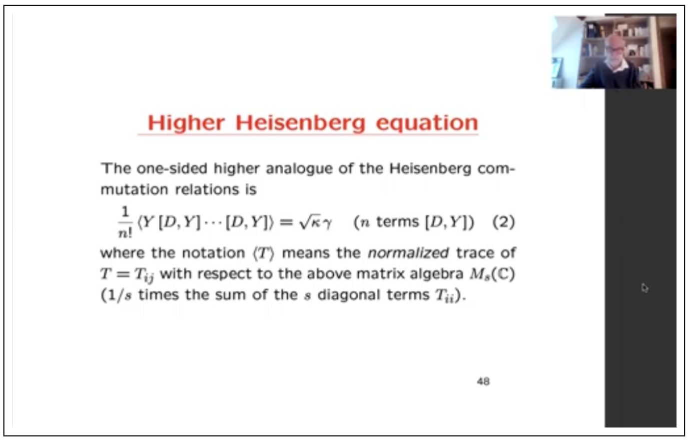
[2]Chamseddine

But this is what we did in our work with Mukhanov. The new idea that came up there is that now, we are not only going to encode, if you want, the full momenta together, assemble all the momenta together, as Dirac did, you know, using the Dirac operator, which was blending together all the components of the momentum to a single entity, but we investigated what would happen if we did the same thing with the coordinates. And what we obtained is an higher analogue of the Heisenberg commutation relations.

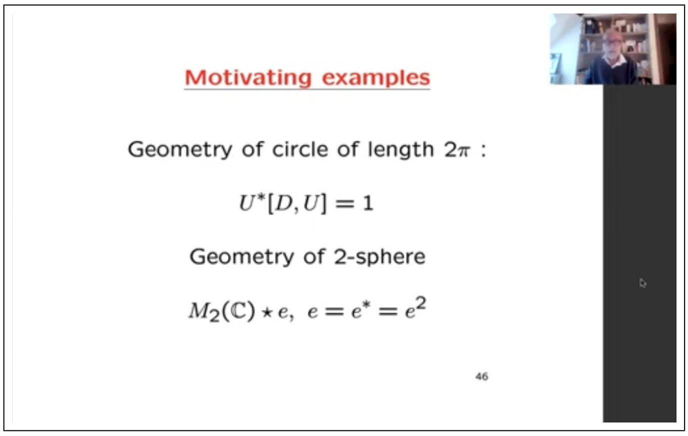
What we first investigated, as I will explain, we wanted to blend together the coordinates to a single operator, so we started of course with the



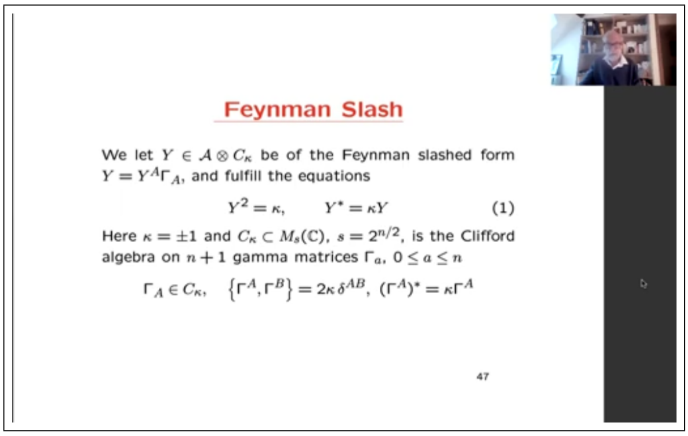
Feynman slash, okay ? We wanted to assemble them into a single entity, and what we found very quickly is that the right condition to ones assemble, because of the Clifford matrices and the relations they were fulfilling, was to have a self-adjoint Y, or ?-adjoint depending on the value of κ that is plus or minus one, that satisfies $Y^2 = \kappa$ (either is self-adjoint of ???-adjoint). This is based on gamma matrices, and at first it is very reminiscent of the sphere because then, the components Y ? A has to satisfy that the sum of their squares is equal to 1.



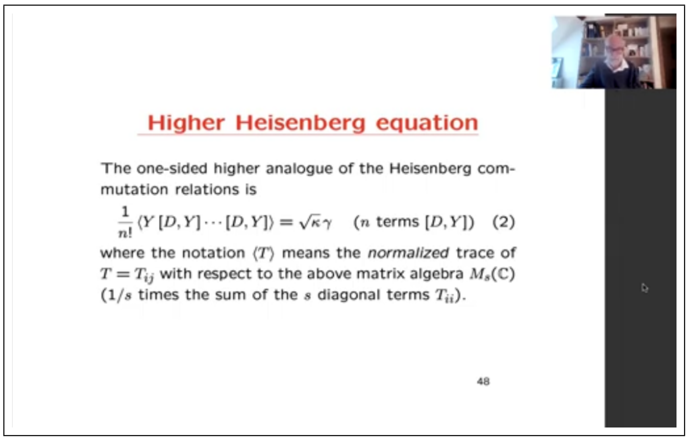
So at first, we wrote a sort of Heisenberg higher type of equation, which is like a commutation relation : if you want to understand what is behind this, I have to come back to a much simpler example,



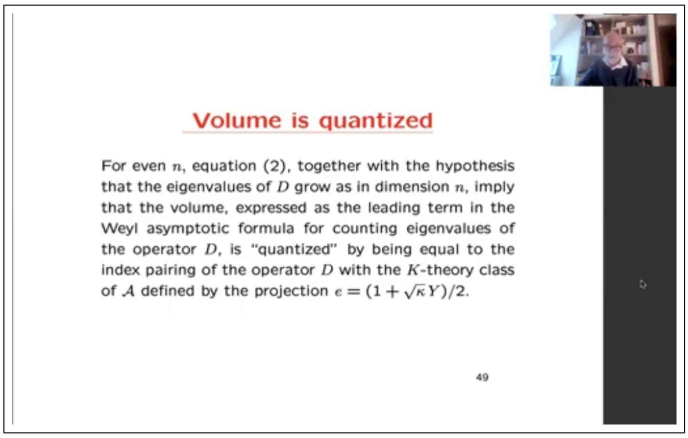
which is the geometry of the circle of length 2π. It's an easy exercize to show that if you look at the geometry of the circle of length 2π, it is uniquely specified by an equation, an operator theoretic equation, which is $U^*[D, U] = 1$. U is a unitary operator, D is a self-adjoint operator, and there is a unique, essentially up to a parameter that plays no role in the metric, irreducible representation of these relations into Hilbert space operators. And when you compute, you find that the spectrum of U has to be the circle and the operator D defines the metric and you find that the corresponding circle has length 2π. And of course, you find this geometry in the right way. Similarly we had started in fact many years ago with Gianni Landi to do the geometry of the 2-sphere in a similar manner by combining 2×2 matrices with projections.



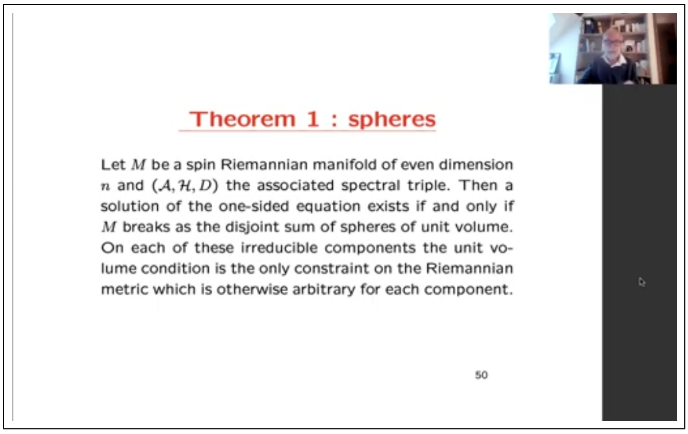
So, with Chamseddine and Mukhanov, we first started with a Feynman slash, we wrote down this higher Heisenberg equation,



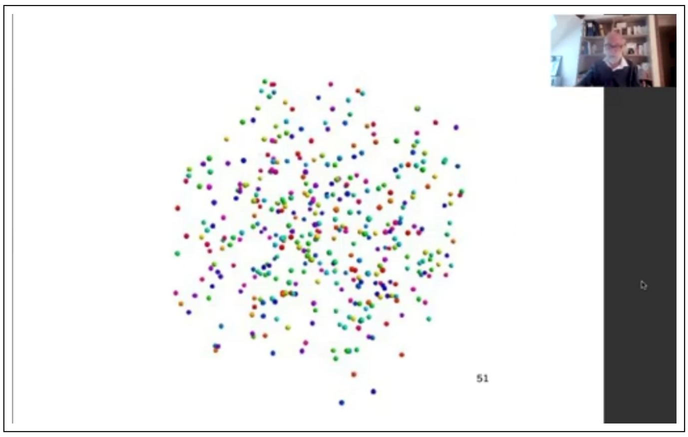
which resembles the equation for the circle, except that now, the commutator D commutator Y is raised to the power, which is the dimension of the space. So we wrote down this equation and we investigated this equation. And one of the first things that we found



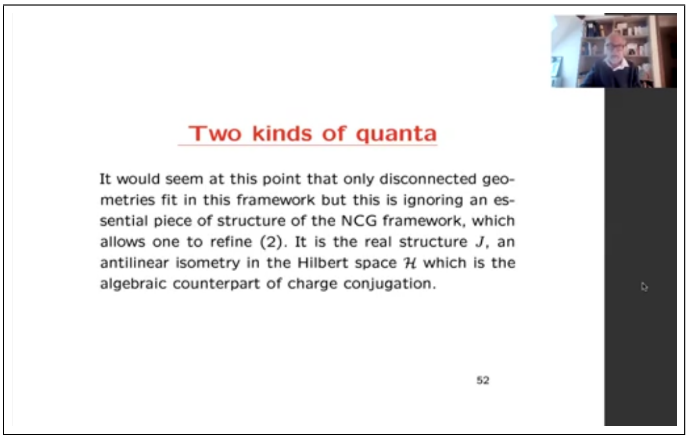
is that this equation, exactly like, in the case of the circle, it was giving you the length 2π, well, it quantizes the volume. So the volume which is given by the growth of the eigenvalues, or if you want, by the logarithmic divergencies of the trace of the right power, is quantized.
So this is the first thing.



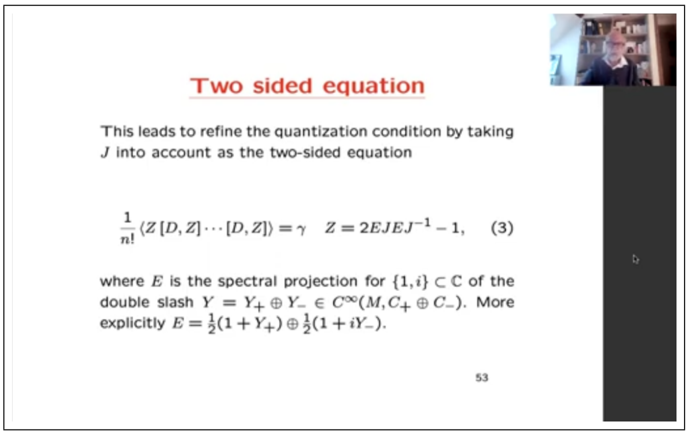
But then, we were a little bit disappointed because what we found is that when you have a solution to this equation, then, automatically, the solution, the manifold, will break as a disjunction of some spheres of unit volume. And if you work in physical units, you find that this unit volume is like the Planck volume.



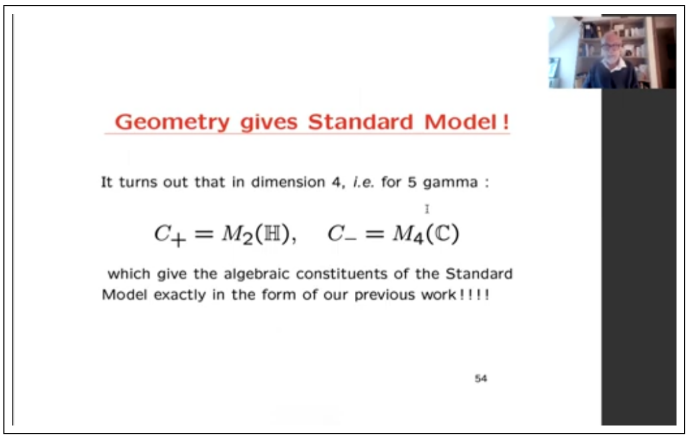
So at this point we were quite disappointed because we said "ok look. Space-time, euclidean or not, it doesn't look like that : it's not a union of spheres, very tiny little spheres." But we have forgotten the essential piece of structure, which is the J, which is charge conjugation, the real structure J.



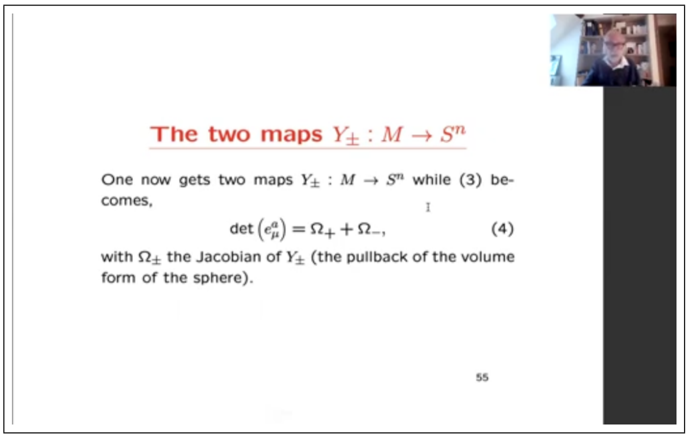
And when you incorporate the real structure J, what you find is that it automatically forces you to refine the higher Heisenberg equation. And because of this issue, KO-dimension 6, and so on and so forth, what do you find ? You find that you are forced to refine the equation by involving the J



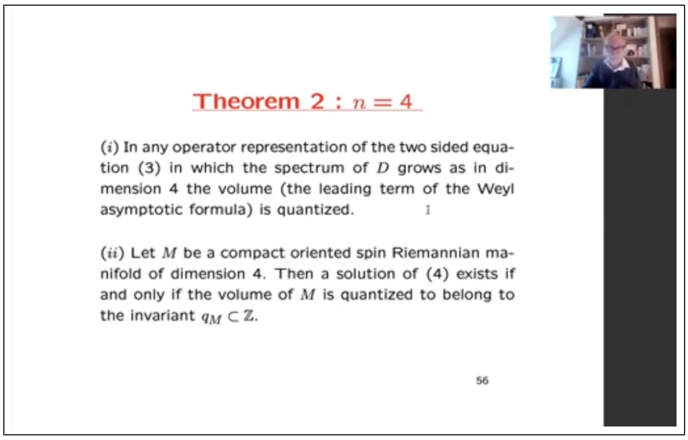
and the J is now involved by passing the projection coming from Y to the commutant. The equation becomes this equation. Now, what really came out of the blue is that all of this, of what I'm saying now, was inspired by the wish of trying to present the geometry in the simplest possible way, having this kind of pairing between the Dirac, and what you obtain by assembling the coordinates into a single operator.



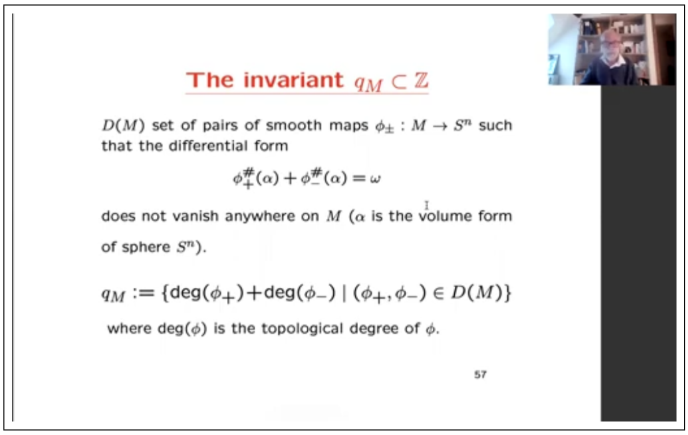
And now we looked at exactly what are the needed Clifford algebras in order to obtain this equation, to obtain the solution of this equation. We looked at the table of Clifford algebras, to find, in the case of dimension 4, so when you take 5 Gamma matrices, then you find that in order to write this, you have two Clifford algebras which appear, irreducibly. And the first one gives you in fact $M_2(\mathbb{H})+M_2(\mathbb{H})$ but because you want to take an irreducible piece, you have $M_2(\mathbb{H})$. And the second is $M_4(\mathbb{C})$ here and they appear all together. They appear if you want as the sum of these two pieces C_+ and C_-.



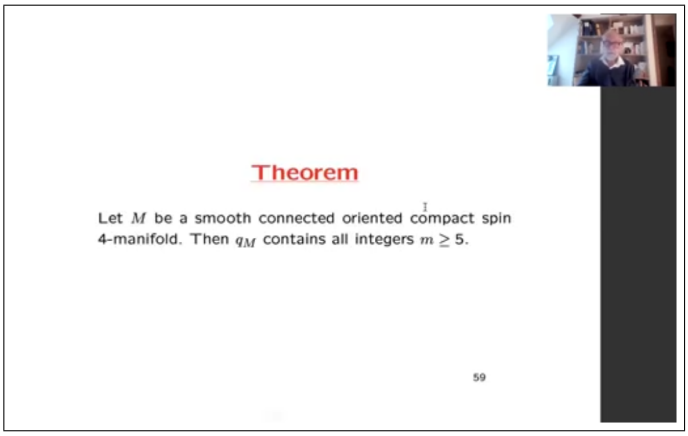
So in fact, out of the purely geometric problem, we found exactly the algebra that was as a sort of put by hand, you know, in our previous work, as a kind of bottom-up story. This was quite amazing but then of course, we had to go further, and we had to prove that we could obtain all possible spin-manifolds, from this construction and no longer a disjoint union of spheres. So what happens is that instead of having a single map from the manifold M to the sphere, and you know, because the sphere is simply connected in higher dimensions, you couldn't escape M to be itself a collection of spheres, but now you have two maps



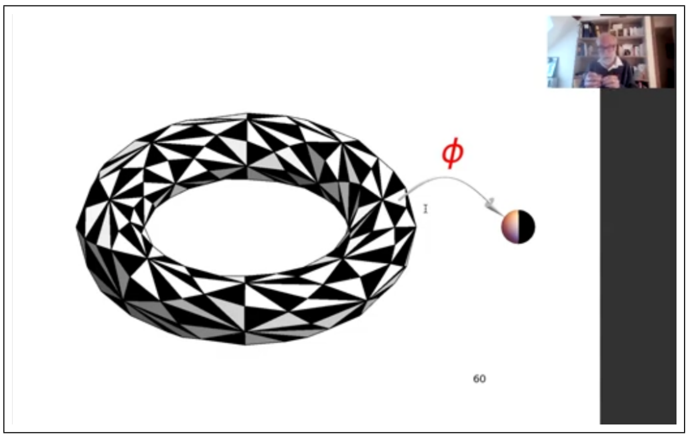
Y_+ and Y_- to the sphere, and the only condition is that when you pullback the volume form of the sphere by plus and by minus, it's not that individually they don't vanish, no, it's that their sum never vanishes. Their sum has to define a differential form that never vanishes, not individually, which of course is not possible, unless you are a sphere.
So very quickly, we obtained two results : we obtained the fact that the volume was quantized, I will come back briefly to that, but we obtain a much more precise fact, that is that if you take a compact oriented riemannian spin-manifold of dimension 4, then a solution of this equation exists if and only if the volume is quantized to belong to a certain invariant,



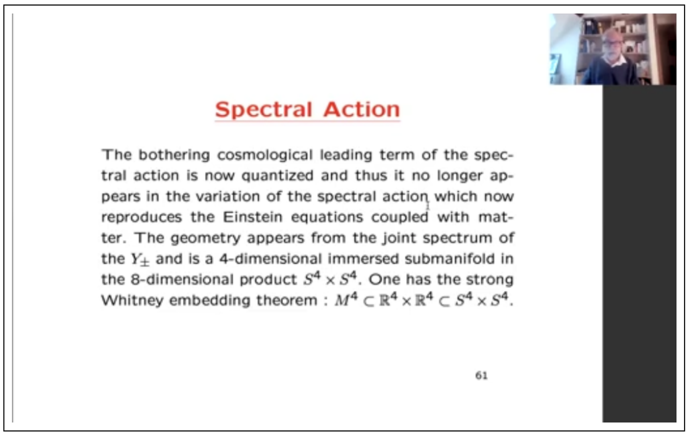
and this invariant is simply the sum of the degrees of this map ϕ_+ and ϕ_- which fulfills the condition that when you pullback the volume, you get something that doesn't vanish. Now after a lot of work, a lot of geometric work, which was using the existence of ramified covers of the sphere and alsousing the full power of the immersion theory which goes back to Smale, Milnor and Poenaru, in fact, a theorem of Poenaru that you have an open oriented manifold of dimension n, then you can immerse it in $\mathbb{R}^n$. Then we were able to prove that, in the case of dimension 4, for any spin-manifold, this invariant will contain all integers n bigger than 4. The case of dimension 2 and 3 is much easier by general transversality arguments but the case where $n = 4$ is much more difficult.



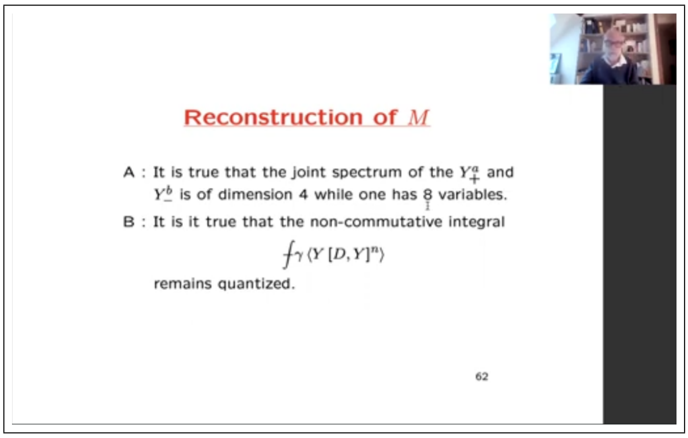
What happens is that now you can obtain any spin-manifold of any arbitrary volume. If you take a smooth connected oriented compact 4-manifold, then this invariant contains all integers bigger than 5 and what does it mean ?



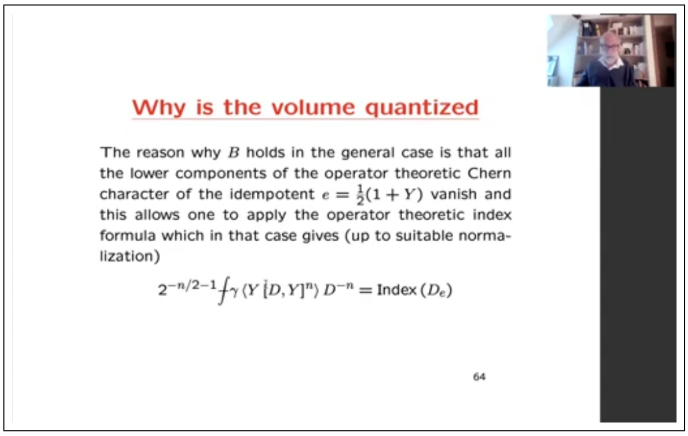
It means that you can sort of obtain this manifold from two little spheres of Planck size but of course, the manifold itself will sort of develop, and it will develop to arbitrary size. This is why we entitled the paper that we wrote with Ali Chamseddine and Mukhanov "*Quanta of Geometry*" because it's really what is going on. There are little quanta that mesh together to form this huge manifold. What happens also is that,



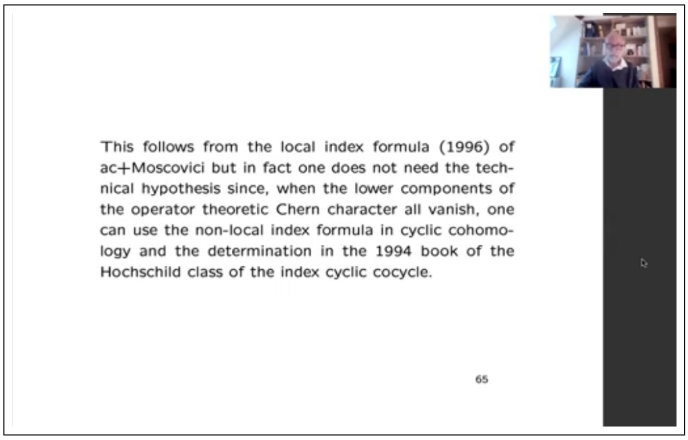
now because the volume is quantized, when you write down the spectral action as we had written with Ali, as I said, in the spectral action, what you have is that there is a cosmological term which is huge, and which is quite bothering. But now, because the volume is quantized, this cosmological term, which is the leading term of the spectral action, plays no role, when you write down the variational equation. And so, when you write down the variational equation, you really reproduce the Einstein action coupled with matter. The geometry is reconstructed as a joint spectrum, and it's a 4-dimensional sub-manifold of this 8-dimensional product of two very little spheres,



and there are rather general facts which are key, in order to do this reconstruction : there is the fact that the joint spectrum will be of dimension 4, this relies on a deep result of Dan Voiculescu, and also, it relies on the fact that the index theorem, will tell you that the volume will remain quantized.

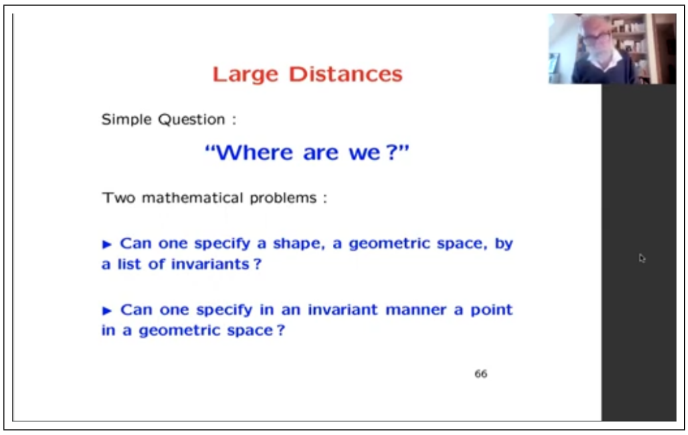


The fact that the volume remains quantized follows from the fact that you have the Heisenberg higher condition which gives you that these quantities will be equal to gamma, so gamma squared is 1, so they cancel on. But this is also an index. So the reason why it is an index relies on an index theorem that we proved

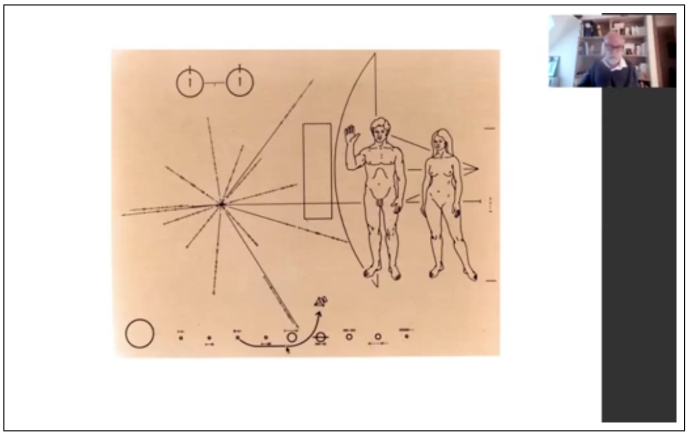


with Henri Moscovici back in 1996, but in fact, one can use a less general result, because it turns out that the components of the Chern characters of the Y automatically vanish, the lower components. So in fact, one doesn't need, if you want, in cyclic cohomology, the full understanding of the index, one just needs the understanding of the Hoschildt class of the index. Of course, this is very instrumental, in proving this result. So I hope I have convinced you... So I just want to add one thing, that some physicists will dismiss this, because at some point, we had made a wrong prediction, which was about the Higgs mass, but there is a very interesting story, which is that with Ali Chamseddine, we wrote a survey paper in 2010 in which we were explaining the theory, and in that paper we had a scalar field, which in fact we ignored, when we did the renormalization group calculations. This scalar field was opering as coefficients for the neutrino and so on, okay, this paper is published in 2010, now what happens is that in 2012, Ali wrote to me an email and he told me "you know, it's amazing because there are three independent groups of physicists who have shown that if you add a scalar field to the Standard Model, then you can recover the stability of the Higgs scattering parameter, the positivity of the Higgs scattering parameter at unification, which is exactly what was, if you want, contradicting our prediction, the fact that it was no longer positive in the usual model. I couldn't believe my eyes, I didn't believe Ali, and I checked and all the signs were correct, our scalar field was exactly the right one. So that, it could correct the prediction and make it compatible with the actual value of the Higgs mass. So the model is not at all disproved by this.

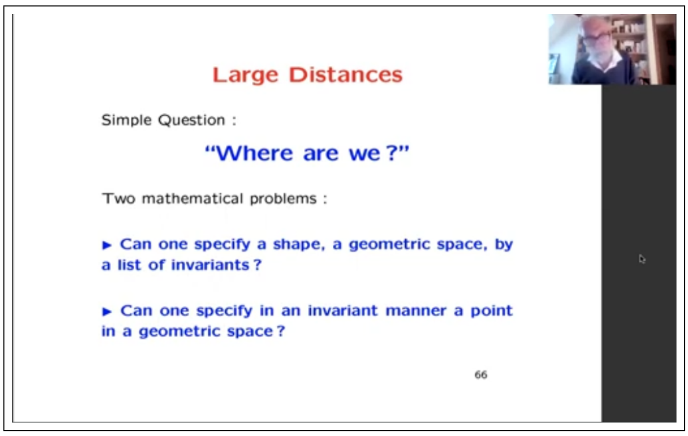
So now let me come to large distances.



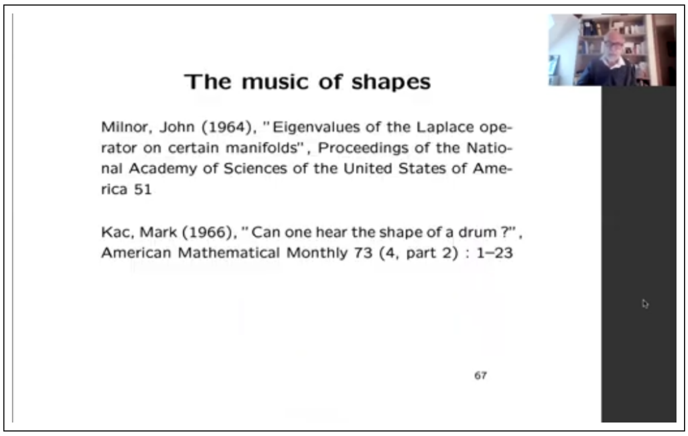
Riemann was very very careful in his inaugural talk, to distinguish between large distances and small distances. So I hope I made the point about small distances and now, when you look at large distances, I want to explain that the spectral point of view is equally relevant. For that, I will ask a simple question, which is "Where are we ?". By this I mean, how can we try to specify the Earth, if we send for instance a probe in outer space, how can we specify where this probe is coming from.



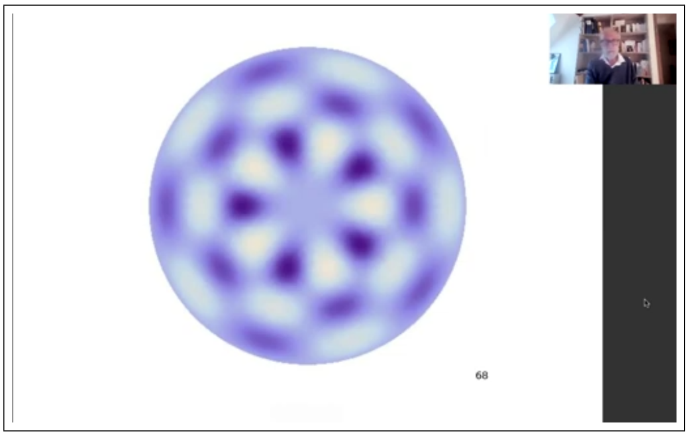
Of course, you can show the solar system, with our planet, you can show what we look like, but there is something that is much closer to the answer I want to explain, and which is this picture, when you have all those straight lines which all group in the same point, and on each of them you have a frequency, which is indicated.



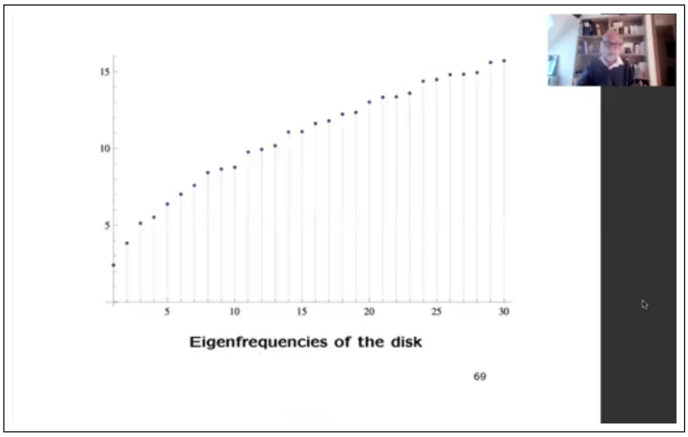
Now this gives birth to two mathematical problems, the first one is "can one specify a geometric shape by a list of invariants ?" So if you try to specify the universe or whatever space, by giving a chart, coordinates system, this is ridiculous, because if for instance you give what is the coordinates system, you have to specify the origin, so the question is completely circular. So what you have to do is first you have to specify the shape, by geometric invariants, by a list of invariants, and then, "can now specify in an invariant manner a point in the geometric space ?". So these are two mathematical problems, and the answer relies on two papers.



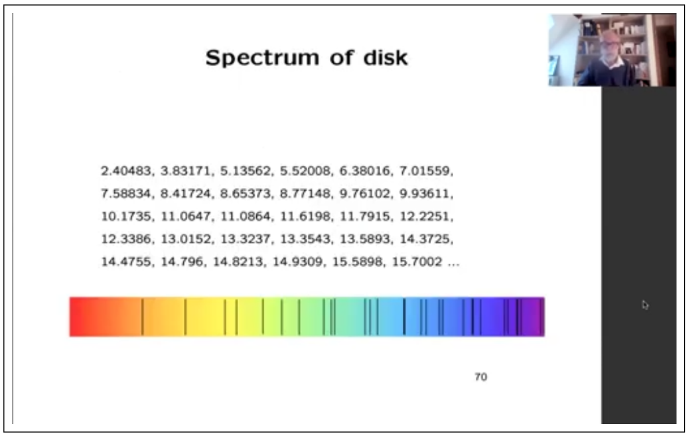
There is paper of Milnor in 1964 : he showed that when you take a space, when you take the eigenvalues, when you take the spectrum, the eigenvalues of the Laplace operator or of the Dirac operator, that doesn't matter for this, then it turns out that this is not a complete invariant of the geometry. He exhibited two spaces in dimension 16, that had the same eigenvalues and the reason is just modular forms and theta functions. And then, there is another paper, which is by Mark Kac in 1966 which is "can one hear the shape of a drum ?".



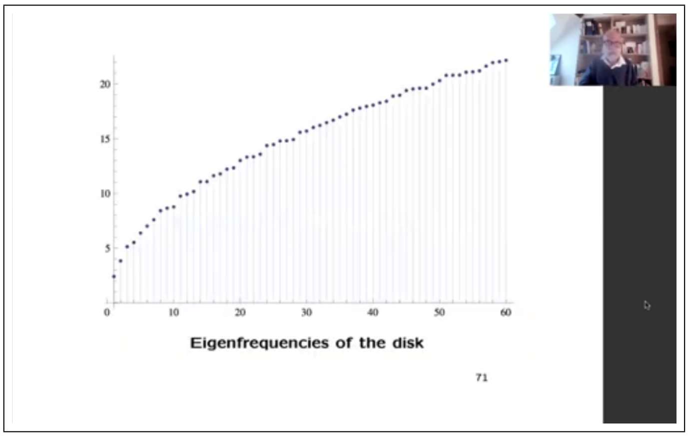
So if you take a drum, it will vibrate, it will have many forms of vibration, wihch depend on how many variations you have when you go around, and how many vibrations there are when you sort of go from the center to the external of the drum ; so it has sequences of eigenvalues.

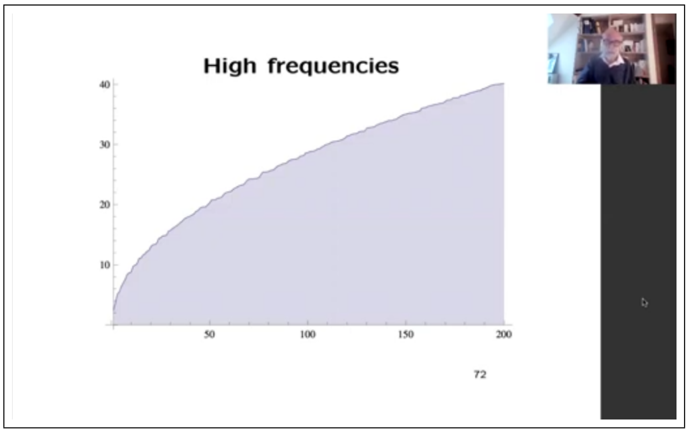


they grow,

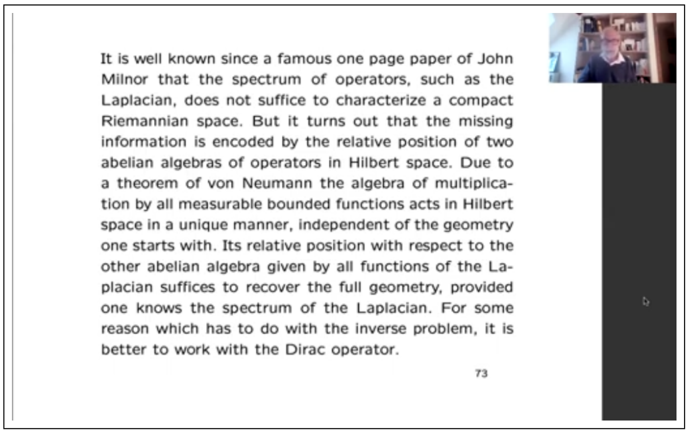


they are computable as zeros of Bessel functions, they form a kind of spectrum, and when you look at them for higher and higher frequencies, they form a parabola.

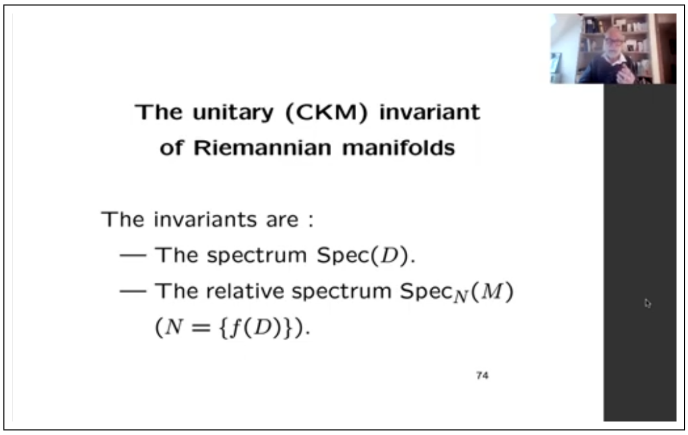




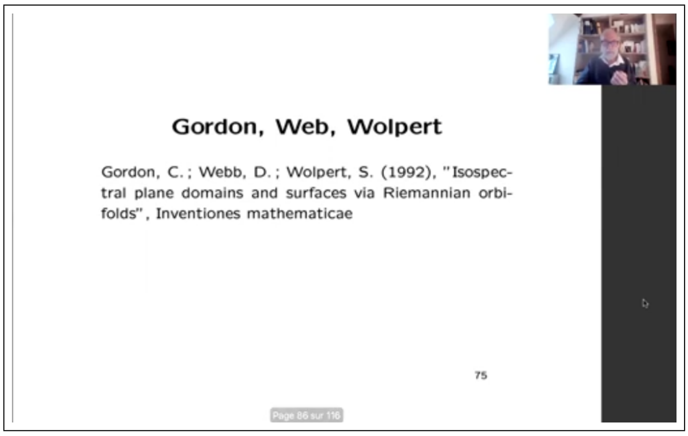
And this parabola indicates that you are handling a form of dimension 2. This is a result of Hermann Weyl which will be quite instrumental later.



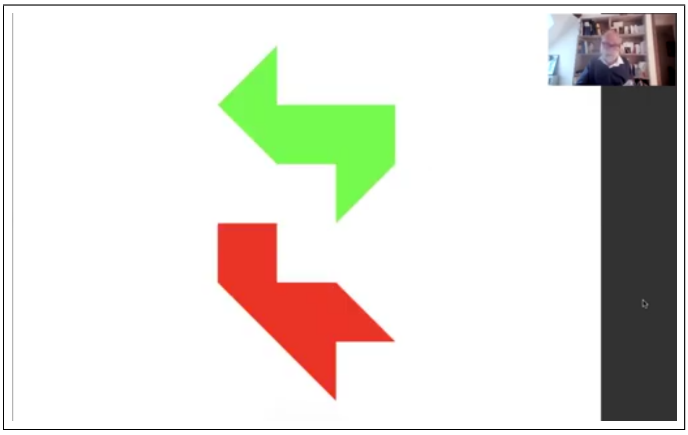
Now, the answer which I want to explain is what is missing when you only have the spectrum, when you only have what they will call the scale if you want, because if you could do music with the shape, you would have a scale which will be forced upon you and which will be the very specific frequencies which are given there. Now it turns out that the missing information that you need, that you are missing in order to reconstruct the space, the geometry with all its properties, is in fact given by the relative position of two abelian algebras of operators in Hilbert space. There is of course the Dirac operator, that by its spectrum is uniquely embeddable in Hilbert space, but there is another, and this comes from a theorem of von Neumann, because the result of von Neumann proves that if you take two manifolds of the same dimension, it turns out that... von Neumann algebra multiplicated by functions, by measurable functions on the manifolds... they are isomorphic in their action... but not only isomorphic as algebras but also isomorphic by the way they are acting on the Hilbert space. So if you want the pair which is given by the algebra and the Hilbert space is unique. The pair which is given by the Hilbert space and the Dirac operator is given by the spectrum. The only thing you are missing is what is their relative position. And this relative position let me to define



an invariant, which is rather subtle to define, but which I can illustrate very simply on an example, which I called the CKM invariant and the reason for which I called it CKM is because of Cabibbo-Kobayashi-Maskawa who are using a similar invariant when they define their... you know, the thing that was actually breaking the CP, you know, in the Standard Model. So the invariants are given by the spectrum of the Dirac operator, but by something which is like giving the possible chords, on this spectrum.

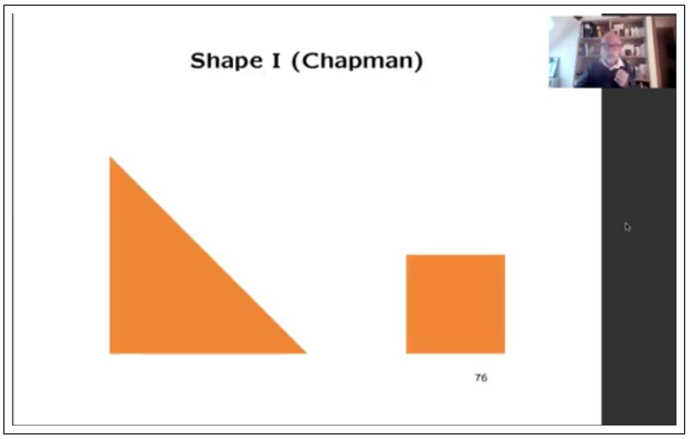


And to illustrate this, I will show you this on a very simple, of course very naive example, what it is. And for that, I will work in dimension 2, because thanks to the work of Gordon, Web and Wolpert for instance, one has beautiful example, of isospectral shapes, in dimension 2.

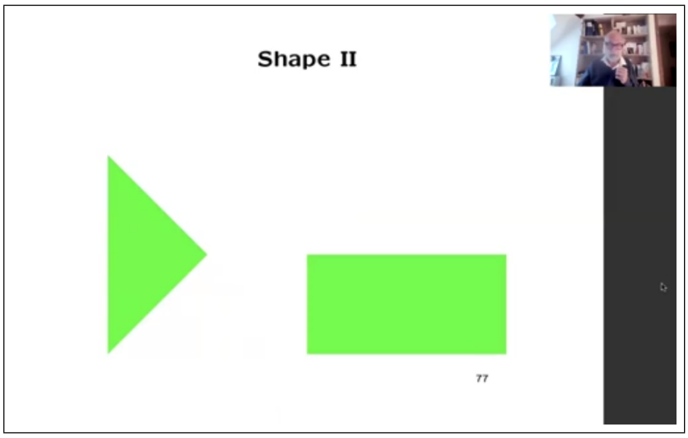


So these two shapes for instance, they have exactly the same spectrum, and of course, they are not the same, here you have this protube of the little square, that doesn't appear in the other shape.

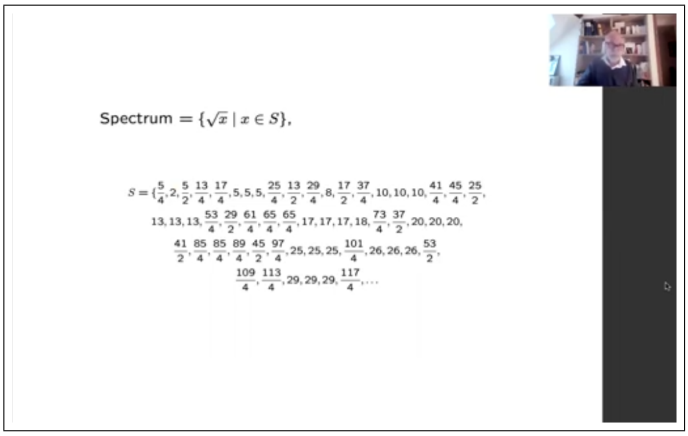
So there is another example which I will use which is due to Chapman, and the two shapes I will use are not connected.



So the first shape is this union, of this isosceles triangle and this little square,

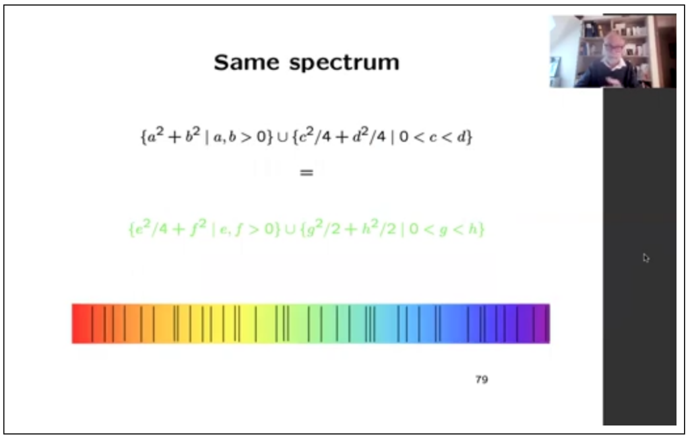


and the second shape is the union of this isosceles triangle with this little rectangle, now it turns out that when you compute, you'll find out that these two shapes have

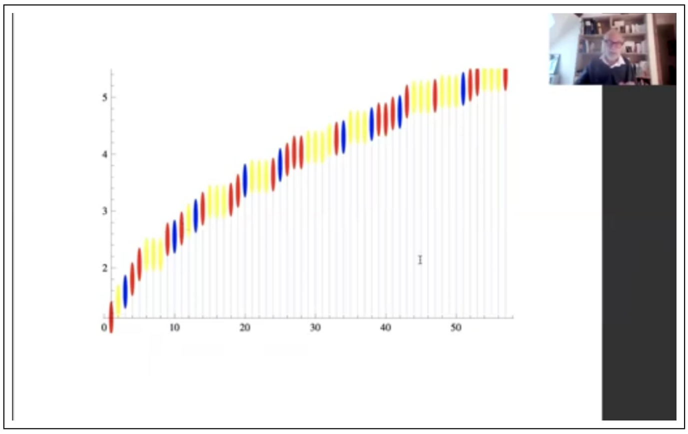


exactly the same spectrum. Each of them is disconnected, but they have the same spectrum. The fact that they are disconnected will help me to show you what is going on.

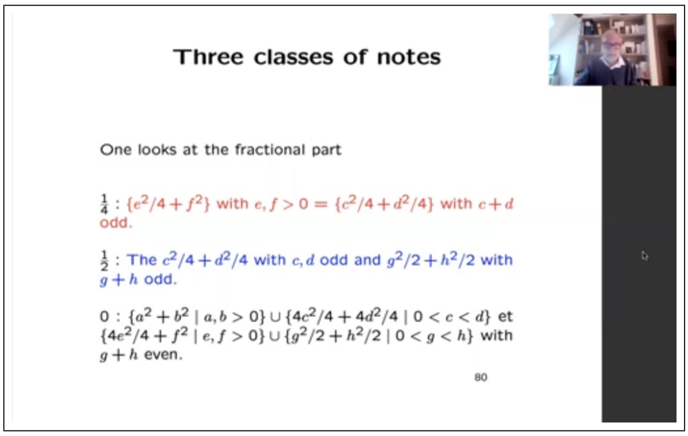
So when you compute the spectrum, you find, when you write the squares of the spectral lines, they are three types, there are three types of nodes in the scale : there are nodes which are of fractional parts 1/4, there are nodes which have 1/2 as fractional part, and there are nodes which are full integers.



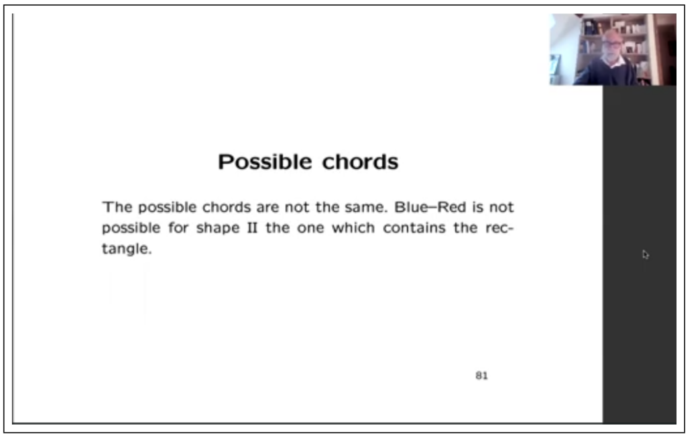
So in fact, this forms a kind of a scale like this, where you are like on a piano, you have the black and the white keys, here you have the blue, the red and the yellow. So you have three types of notes. Now as I said

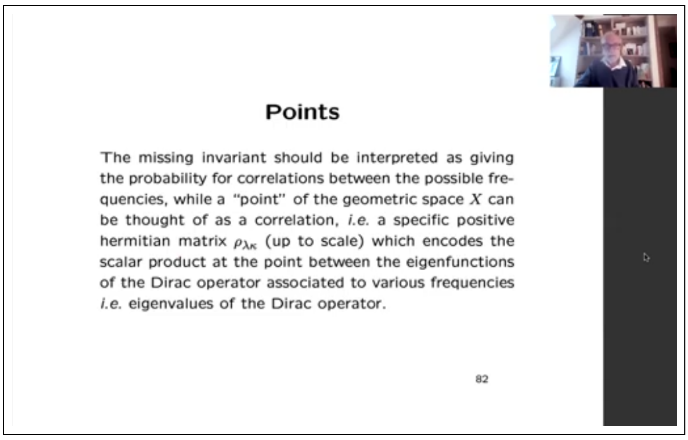


they have the same spectrum, spectrum looks like this, there are these three classes of nodes,

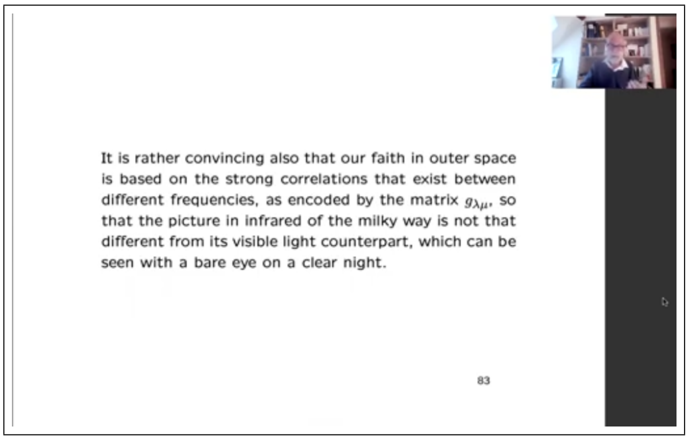


and now, the two shapes are not the same, why ? Because the possible chords are not the same. What you find out, you have to think a bit, is that the chord blue-red is not possible for shape II, the one which contains the rectangle, but that this chord blue-red is possible for shape number I. Okay you have to define what you mean by a chord, and so on.



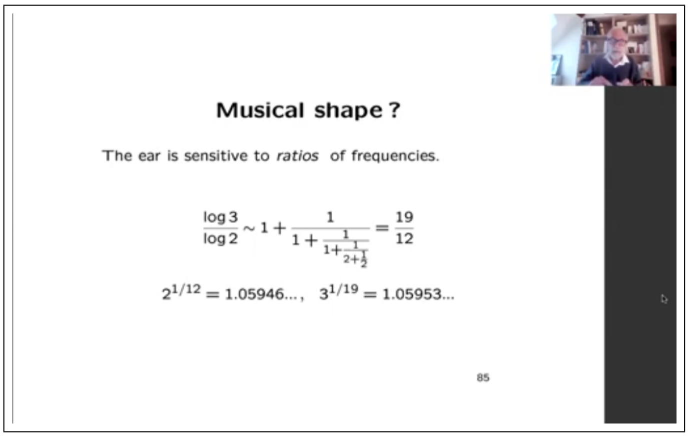


But what it means in general is that the idea of a point also emerges from this type of thinking. The invariant is quite difficult, quite delicate to define. What emerges also is the idea of a point. The idea is that a point in a geometric space should be thought of as a correlation. In fact, it's given there as a specific Hermitian metrics, but what it encodes is the scalar product at the point of the eigenfunctions of the Dirac operator, but what it encodes if you want is the correlation between various frequences.



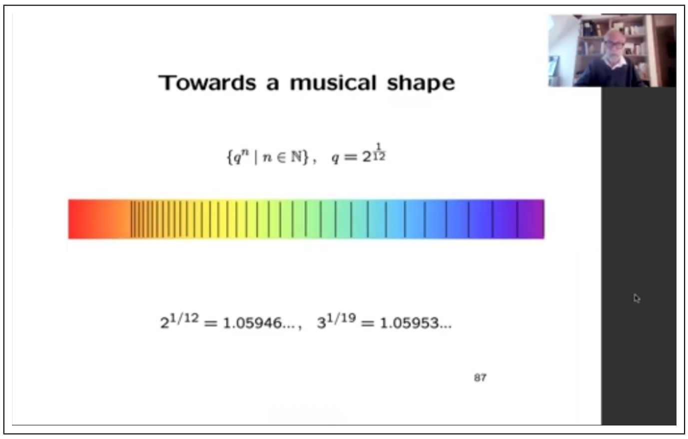
And this is very convincing since our faith in the existence of the outer space is based on the strong correlation which exists between different frequences. For instance, when we look at the milky way, we can look at it in visible light, but we can also look at it in other frequences like X-ray, or infrared, and so on. And it's crucial that all these various pictures that we get in different frequences are actually correlated to each other.

So this is what this additionnal invariant is telling. Now to make a little break, when I was playing with these various shapes and with their scales, I was wondering "is there one that would allow us to do music as we like it ?", like the notes like on a piano. And of course, you have to know the minimal amount of music which is that the ear is sensitive not to adding one, like you would get in an arithmetic progression, not at all, the ear is sensitive to ratios of frequences : if you multiply a frequency by 2, it's like when you play on a piano, you play an A, and now if you play the same A one octave up, you are in fact just doubling the frequency, and the ear is very sensitive to that. Now it's also sensitive to multiplication by 3,

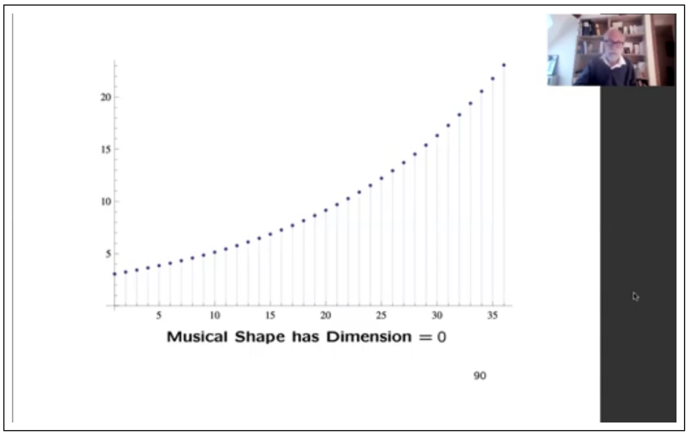


and one mathematical fact which is extremely used in music is the fact that when you look at 2^{19}, it's almost 3^{12}. Of course, they can be equal, because one is even and the other is odd, but what it means is that if you take $\frac{\log 3}{\log 2}$, it's very closed to $\frac{19}{12}$. In fact $\frac{19}{12}$ appears in the continuous fractions expansion. So in fact, the twelve-th root of 2 is very closed to the nineteen-th root of 3, and it turns out that the correct musical shape is the one that you can see on a guitar. You see, when you look at a guitar,

you will find out that the frettes on the guitar which are like here, they do not form at all an arithmetic progression, they are not equally spaced, no. If you think a bit, you have to think, and then you compare it, you make some measurements and so on, you find that these frettes are exactly the powers of this number q which is $2^{1/12}$.

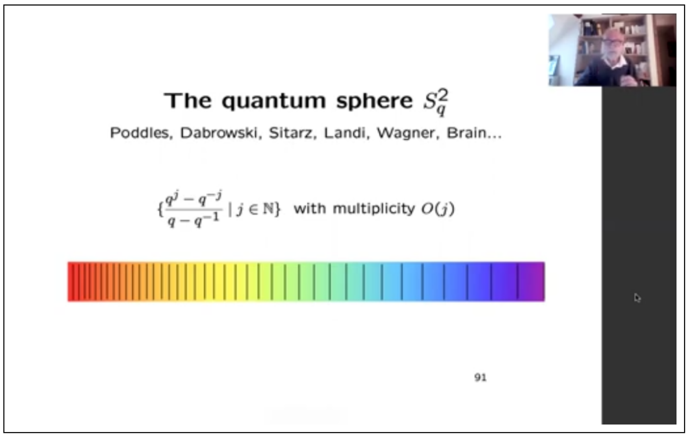


And the spectrum we are looking at, this musical shape, what it should be is exactly what happens with the frettes, namely it should be the powers of this number. Now, you can look at the shapes we know, we are used to, and try to find this spectrum. You get nowhere : if you get the sphere



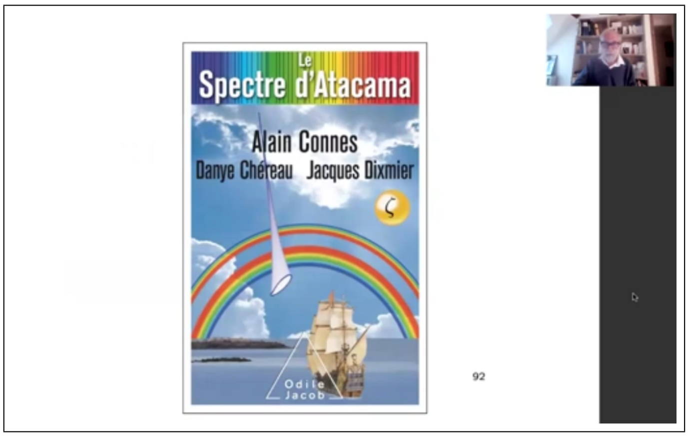
for instance, the 2-sphere, well of course, you know, the high frequencies look like a parabola, but you get nowhere, why ?

You get nowhere because this musical shape when you look at the spectrum, it grows exponentially fast, of course, because it's a geometric series. So when you look at its dimension, I mean, involving the previous ideas that you have to use the Hermann Weyl theorem and so on, you find that it has dimension 0. Dimension 0, it means that it's sort of hopeless to find it among the shapes that we know. But amazingly, it does exist in the non-commutative world, and it is the quantum sphere,

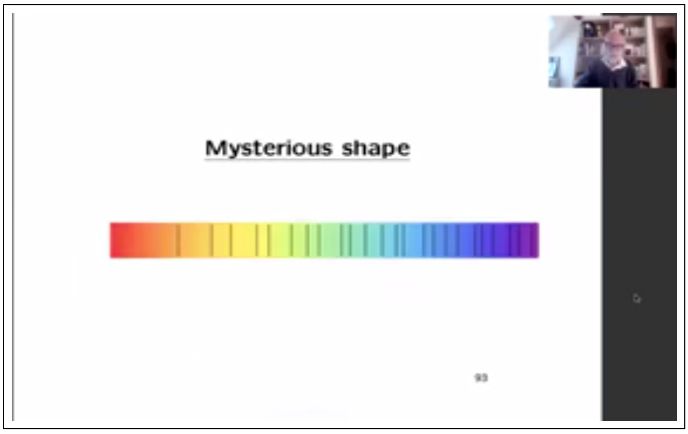


which is a deformation of the sphere, and whose beauty is due to the fact that not only, it has the spectrum that we would like to have, but also, it has the symmetries we would like to have, namely, like a sphere, which has the full group of symmetries which are acting transitively, the quantum sphere has a quantum group of symmetries, which is acting transitively in the suitable sense.

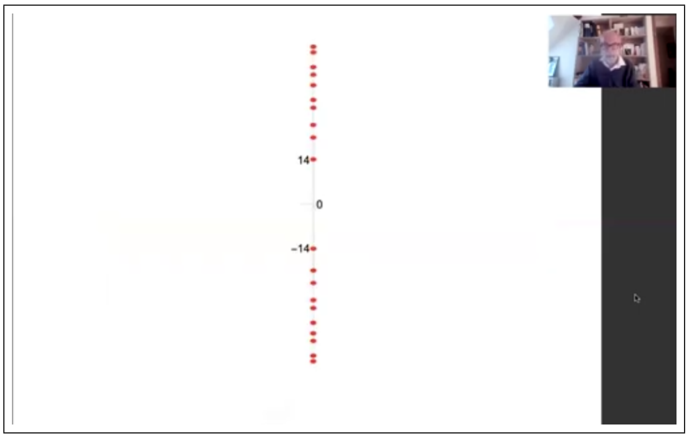
Let me now come to a very important topic which will be like ending of my talk.



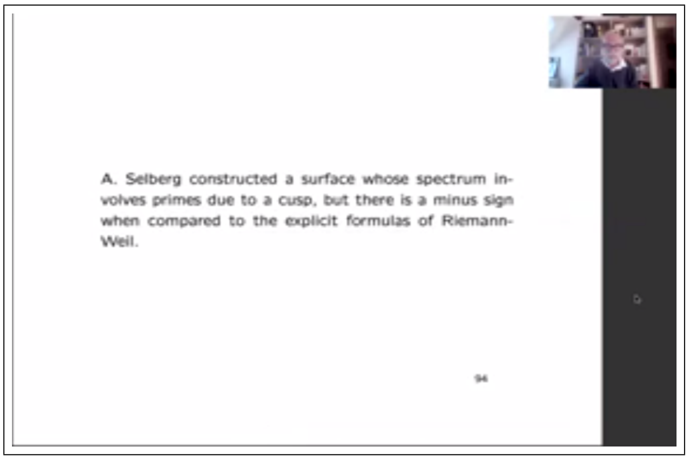
There is a book which I have written with my wife and with Jacques Dixmier, which is a kind of Prelude to this last topic.



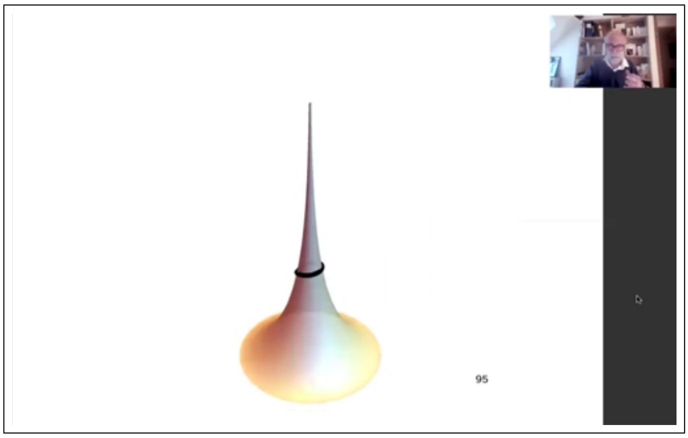
So this last topic is a mysterious shape, there is a fairly mysterious shape, which I am showing you, as far as this spectrum is concerned, and this is how it appears,



when you first look at it, and what is it ? Well, people who know a little bit of number theory will have recognized the zeroes of the Riemann zeta function. Now this is a very mysterious shape, and you have to admit that it looks a bit like a kind of spectrum of some Dirac operator, this remark was made to me by Atyiah, and you know

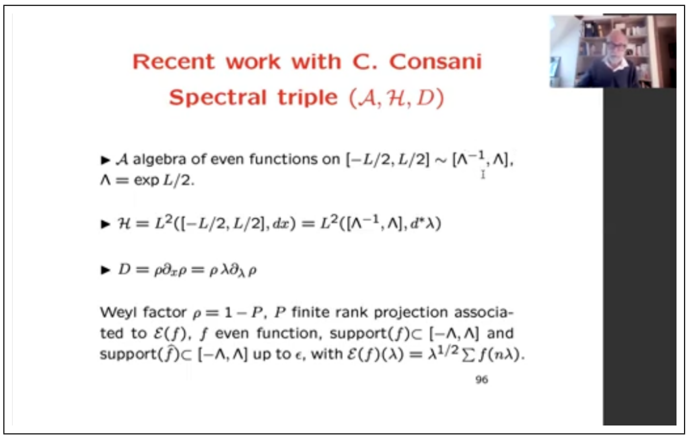


Selberg tried to find, to construct a surface whose spectrum could be quite related to this. He constructed a surface, which, because of a cusp, was related to primes, but when you compute with this, you find that there is a minus sign which appears, and when you compare to the explicit formulas of Riemann-Weil.

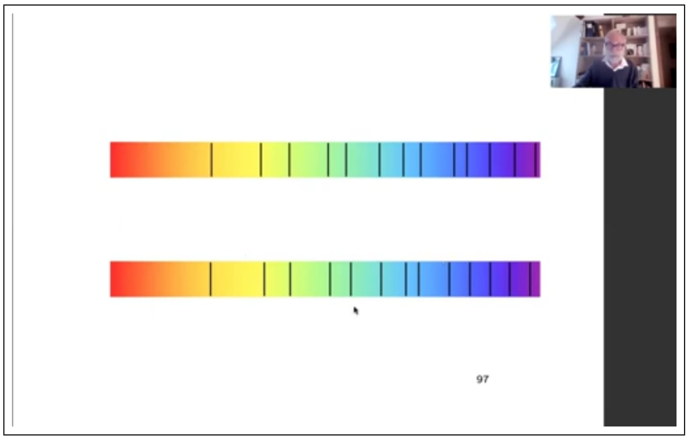


So this sort of didn't quite work, and this is the type of cusp that Selberg was getting and which was coming rather close.

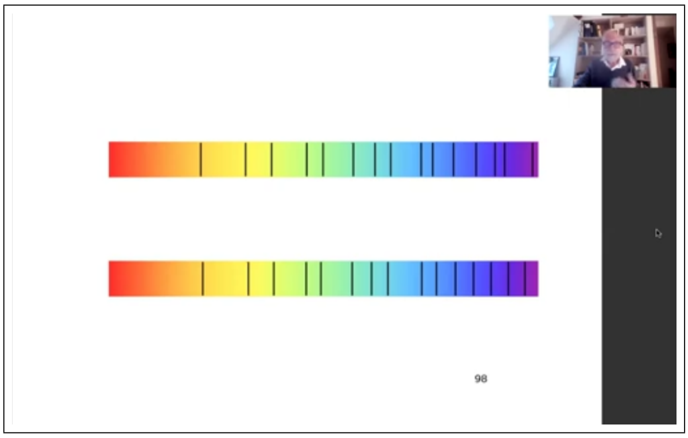
Now the reason why I mention that is that a very recent work which was done in the last month with Katia Consani, what we have found, is a non-commutative geometry, but this non-commutative geometry is of course given by a spectral triple, but the algebra is commutative.



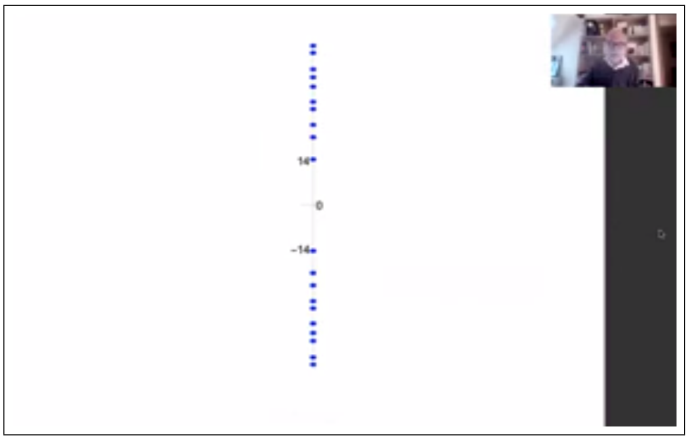
so the algebra is just the algebra of ordinary even functions on the interval $[-L/2, L/2]$, I think you better think of it multiplicatively ($[\Lambda^{-2}, \Lambda]$, this is a bit better. So you think of it as functions from lambda length to lambda by using the exponential. Okay. The Hilbert space is equally simple, it's the Hilbert space of all L^2-functions, on this interval, to dx or to Haar measure multiplicative $d^*\lambda$ when you think multiplicatively, and what about the Dirac operator ? The Dirac operator is a very tiny perturbation of the ordinary Dirac. And the ordinary Dirac is given by D by ∂x or by λD by ∂_λ when you work with the exponential, and this tiny perturbation is by a Weyl factor : it turns out in general that you can write the Dirac operator from a given metric to the Dirac operator from the new metric obtained by just introducing a Weyl factor by the formula $\rho\partial\rho$. So this is the formula I am using here, and the Weyl factor couldn't be simpler, it's of the form $1 - P$ where P is the finite rank projection, which is associated to even functions whose support is between $-\Lambda$ and Λ, now I am in the multiplicative framework, and whose support of the Fourier transform is also in $[-\Lambda, \Lambda]$. So one has to be very careful, I mean, this is impossible, but it's possible up to ε and this leads to prolate functions. Okay. So what we did with Katia, we defined this spectral triple, it depends on the length of the interval, and we were able to compute the corresponding spectrum of Dirac operator, only in very simple cases because the formulas for the prolate functions are quite complicated. So we did compute it for small values of L and to our amazement, you know,



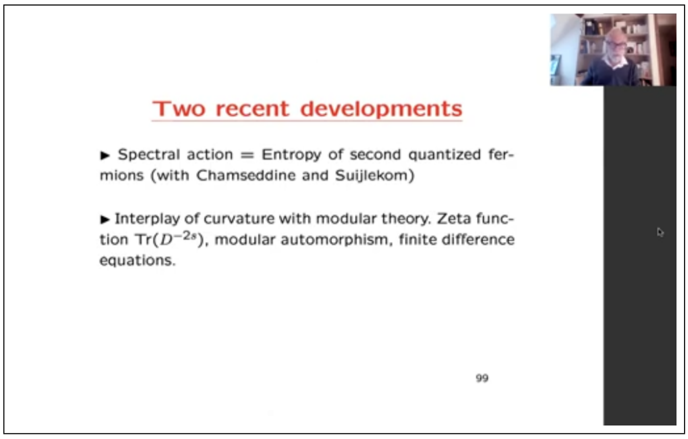
I mean, this thing is the spectrum of the zeroes of zeta, and this is the spectrum that we found, in the first example, and in the second example, we had an even better coincidence between the two.



So we are now exploring this coincidence, trying to understand in which sense, in the limits the two spectra coincide, and that is just the tip of an iceberg in a huge program that we are pursuing with Katia Consani on the Riemann zeta function and which of course, I mean, has many connections to non-commutative geometry but finally, if you want, it has a connection with the spectral point of view. It has many connections with other sides of non-commutative geometry, in fact, to the singular spaces, like the adele class spaces, and also to topos theory of Grothendieck, and so on and so forth.



So this is the spectrum we obtained which is so similar, the spectrum of our Dirac operator, so similar to the zeroes of zeta,



and I want to end my talk by mentioning two recent very active developments. There is one development which I like very much which is that when we developed the spectral action, with Chamseddine, the spectral action is depending on a function, it depends very little of this function because you just give the asymptotic expansion but we had no way to choose the function. Now it turns out that with Ali Chamseddine and Walter van Suijlekom, we showed that in fact, the spectral action is equal to the entropy of the second quantized fermions, but for a very very specific test function, which is related to the Riemann zeta function. And the other development, which I didn't have time to mention, is the interplay of the curvature, which is typically a Riemannian curvature, an extension of the Riemannian curvature to the non-commutative case, there is a fantastic interplay between this curvature and the modular theory, the modular operators, that I mentioned with respect to time evolution. So this theory is amazing in the sense that one has to compute asymptotic expansion, and as far as I am concerned, you know, I started to work on that with Paula Tretkoff, Paula Cohen at the end of the 1980's, and this was revised more recently to prove the Gauss-Bonnet theorem in the non-commutative setup, this Gauss-Bonnet theorem was proved in a particular case, but then was proved by Masoud Khalkhali and his collaborators in the general case, and this work, as far as I am concerned, really acquired incredible substance in my collaboration with Henri Moscovici. And what we found in particular is that the formulas were fulfilling certain finite differential equations that were allowing you to compute the functions of several variables which were occurring in the interplay of curvature with modular theory and these things can be put on computer and can be checked, we were absolutaly amazed that the theory was predicting some very complicated relations which were actually fulfilled and an higher case of this was done in my collaboration with Farzad Fathizadeh, when we computed the a_4 terms in the asymptotic expansion.

I will end with a diagram which shows the relations between non-commutative geometry and other branches of mathematics. So I mean non-commutative geometry fits itself tremendously on its relation with physics, high energy physics as I explained, its relation with number theory, with the space underlying the adele classes and so on, with operator algebras of course, from the very start, with K-theory, Index theory and fantastic KK-theory of Kasparov which is one of the key tools, with of course algebraic topology, geometric group theory and so on and so forth, and also with differential geometry, because in all these cases, there is a feedback, for instance in differential geometry what Skandalis and his collaborators have shown is how much it is relevant, not only to study manifolds but to study smooth groupoids, and to study smooth groupoids, you need non-commutative geometry. So okay, I think I will end here, and I will thank you for your patience, thanks a lot.