Ups and downs (Denise Vella-Chemla, 12.5.2018)

We try to demonstrate Goldbach's conjecture. Let us define 4 variables :

$$X_a(n) = \#\{p+q=n \textit{ such that } p \textit{ and } q \textit{ odd},\ 3 \leqslant p \leqslant n/2,\ p \textit{ and } q \textit{ prime}\}$$

$$X_b(n) = \#\{p+q=n \textit{ such that } p \textit{ and } q \textit{ odd},\ 3 \leqslant p \leqslant n/2,\ p \textit{ compound and } q \textit{ prime}\}$$

$$X_c(n) = \#\{p+q=n \textit{ such that } p \textit{ and } q \textit{ odd},\ 3 \leqslant p \leqslant n/2,\ p \textit{ prime and } q \textit{ compound}\}$$

$$X_d(n) = \#\{p+q=n \textit{ such that } p \textit{ and } q \textit{ odd},\ 3 \leqslant p \leqslant n/2,\ p \textit{ and } q \textit{ compound}\}$$

In the following, let us note $E(x)$ the integer part of x (i.e. $\lfloor x \rfloor$).

One will assume that $X_a(2n) = 0$ while $X_a(n)$ is not zero and try to end up at a contradiction.

One tries a first case, helping visually oneself with small drawings ; different parts are numbered to "trace" them while passing from n's drawing to $2n$'s drawing.

Some elements must be specified :
- gray color is used to code compound numbers, white color is used to code prime numbers ;
- rectangles down the drawing represent small sommants while rectangles up the drawing represent big sommants in decompositions ;
- decompositions are "melted" : we exchange columns containing each an integer x in the line down the drawing and its complementary $n-x$ in the line up the drawing in such a way that every same type columns (i.e. containing decompositions of the form $prime + prime, prime + compound, compound + prime$ and $compound + compound$) are juxtaposed to allow the counting by $X_a(n), X_b(n), X_c(n), X_d(n)$ variables by contiguity.
- it's important to have in mind that while passing from n to $2n$, every number that intervenes in n's decompositions (between 3 and $n-3$), that is in one of the two lines of numbers represented in figure 1, must be in the line down in the drawing of figure 2 (they are the small sommants of $2n$ that are between 3 and n *).

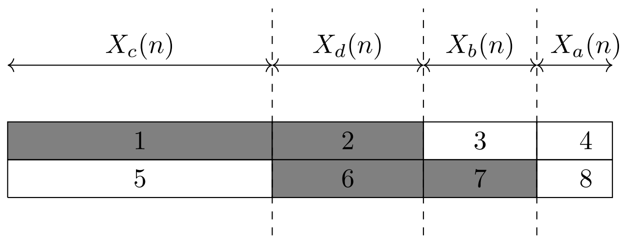

FIGURE 1 : n's decompositions

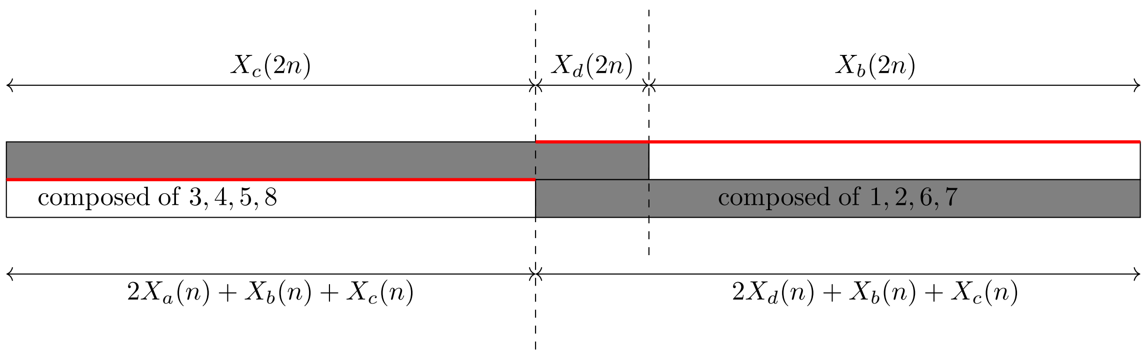

FIGURE 2 : $2n$'s decompositions

On figure 2 above, we overline in red color the segments whose length is used to equal the total size of the figure that is known.

∗. We should note $n-3$ in place of n but we forget limit cases here.

For figure 2, since we assumed as hypothesis that $X_a(2n) = 0$, we have :

$$X_b(2n) + X_d(2n) + 2X_a(n) + X_b(n) + X_c(n) = \frac{2n}{4}$$

Explanation concerning the origin of the equality above : $X_b(2n)+X_d(2n)$ is the length of the red segment at the up right side of the figure while $2X_a(n) + X_b(n) + X_c(n)$ is the length of the red segment at the down left side of the figure.

In figure 1, we have :

$$X_a(n) + X_b(n) + X_c(n) + X_d(n) = \frac{n}{4}$$

From those 2 equalities, we can deduce substracting the second one to the first :

$$X_b(2n) + X_d(2n) + X_a(n) - X_d(n) = \frac{n}{4}$$

1) Let us suppose $X_d(n) > X_a(n)$; then $X_a(n) - X_d(n)$ in negative from one side and $2X_d(n) > X_d(n) + X_a(n)$ from the other side.

But this leads to a contradiction since we obtain † :

$$X_d(n) + X_a(n) + X_b(n) + X_c(n) + X_a(n) - X_d(n) = \frac{n}{4}$$

while $X_a(n) + X_b(n) + X_c(n) + X_d(n) = \frac{n}{4}$ is figure 1's size that shouldn't be equal to $\frac{n}{4}$ if we add a negative number to it $(X_a(n) - X_d(n))$.

2) Let us suppose now that $X_d(n) \leqslant X_a(n)$.

Contradiction comes from the inequality :

$$X_d(n) - X_a(n) = E(n/4) - \pi(n) + \delta(n) \qquad (1)$$

$X_d(n) - X_a(n)$ is strictly positive for $n \geqslant 122$. $\delta(n)$ is a negligible variable that equals 0, 1 or 2.

For $n = 122$, $X_d(n) = X_a(n)$.

Above this number, since $E(n/4)$ regularly grows each 4 integers, $\pi(n)$ is only growing at each prime, and so the difference $X_d(n) - X_a(n)$ is always growing more and more.

We proved by recurrence in https://hal.archives-ouvertes.fr/hal-01109052 properties from which $X_d(n) - X_a(n) = E(n/4) - \pi(n) + \delta(n)$ can be obtained.

Alain Connes gave a very simple justification for $X_d(n) - X_a(n) = E(n/4) - \pi(n) + \delta(n)$ that we copy below :

[Indeed, for n fixed, let us call J the set of odd numbers between 1 and n/2 and let us consider the two subsets of J : $P = \{j \in J \mid j \textit{ prime}\}, Q = \{j \in J \mid n - j \textit{ prime}\}$.

Then I claim that (1) comes from the very general fact on any subset and intersection and union cardinalities :

$$\#(P \cup Q) + \#(P \cap Q) = \#(P) + \#(Q) \qquad (2)$$

Here (neglecting limit cases that contribute to $\delta(n)$), we see that
(a) $\#(P \cap Q)$ corresponds to $X_a(n)$.
(b) $\#(P \cup Q)$ corresponds to $E(n/4) - X_d(n)$.
(c) $\#(P) + \#(Q)$ corresponds to $\pi(n)$.

So we have a very simple proof of (1) as a consequence of (2).]

†. To understand how we obtain this equality, look at the two expressions associated to the red segment at the up right side of the figure 2, one at the top and one at the bottom of the figure.