THEOREM 2.18. *The dimension of the near intersection of Q_Λ with N_E is given by*

$$\mathrm{Tr}(Q_\Lambda N_E) = \frac{4E}{2\pi}\log\Lambda - 2(\langle N(E)\rangle - 1) + o(1), \quad \textit{for} \;\; \Lambda \to \infty. \tag{2.132}$$

This still requires justifying the fact that we applied the result of Theorem 2.6 to the function $h_E \notin \mathcal{S}(\mathbb{R}^*)$. This is discussed in §5.1 below. We also give in Remark 2.23 below an explanation for the additive 2 that appears in the expression $-2\langle N(E)\rangle + 2$ in (2.131) and (2.132).

5.1. Quantized calculus.

In order to prove Theorem 2.18 and refine the analysis of §5, we use the quantized calculus developed in [**68**]. In particular, we analyze here the relative position of the three projections P_Λ, $\widehat{P}_\Lambda$, and N_E, using identities involving the quantized calculus, as proved in [**72**]. The method used here is based on the idea of Burnol [**37**] which simplifies the original argument of [**71**].

First recall from §IV of [**68**] that the main idea of quantized calculus is to give an operator-theoretic version of the calculus rules, based on the operator-theoretic differential

$$đf := [F, f], \tag{2.133}$$

where f is an element in an involutive algebra $\mathcal{A}$ represented as bounded operators on some Hilbert space $\mathcal{H}$, and the right-hand side of (2.133) is the commutator with a self-adjoint operator F on $\mathcal{H}$ with $F^2 = 1$.

In particular, we recall the framework for the quantized calculus in one variable, as in §IV of [**68**]. We let functions $f(s)$ of one real variable s act as multiplication operators on $L^2(\mathbb{R})$, by

$$(f\,h)(s) := f(s)\,h(s)\,, \quad \forall h \in L^2(\mathbb{R}). \tag{2.134}$$

We let $\mathbf{F}_{e_\mathbb{R}}$ denote the Fourier transform with respect to the basic character $e_\mathbb{R}(x) = e^{-2\pi i x}$, namely

$$\mathbf{F}_{e_\mathbb{R}}(h)(y) := \int h(x)\, e^{-2\pi i x y}\, dy. \tag{2.135}$$

We also introduce the notation

$$\Pi_{[a,b]} := \mathbf{F}_{e_\mathbb{R}}\, \mathbf{1}_{[a,b]} \mathbf{F}_{e_\mathbb{R}}^{-1}, \tag{2.136}$$

for the conjugate by the Fourier transform $\mathbf{F}_{e_\mathbb{R}}$ of the multiplication operator by the characteristic function $\mathbf{1}_{[a,b]}$ of the interval $[a,b] \subset \mathbb{R}$.

DEFINITION 2.19. *We define the* quantized differential *of f to be the operator*

$$đf := [H,\, f] = H\,f - f\,H, \tag{2.137}$$

where H is the Hilbert transform $H = 2\,\Pi_{[0,\infty]} - 1$ given by

$$(H\,h)(s) := \frac{1}{i\pi}\int \frac{h(t)}{s-t} dt. \tag{2.138}$$

Thus, the quantized differential of f is given by the kernel

$$k(s,t) = \frac{i}{\pi}\,\frac{f(s)-f(t)}{s-t}. \tag{2.139}$$

We follow [**281**] and [**37**], and use the classical formula expressing the Fourier transform as a composition of the inversion

$$I(f)(s) := f(s^{-1}) \tag{2.140}$$

with a multiplicative convolution operator. We use the unitary identification

$$w : L^2(\mathbb{R}, ds)^{\text{even}} \to L^2(\mathbb{R}_+^*, d^*\lambda),\ \ w(\eta)(\lambda) := \lambda^{1/2}\,\eta(\lambda), \forall \lambda \in \mathbb{R}_+^* \tag{2.141}$$

whose inverse is given by

$$w^{-1} : L^2(\mathbb{R}_+^*, d^*\lambda) \to L^2(\mathbb{R}, ds)^{\text{even}},\ \ w^{-1}(\xi)(x) := |x|^{-1/2}\,\xi(|x|)\,. \tag{2.142}$$

Also we define the duality $\langle \mathbb{R}_+^*, \mathbb{R}\rangle$ by the bicharacter

$$\mu(v,s) = v^{-is}\,, \quad \forall v \in \mathbb{R}_+^*, s \in \mathbb{R}\,, \tag{2.143}$$

so that the Fourier transform $\mathbf{F}_\mu : L^2(\mathbb{R}_+^*) \to L^2(\mathbb{R})$ associated to the bicharacter μ is

$$\mathbf{F}_\mu(f)(s) := \int_0^\infty f(v) v^{-is} d^*v\,. \tag{2.144}$$

LEMMA 2.20. *On $L^2(\mathbb{R})^{\text{even}}$ one has*

$$\mathbf{F}_{e_\mathbb{R}} = w^{-1} \circ I \circ \mathbf{F}_\mu^{-1} \circ u \circ \mathbf{F}_\mu \circ w, \tag{2.145}$$

where u is the multiplication operator by the function

$$u(s) := e^{2\,i\,\theta(s)}, \tag{2.146}$$

where $\theta(s)$ is the Riemann-Siegel angular function of (2.24).

PROOF. First $\mathbf{F}_{e_\mathbb{R}}$ preserves globally $L^2(\mathbb{R})^{\text{even}}$. One has, for $\xi \in L^2(\mathbb{R}_+^*)$,

$$\begin{aligned}(w \circ \mathbf{F}_{e_\mathbb{R}} \circ w^{-1})(\xi)(v) &= v^{1/2} \int_\mathbb{R} |x|^{-1/2}\,\xi(|x|)\,e^{-2\pi i x v}\,dx \\ &= v^{1/2} \int_{\mathbb{R}_+^*} u^{1/2}\,\xi(u)\,(e^{2\pi i u v} + e^{-2\pi i u v})\,d^*u\,.\end{aligned}$$

This gives

$$\begin{aligned}(I \circ w \circ \mathbf{F}_{e_\mathbb{R}} \circ w^{-1})(\xi)(\lambda) &= (w \circ \mathbf{F}_{e_\mathbb{R}} \circ w^{-1})(\xi)(\lambda^{-1}) \\ &= \lambda^{-1/2} \int_{\mathbb{R}_+^*} (e^{2i\pi u\lambda^{-1}} + e^{-2i\pi u \lambda^{-1}})\,u^{1/2}\xi(u) d^*u\end{aligned}$$

We thus obtain

$$I \circ w \circ \mathbf{F}_{e_\mathbb{R}} \circ w^{-1} = C \tag{2.147}$$

where the operator C on the Hilbert space $L^2(\mathbb{R}_+^*, d^*\lambda)$ is given by convolution by

$$v \mapsto 2v^{-1/2}\cos(2\pi v^{-1}). \tag{2.148}$$

By construction C is unitary and commutes with the regular representation of $\mathbb{R}_+^*$. Thus $C = \mathbf{F}_\mu^{-1} \circ u \circ \mathbf{F}_\mu$ where u is the Fourier transform

$$u(s) = \int_0^\infty 2v^{-1/2}\cos(2\pi v^{-1}) v^{-is} d^*v. \tag{2.149}$$

This is defined in the sense of distributions. To compute (2.149) one can let $v = e^{-t}$ and use

$$\int_{\mathbb{R}} e^{t/2} e^{-ze^t} e^{ist}\, dt = \; z^{-(1/2+is)}\Gamma(1/2+is)\,, \quad \forall z\,,\; \Im z > 0\,. \tag{2.150}$$

This gives

$$u(s) = \; 2\cos((1/2+is)\pi/2)(2\pi)^{-(1/2+is)}\Gamma(1/2+is) \tag{2.151}$$

and the duplication formula

$$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1+z}{2}\right) = \pi^{1/2}2^{1-z}\Gamma(z) \tag{2.152}$$

shows that $u(s)$ is given by

$$u(s) = \; \frac{\pi^{-z/2}\Gamma(z/2)}{\pi^{-(1-z)/2}\Gamma((1-z)/2)}\,,\; z = 1/2+is\,, \tag{2.153}$$

which, using (2.24) gives (2.146). □

The following lemma relates the quantized calculus to the analysis of the geometry of the three projections P_Λ, $\widehat{P}_\Lambda$ and N_E.

Lemma 2.21. *For any Λ there is a unitary operator*

$$W = W_\Lambda : L^2(\mathbb{R})^{\rm even} \to L^2(\mathbb{R}), \tag{2.154}$$

such that, for any functions $h_j \in \mathcal{S}(\mathbb{R}_+^*)$, $j = 1, 2$, one has

$$\begin{aligned} & W\vartheta_{\rm a}(\tilde h_1)\widehat{P}_\Lambda P_\Lambda \vartheta_{\rm a}(\tilde h_2)W^* = \\ & \qquad \hat h_1\left(\tfrac{1}{2}u^{-1}{\rm đ}u\Pi_{[-\infty,\frac{2\log\Lambda}{2\pi}]} + \Pi_{[0,\frac{2\log\Lambda}{2\pi}]}\right)\hat h_2 \end{aligned} \tag{2.155}$$

Here $\tilde h_j(\lambda) = \lambda^{-1/2}h_j(\lambda)$, the operator ${\rm đ}u$ is the quantized differential of the function u of (2.146) *and $\hat h_j$ is the multiplication operator by the Fourier transform $\mathbf{F}_\mu(h_j)$.*

Proof. We let $\vartheta_{\rm m}$ be the regular representation of $\mathbb{R}_+^*$ on $L^2(\mathbb{R}_+^*)$

$$(\vartheta_{\rm m}(\lambda)\,\xi)(v) := \; \xi(\lambda^{-1}\,v)\,, \quad \forall \xi \in L^2(\mathbb{R}_+^*)\,. \tag{2.156}$$

One has

$$w\,\vartheta_{\rm a}(\tilde h_j)w^{-1} = \vartheta_{\rm m}(h_j), \quad \text{and} \quad \mathbf{F}_\mu\vartheta_{\rm m}(h_j)\mathbf{F}_\mu^{-1} = \widehat{h}_j\,. \tag{2.157}$$

Now we have $P_\Lambda = \mathbf{1}_{[-\Lambda,\Lambda]}$, so that $wP_\Lambda w^{-1} = \mathbf{1}_{[0,\Lambda]}$, and we obtain

$$\begin{aligned} \widehat{P}_\Lambda P_\Lambda = \mathbf{F}_{e_{\mathbb{R}}} P_\Lambda \mathbf{F}_{e_{\mathbb{R}}}^{-1} P_\Lambda &= w^{-1} I \mathbf{F}_\mu^{-1} u \mathbf{F}_\mu \mathbf{1}_{[0,\Lambda]} \mathbf{F}_\mu^{-1} u^{-1} \mathbf{F}_\mu I \mathbf{1}_{[0,\Lambda]} w \\ &= w^{-1} \mathbf{F}_\mu^{-1} u^{-1} \mathbf{F}_\mu \mathbf{1}_{[\Lambda^{-1},\infty]} \mathbf{F}_\mu^{-1} u \mathbf{F}_\mu \mathbf{1}_{[0,\Lambda]} w, \end{aligned} \tag{2.158}$$

where we used the identity $I\mathbf{F}_\mu^{-1}u\mathbf{F}_\mu = \mathbf{F}_\mu^{-1}u^{-1}\mathbf{F}_\mu I$, which follows from the symmetry $\theta(-s) = -\theta(s)$. We also used the identity $I\mathbf{1}_{[0,\Lambda]}I = \mathbf{1}_{[\Lambda^{-1},\infty]}$.
We now set

$$W_\Lambda := \mathbf{F}_\mu\,\vartheta_{\rm m}(\Lambda)\,w, \tag{2.159}$$

with $\vartheta_{\rm m}(\Lambda)$ as in (2.156). One has

$$W_\Lambda\,\vartheta_{\rm a}(\tilde h_1)\widehat{P}_\Lambda P_\Lambda \vartheta_{\rm a}(\tilde h_2)W_\Lambda^{-1} = \tag{2.160}$$

$$\mathbf{F}_\mu \vartheta_{\rm m}(\Lambda)\vartheta_{\rm m}(h_1)\,\mathbf{F}_\mu^{-1}u^{-1}\mathbf{F}_\mu \mathbf{1}_{[\Lambda^{-1},\infty]}\mathbf{F}_\mu^{-1}u\mathbf{F}_\mu \mathbf{1}_{[0,\Lambda]}\vartheta_{\rm m}(h_2)\vartheta_{\rm m}(\Lambda)^{-1}\mathbf{F}_\mu^{-1}$$
$$= \widehat{h}_1 u^{-1}\mathbf{F}_\mu \vartheta_{\rm m}(\Lambda)\mathbf{1}_{[\Lambda^{-1},\infty]}\vartheta_{\rm m}(\Lambda)^{-1}\mathbf{F}_\mu^{-1}u\mathbf{F}_\mu \vartheta_{\rm m}(\Lambda)\mathbf{1}_{[0,\Lambda]}\vartheta_{\rm m}(\Lambda)^{-1}\mathbf{F}_\mu^{-1}\widehat{h}_2\,.$$

Here we use the identities (2.157) and the fact that $\vartheta_{\rm m}(\Lambda)$ commutes with multiplicative convolution operators such as $\mathbf{F}_\mu^{-1}u^{-1}\mathbf{F}_\mu$. Next we see that

$$\vartheta_{\rm m}(\Lambda)\mathbf{1}_{[\Lambda^{-1},\infty]}\vartheta_{\rm m}(\Lambda)^{-1} = \mathbf{1}_{[1,\infty]}\,,\ \vartheta_{\rm m}(\Lambda)\mathbf{1}_{[0,\Lambda]}\vartheta_{\rm m}(\Lambda)^{-1} = \mathbf{1}_{[0,\Lambda^2]}\,. \tag{2.161}$$

In order to use the quantized calculus on functions on $\mathbb{R}$ as in Definition 2.19, we use the isomorphism of abelian groups

$$t \in \mathbb{R} \mapsto e^{2\pi t} \in \mathbb{R}_+^* \tag{2.162}$$

and note that $\mathbf{F}_\mu$ and $\mathbf{F}_{e_\mathbb{R}}$ are conjugate by this isomorphism since the bicharacter $\mu(v,s) = v^{-is}$ of (2.143) fulfills

$$\mu(e^{2\pi t}, s) = e^{-2\pi i st} = e_\mathbb{R}(st)\,. \tag{2.163}$$

Thus we get from (2.136),

$$\mathbf{F}_\mu \mathbf{1}_{[a,b]}\mathbf{F}_\mu^{-1} = \Pi_{[\frac{\log a}{2\pi},\frac{\log b}{2\pi}]}\,, \tag{2.164}$$

and we obtain, using (2.160)

$$W_\Lambda\, \vartheta_{\rm a}(\tilde{h}_1)\widehat{P}_\Lambda P_\Lambda \vartheta_{\rm a}(\tilde{h}_2)W_\Lambda^{-1} = \widehat{h}_1 u^{-1}\Pi_{[0,\infty]}\, u\, \Pi_{[-\infty,\frac{2\log\Lambda}{2\pi}]}\widehat{h}_2\,. \tag{2.165}$$

We then use

$$\frac{1}{2}\,đu\, = [\Pi_{[0,\infty]}, u]\,, \tag{2.166}$$

which completes the proof of (2.155). □

5.2. Proof of Theorem 2.18.

As a first application of Lemma 2.21 we now complete the proof of Theorem 2.18. One has $N_E = \vartheta_{\rm m}(h_E)$, where $\widehat{h}_E = \mathbf{1}_{[-E,E]}$. Thus, $N_E\widehat{P}_\Lambda P_\Lambda$ is unitarily equivalent to

$$\mathbf{1}_{[-E,E]}\left(\frac{1}{2}u^{-1}đu\,\Pi_{[-\infty,\frac{2\log\Lambda}{2\pi}]} + \Pi_{[0,\frac{2\log\Lambda}{2\pi}]}\right). \tag{2.167}$$

The trace of $\mathbf{1}_{[-E,E]}\Pi_{[0,\frac{2\log\Lambda}{2\pi}]}$ is equal to $2E\frac{2\log\Lambda}{2\pi}$ and gives the leading term in the formula for $\mathrm{Tr}(N_E\widehat{P}_\Lambda P_\Lambda)$. If we replace $\Pi_{[-\infty,\frac{2\log\Lambda}{2\pi}]}$ by 1 the other term gives

$$\mathrm{Tr}\left(\mathbf{1}_{[-E,E]}\left(\frac{1}{2}u^{-1}đu\right)\right) = \int_{-E}^{E} k(s,s)ds, \tag{2.168}$$

where $k(s,t)$ is the kernel representing $\frac{1}{2}u^{-1}đu$. Its diagonal values are

$$k(s,s) = -\frac{1}{\pi}\frac{d\theta}{ds}, \tag{2.169}$$

where we use (2.139) and (2.146). Thus, the integral gives

$$\mathrm{Tr}\left(\mathbf{1}_{[-E,E]}\left(\frac{1}{2}u^{-1}đu\right)\right) = -\frac{2}{\pi}\theta(E) = -2(\langle N(E)\rangle - 1). \tag{2.170}$$

The remainder in the formula

$$\operatorname{Tr}(N_E \widehat{P}_\Lambda P_\Lambda) = 2E\frac{2\log\Lambda}{2\pi} - 2(\langle N(E)\rangle - 1) + r(E,\Lambda) \tag{2.171}$$

is therefore given by

$$r(E,\Lambda) = \frac{1}{2}\operatorname{Tr}(\mathbf{1}_{[-E,E]}u^{-1}\eth u\,\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}). \tag{2.172}$$

For $\Lambda\to\infty$ one has $\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}\to 0$ strongly. However, one needs to be a bit careful about the operator $T = \mathbf{1}_{[-E,E]}u^{-1}\eth u$, since it is unclear that it is of trace-class. One can check that the operator TT^* is a Hilbert-Schmidt operator. Also $\mathbf{1}_{[-E,E]}u^{-1}\eth u\, f$ is of trace-class, for any compactly supported function f, since θ is smooth. With f smooth, compactly supported and identically equal to 1 on $[-E,E]$ one has

$$\operatorname{Tr}(\mathbf{1}_{[-E,E]}u^{-1}\eth u\,\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}) = \operatorname{Tr}(\mathbf{1}_{[-E,E]}u^{-1}\eth u\,\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}\,f). \tag{2.173}$$

In fact, $\mathbf{1}_{[-E,E]}u^{-1}\eth u\,\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}$ is of trace-class by Lemma 2.21.
Moreover, the commutator $[\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}, f]$ is the conjugate of $\eth f$ by the function $s\mapsto\Lambda^{is}$. Thus, it is of trace-class and converges weakly to 0 as a family of Hilbert-Schmidt operators for $\Lambda\to\infty$. Since $\mathbf{1}_{[-E,E]}u^{-1}\eth u$ is Hilbert-Schmidt, we see that we can permute f with $\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}$ without affecting the limit. Since $\Pi_{[\frac{2\log\Lambda}{2\pi},\infty]}\to 0$ strongly, we obtain

$$r(E,\Lambda)\to 0 \quad \text{for} \quad \Lambda\to\infty. \tag{2.174}$$

This completes the proof of Theorem 2.18, making use of (2.66) to control the difference between Q_Λ and $\widehat{P}_\Lambda P_\Lambda$ in the statement.

REMARK 2.22. One can use Lemma 2.21 to estimate the angle of the projections Q_Λ and N_E. Indeed, we obtain

$$[N_E, \widehat{P}_\Lambda P_\Lambda] \sim [\mathbf{1}_{[-E,E]}, \frac{1}{2}u^{-1}\eth u\,\Pi_{[-\infty,\frac{2\log\Lambda}{2\pi}]}] + [\mathbf{1}_{[-E,E]}, \Pi_{[0,\frac{2\log\Lambda}{2\pi}]}]. \tag{2.175}$$

The second commutator on the right hand side of (2.175) is of the order of $\log(E) + \log(\log(\Lambda))$ by the analysis of §3.2. The limit for $\Lambda\to\infty$ of the first commutator on the right hand side of (2.175) has Hilbert-Schmidt norm of the order of $\sqrt{\log(E)}$, as one gets from the estimate

$$\int_{|s|>E}\int_{t\in[-E,E]}\left|\frac{e^{2i\theta(s)} - e^{2i\theta(t)}}{s-t}\right|^2 dt\,ds = O(\log(E)). \tag{2.176}$$

6. The map $\mathfrak{E}$

Notice that the first term in (2.132) of Theorem 2.18 above, of the form $\frac{4E}{2\pi}\log\Lambda$, is in fact the symplectic volume $v(W(E,\Lambda))$ of the box

$$W(E,\Lambda) = \{(\lambda,s)\in\mathbb{R}_+^*\times\mathbb{R} \,:\, |\log\lambda|\le\log\Lambda\,,\ \text{and}\ |s|\le E\} \tag{2.177}$$

as in (2.46). The symplectic volume is computed in the symplectic space given by the product of the group $\mathbb{R}_+^*\sim\mathbb{R}$ by its dual $\mathbb{R}$ under the pairing $(\lambda,s)\mapsto\lambda^{is}$, with the symplectic form given by the product of the Haar measure by the dual one.