Transcript of a conference given by Alain Connes "Duality between shapes and spectra", given at the Collège de France, on October 13, 2011
The conference can be viewed here :
http://www.college-de-france.fr/site/colloque-2011/symposium-2011-10-13-10h15.htm

So then, my talk will focus on the duality that exists between the shapes and their spectrum, that is, if you like, the range that is associated with a shape. That, of course, partially answers the question of the relationship between shapes and time, since the vibrations of a shape take place over time.

And I will start my presentation by explaining why it's absolutely fundamental to ask the following question "how can we define invariants of a shape ?". Imagine you have a shape in the very naive sense of the term ? You can talk about its diameter, you can talk about its size and volume, things like that, but of course, to get the fully shape, you need invariants much more subtle than that.

And among these invariants, there are precisely the frequencies, the possible range produced by a shape, this is what we're going to talk about. And then, we not only want to know how to characterize a shape, but we also want to know how to characterize the position of a point in relation to this shape. And what we will see is that basically, if you want, a point, it is characterized by a notes chord, in this range. So to present things to you in a somewhat naive way. So we will talk about vibrations, shapes, I will make you hear simple shapes and then, by reference to a famous article by Mark Kac in the 60s, who asked "Can we hear the shape of a drum ?", we will speak of an additional invariant which makes it possible to complete the table, that is to say which allows, if known, to know the shape. And finally, I will finish, I really want this little addition, because in preparing my talk, I realized that I had tried to play *Standing in the moonlight* over the range that is produced by the simplest shapes like a sphere or things like that. And I realized that it gave a result which was not good at all. And I realized that in fact, the range, the real musical scale, the one that is sensitive to the ear, well, there is no simple shape to which it corresponds, i.e. a shape whose frequencies correspond to the musical range as we know it. And I had fun looking for an object and there is a really interesting object that seems to answer the question and which I will talk at the end, which is the quantum sphere.

So that's the program.

So what ? So, to start, we will try to think in an intrinsic way to the concept of shape or to the concept of position in relation to a shape, asking a question which is a very simple question and which is "where are we ?".

See, to this question, you can answer “we are at the Collège de France, in the Marguerite de Navarre amphi”. But if you want to pass this information to another civilization will be inaudible.

How can we transmit where we are in an intrinsic manner ? So, of course, the men tried and sent the Pioneer probe in the space. And on this probe, they gave a certain amount of information. What are those informations ?

Well, of course they showed what they looked like. This is the drawing that is there.

They also gave a small overview of the solar system. We see, below, the Sun, we see a first planet Mercury. We see a second planet, Venus. We see the third from which the probe left. That’s why they put the little one drawing, and so on.

But it is obvious that for the moment, you have information which is almost zero because there will be an infinity of planetary systems which will have little almost the same pace. And so, in fact, you will have absolutely no idea where you are.

So, in fact, there is, in the drawing that they sent, something which is much more interesting, much more cryptic and much more informative, and that is what which is in the middle, on the left, you see.

And what is it ?

These are the directions from the Sun relative to 14 pulsars and to the center of the galaxy. And in addition, they indicated for each of these directions the corresponding frequency laying. So we will see that this is very, very, very close to the answer we are going to get from an abstract mathematical reflection on the abstract problem. And the abstract problem, it can be formulated as follows, it can be formulated in the form of two questions, excuse me, from time to time, I will put slides in English because I know there is a simultaneous translation and I take advantage of it, So, to put some slides in English, the deal is... so, the first question is “can we find complete invariants for geometric spaces or, if you like, shapes, more generally ?”

And second, “can we invariantly specify where is a point with respect to a shape ?”. So the essential thing, we can see it in the example that I gave you, when we wanted to give our position relative to the universe, is that if you see someone very learned, you would say “But to give your position in the universe, you just have to give your coordinates in relation to a reference system.”. Yes, but where is the origin of the reference system ?

You have to say where it is. And to do that, you have exactly the same problem as in the original problem. And so on. So you see, it is not at all simple. It is not at all obvious. We could you say “Well, I know general relativity. I know a point is specified by his contact details, all that...”. But these answers are null and void compared on the invariant and intrinsic side of the problem.

So the important thing, therefore, the important thing is that in fact, in a shape, therefore, there is a whole series of invariants already. And these invariant is, if you want, the range of the shape, then this is where we will see if the sound works, I hope that it will work. So I'm going to make a try. This morning, when I woke up, my computer had rebooted and therefore nothing was working. And like the program takes a very long time to get started, I was really scared. We'll see if it works. It works, so we hear the sound, so I'll start by the most elementary shape, the most elementary shape that exists, it is the interval.

If you want, it's a string that will vibrate, like a violin string, and it will vibrate. It will have a fundamental sound. And then it's going to have the multiples of this sound, like vibrations. The corresponding range will be extremely simple.

And we're going to have fun playing a little with this range. Okay, so if I do that. *(He clicks the numbered buttons 1 to 20 and means associated sounds.)* It sounds weird on the 7th. Well, I pretend that if you try on that range of play *Standing in the moonlight*, the first note must be 131.

So you see that it looks... Naively, we say to ourselves “But that's the range ! Of course ! Since these are the multiples of an integer...”. No that's not true. It is not true at all. Big mistake. First naive mistake we would make. So that, it is for the simplest object. A slightly more complicated object, but even with this object, if you will. It has an extremely simple spectrum. When we want visualize frequencies, they can be represented in their visual shape, that is to say in their shape from the spectrum. And then, we can also represent them in the shape of a graph. The graph is interesting because we will see the multiplicity of a proper value in the graph. So now let's move on to a shape that is already more evoluated, which is the disc.

So what does disc mean ? The sounds produced by the disc, that means you take a round drum, you tap on that drum. There is going to be its fundamental sound. There is going to be exactly as in the case of the vibrating string. There are going to be harmonics, there are going to be other sounds. So the drum will produce a whole series of sounds that will no longer be as simple as the integers whose I spoke earlier, and that will give you a range.

And this range will still be extremely informative on the drum, that is, the lowest note will give you the diameter, will give you immediately the diameter measurement. And then the behavior, for example, of much larger notes, will give you the size of the drum, etc., etc.

So I will give you a bibliography at the end. I mean, for all mathematicians who have been involved in this kind of thing, but I'm not going at all to tell you "this is due to x or this is due to y". I will give you the bibliography at the end, but let's listen to the drum a little.

(Clicks do not produce any sound.) So there, I did not put sounds precisely, so I didn't take long because there are sounds that are very sharp, let's look how it's vibrating right now.

Okay, you see, I hope. You see how it vibrates : whenever you have a picture like that. The drawing is not at all as simple as it would appear, because the functions that are involved are what are called Bessel functions. And if you want, precisely, when you type more or less, to a sufficient place on the drum, etc. We're going to make it vibrate. According to one of his harmonic frequencies. We will listen to them, let's listen to them. So we can calculate them. These are numbers that are not at all trivial. They are not at all numbers like integers. These are zeros of a function which is quite complicated that we call the Bessel function, which are parameterized by two integers and that can be calculated. We can calculate them with as many decimal places as we want. But these are not simple numbers and that's the range of the disc. So the disc has a range like that.

I show you the first notes. It has a spectrum that is like that and now, we will hear it. *(AC varies the values of the two sliders, and we hear more or less acute notes, and we see at the same time, the circle colored in blue gradations, having more or less numerous radial and angular colored divisions.).*

So, I hope it's a note not too high, because I didn't want to make you hear too sharp notes..., I was nice, I did not put notes too acute. It goes on of course, as much as we want, etc.

And so we get, you want, so, a spectrum which is the spectrum of the disk that looks like this, so it continues indefinitely, it continues indefinitely.

And you can see, of course, that it has nothing to do with the spectrum we had just now for the interval. So now let's go a little further.

Take an object always of dimension 2, let us take an object which is a square now. It's like taking a piece of skin, stretching it out in between if you want a frame, like that, a square drum, and you tap on it.

And now the vibrations you get have the following look. It will do noise twice before giving it the right amount. We're going to climb a little higher. *(Square sounds).* Okay, so we see a spectrum again, the spectrum looks like this.

It is very, very different from what was happening in the case of the disc, because if you want for the case of the square, it is not very difficult to do the calculation. We realize that the corresponding frequencies are the square roots of the sums of two squares, so the numbers are of squareroot form of n^2 plus m^2. So this is something which is very simple to understand, which is much simpler to understand than the numbers that came up for the record. They are very different, but they have, if you want the same sort of infinite distribution. We can change the color if you want. But now, let's come to the sphere, so, all of this, these are 2-dimensional shapes and a priori, they are very banal shapes.

When I speak of the disc, when I speak of the square or when I speak of the sphere, I mean the title of the conference, it's *La vie des formes.* So you have to make them live.

And to make them live, you have to make them vibrate. And from the moment we make them vibrate, we realize that, although they are shapes that look extremely simple, extremely common, extremely elementary, when vibrated, the vibrations themselves decorate these shapes in an extremely harmonious and extremely non-trivial manner. So if we take the 2-sphere, if we take the round sphere, its spectrum, this time, is very, very simple. It is also formed by integers, exactly as in the case of a string.

But these integers appear this time with a certain multiplicity, that is to say that it's not exactly integers. It is more exactly the root of $J(J+1)$. It's practically an integer, so it looks a lot like what was going on in the case of the circle or the interval. But they appear with a multiplicity.

So now, if we take the sphere, then there, I am afraid that it is too acute.

See what happens, what happens is that if I take for example the Spin $= 6$, there is a certain number of frequencies, how to say, of what are called eigenfunctions that exist, but that have exactly the same, the same frequency.

How to say ? The shapes on the sphere are different, the sound we hear is the same. And that is what is called spectral multiplicity, that is to say that in the spectrum, what will happen is that we will have the same value, but it will happen multiple times. So this is what happens to the sphere... I will come back to this for the musical shape, that, we will see that later.

So now I'm going to go to the normal course starting from the pdf. So I go and do that.

So we have these two questions, we have these two questions, to define a full invariant of a shape.

So in fact, we have known since a famous article by John Milnor in the 60s, that the spectrum of a shape is not sufficient to characterize this shape.

It's a wonderful math article. It is one of the rare articles of math that has only one page. And what Milnor did was something remarkable. He used a Witt result to see that there are toroids, these toroids are of fairly large dimensions, they are not toroids of low dimensions. But there are toroids, which are geometrically different, but which have exactly the same range. And that comes from a result of number theory. Because, of course, the range associated with a shape with all its subtlety, as we have just seen in the examples that I have shown you, this range has of course a very deep relationship with arithmetic, arithmetic in the most naive sense, the arithmetic of the first range of all these integers. But basically, each shape has an arithmetic associated with it, and that is the arithmetic of the range it gives us naturally through the sounds it produces.

So, what is very, very interesting is that, as I will explain, the invariant which was missing compared to the spectral invariant, it is an invariant which one will see that is connected, in fact, to what physicists call the Cabibbo-Kobayashi-Maskawa matrix. And it's an invariant that actually measures an angle between two algebras and which generalizes a little what physicists did when they looked at what's going on with quarks. And that I will talk about.

But don't worry about the technical side of this page at all. So, we have considering the vibrations of the disc.

We have seen examples of disc vibrations, we have seen the natural frequencies of the disc which, as I told you, is not as simple as it seems. We saw its spectrum with the start of the spectrum. We saw, now, I put more natural frequencies for the disc and you see that it starts to look like... This curve there, it starts to look like what...

it starts to look like a parabola. And the more frequencies we add, the more we go at high frequencies, the more it will resemble a parabola. See if you take the disc where the eigenvalues are difficult to calculate, the range is difficult to calculate, but if you look at the range, but now from a great distance, i.e. you look at the highs frequencies and you put all these frequencies together, you see that that it seems more and more to a parabola and there is a famous theorem of Hermann Weyl which dates from the 30s and which says that this parabola has an invariant, if you will, which is how it is angulated or not, this invariant gives exactly in the case of surfaces, in the case of 2-dimensional shapes, gives exactly the surface of the shape, okay. So you see, we can measure the area of the shape of dimension 2 simply by looking at this parabola. This is what Hermann Weyl demonstrated.

So we also saw what is happening for the square. I had shown you some vibrations of the square earlier. The spectrum of the square, as I said, is extremely simple. These are the numbers which are the square roots of n^2 plus m^2. That's the vibrations of the square.

Of course, I don't want to bother you with mathematical formulas, but that, it comes from the Helmholtz equation, we look at the Laplacian, and we look at the equation of waves.

So the spectrum of the square, we look at the natural frequencies of the square, see, they look a little bit, from afar, like what was happening just now for the disc. We look at high frequencies, we look at high frequencies, it oscillates a little. And then, now we're going to look at very high frequencies, very high frequencies, you see, it's incredible, we really see a parabola, it stands out with the naked eye, absolutely a parabola. This parabola, precisely, its invariant, is the area of the square.

So, we look at the sphere too. The spectrum of the sphere, as I said, is practically the integers, these are the numbers of the square root form of $J(J+1)$. Here if you look at the natural frequencies of the sphere, that gives you that, the frequencies proper quences of the sphere ? *(We see a parabola, but it has like stairs.)* Why it looks like that, by stages ? Well, it's because you have the same frequency that will repeat itself a bunch of times. Okay, so, it's something that's tiered like that.

So you say to yourself "But that doesn't sound like a parabola at all."

It doesn't sound like a parabola, but it's because you don't watch enough high frequencies. And if you now look at much higher frequencies, you see that there are still small floors, of course.

Okay, but it looks more and more like a parabola and it will give you the area of the sphere. Okay.

Well. So, what does this have to do with the problem we had at the start, so that was the problem to say where we are precisely ?

If we mean where we are. Two things must be said :

We must say in which universe we are and how far from this universe we are. To say in which universe we are in fact, what I pretend, is that what you have to give are precisely the vibration frequencies of this universe, the first thing to give. So how do we do it ? We did it if you will, there is a very interesting thing that happens, and goes back to Mark Kac, and to “Can we hear the shape of a drum ?, etc.”, we are concerned with the equation of the waves and we are concerned with something called an operator that mathematicians call the Laplacian, which is called the Laplacian, which is called Δ but when you write the wave equation, if you want, in fact, we write this equation in the Δ form of a function plus k square times f equals 0. *(AC written on the blackboard* $\Delta f + k^2 f = 0$. *)*

This is the Helmholtz equation. And when we write this Helmholtz equation, we see that the number k that appears, it’s going to be, if you want, what we call eigenvalues of the opposite of Δ, but that is not k, because it is k^2 which is an eigenvalue of $-\Delta$, and therefore in fact, the number k which appeared, in all the examples that I have given, it is a number which is eigenvalue of the square root of $-\Delta$.

So this is called an elliptical differential operator and its square root, it’s not something very pretty.

And then fortunately, there is a physicist, who is Paul Dirac, who found a way to extract a square root from the opposite of the Laplacian in such a way that it is a differential operator in a aesthetical way. This is called the Dirac operator.

So what makes that in all the examples I gave you, in done, it’s much more natural and important to give for a geometrical shape is not to give the spectrum of the Laplacian but to give the spectrum of the Dirac operator, that wouldn’t make difference practically for all the examples I gave you. So this is very important. Well, this is the first thing, that is to say that what we are going to keep back, basically, it’s a square root of $-\Delta$, so it’s not going to change a lot. So we will, we will give all of its eigenvalues. We will give its range, if you want. And now, which is enough extraordinary is that there is a way to find a complementary invariant from this range.

And basically, this a complementary invariant, it's going to be a prescription, we're going to give the possible agreements on this range.

We are going to give a set of possible agreements, but the origin, the origin of this invariant : it comes from physics, and what is called in physics... flavor. If you want, in physics, there are quite complicated phenomena called weak interactions and in weak interactions, people realized that there was what are called currents which allowed to change “flavor”, i.e. to change family. That is to say that, for example, for quarks, you have the quarks that we know that are the up and down, that are the main quarks, that form the neutrons, protons, etc.

But you have other quarks, there are two other families of quarks. Well, there are interactions in physics that allow to change..., that allow to pass from one family of quarks to another family of quarks, this is called “flavor changing neutral current”, and physicists understood that what measured if you want, these currents which allow to change family, it was in fact an angle between two commutative algebras, but very simple in their case. It was first found by Cabibbo, then by Kobayashi and Maskawa. And this is called the Cabibbo-Kobayashi-Maskawa matrix. And what it does is that it measures, if you actually want, the angle between two algebras. And what is quite extraordinary, is that good, in fact, they are algebras in a space of dimension 3. So it is something very simple. And yet a complex number appears and that's it that made the violation of what is called CP in physics.

So now, what is quite fun is that if we generalize this idea, we get a solution to the problem just now, that is, we get another invariant which is not only the spectrum of the Dirac operator. So, the first invariant, if you will, is the spectrum of the Dirac operator.

This is very important *(AC writes Spec D in chalk on the board.)*. This is the range, if you want. Okay. Okay, but there is a second invariant. And what is this second invariant ? Well, this second invariant is also an angle.

It is not a number, it is an angle, it is an angle. It is a notion a lot more complicated. It is an angle between two algebras. Then there is the algebra of functions of the Dirac operator.

It means that you look at all the operators that are diagonal in the same basis as the Dirac operator, which is the basis of the eigenfunctions and another algebra, which is the algebra of functions on the space in which you are, on the shape in which you work.

And then there is a wonderful von Neumann's theorem that dates back to years, from very, very long and who says that the representation of this algebra in Hilbert space is independent of the shape you choose. That is, if you take any shape, be it a sphere, a disc, a shape of dimension higher, etc., well, the function algebra will always act the same way in Hilbert space.

So the only thing missing to complete the picture is going to be the relative position of these two algebras and the relative position of these two algebras, in fact, it is specified by a series of chords.

Okay, this is a continuous series of chords, which are formulated over the range and what happens is that... So, we have these two invariants and how should we interpret a point, so a point, in fact, if we look from this point-of-view, a point is given by correlations between different frequencies. So, you can think these correlations, these are complex numbers. But you can exactly think of these correlations between different frequencies like a chord, we take a chord between these notes and that's it, a point, it's that, okay. So think in your head, and keep a geometric object, a geometric shape is given by its music, by its scale, it is given by its range, and the set of points is given by the set of possible chords and a point is given by a chord.

Okay, so then if we continue, from this point of view, we realize *(we can dim the light, there, it's good).*

So we realize that it's quite astonishing to see how much this point of view which I have just spoken of is close to physical reality. Why ?

Because now, man has evolved, maybe by natural selection, sufficiently to be able to look at the universe. He has an eye. This eye is Hubble. His current eye is the Hubble telescope. With this telescope, man looks at the universe. I advise everyone to log into the NASA website every morning, NASA gives a new image every morning. It is done on a daily basis and each day, you can look at the universe and you will have a different picture of the universe.

And what's amazing, what's really amazing is that the information that comes to us from the universe, it is spectral. And not only did this information provides us information, by a spectrum, on the composition of very, very distant stars or intergalactic clouds, etc., simply by their spectrum, but in addition, it gives us signs on their origin. And how does it tell us about their origin ? Because by Doppler effect, the more things are distant, the more there is what is called redshift, then the redshift, naively, if you are very naive, you think the redshift, you will take the spectrum and you will shift it like that by a translation. But it is not true. Redshift is a multiplication. It's not an offset, it's a multiplication. That is to say that we take all the frequencies and we multiply them by the same number. And what's really amazing about redshift is that now we observe, we observe... So why do we know that this is the same thing we see ? Well, by the fact that the range is the same.

They look alike, of course. So you can see that on the left, you're going to have some provision in the range. It will end up on the right, but it will not be found in the same place and it will be offset, but not offset by a translation, offset by an homothety, that is to say that we multiply all numbers by something. And what is extraordinary is that it is thanks to this redshift that we can go back in time. We are now measuring redshifts which are on the order of 10, but in fact we expect redshifts on the order of 1000, etc., and they correspond, of course, to increasingly distant times.

So it's really amazing, it's really amazing that this view on shapes be as close in point of view as abstract mathematics suggests on the shapes.

And on the other hand, there is another thing that is extremely important : it is that, of course we can't get around at the moment, but probably for always, towards other galaxies. And so, it's an act of faith that we do, to know that these things exist somewhere. And this act of faith, it comes precisely because of the correlations that there are between the different frequencies and the image that I'm going to show you now, this is a picture of the Milky Way.

But it is an image which is not at all taken in the visible. It's an image which is caught in wavelengths that are completely invisible. Okay, so, it's absolutely mind-blowing and incredible that there are so many correlations between these different frequencies that in fact, these images are compatible. And here is an image, therefore, of the Milky Way, taken in frequencies which are absolutely not visible, but which, precisely, are correlated with the images in the visible and therefore ensure us that there is indeed a coherence, okay.

So now, when I was preparing this talk, it took me a long time to prepare this presentation. Why ? Because good, of course, I was given a rule which was that I should not show images like this that were not approved by the author of the image, etc. So, I said to myself "that, maybe I can do it.".

But on the other hand, I didn't have it for all the other images I wanted to show on the spheres. So I stuck doing it on the computer and it took me a lot of time. And then, at one point, I wanted to relax a little and I said to myself "Oh well, I'm going to play *Standing in the moonlight*" on the range of the simplest object, ie the vibrating string.

I tried and that's at that time that I realized that the first note that I had to use was 131. I said to myself "There is something strange".
So I asked myself the problem. I asked myself the problem of finding a musical shape.

So what do I mean by that ? Well I mean that when you make music, we'll see it right away, when you make music, in fact, it is not at all integers 1, 2, 3, 4, 5, etc., as frequencies which are used ? Absolutely not, these are the powers of the same number, the powers of the same number, that is to say we have a number q. And we look at the numbers q^n, that is what counts, because it is the relationships between frequencies that count. And the wonder that makes piano music exist, called *The harpsichord well temperated*, etc., it is the arithmetic fact that exists, which means that if we take the number 2 to the power of a twelfth, if you take the twelfth root of 2, that's very, very close to the nineteenth root of 3.

See, I gave those numbers. You see that the twelfth root of 2 is 1.059..., etc. The nineteenth root of 3 is 1.059... Where does 12 come from ?

The 12 comes from the fact that there are 12 notes when you make the chromatic range. And the 19 comes from the fact that 19 is 12 + 7 and that the seventh note in the chromatic scale, this is the scale that allows you to transpose. So what does it mean ? It means that going to the range above is multiplying by 2 and the ear is very sensitive to that. And transpose is multiplication by 3, except that it returns to the range before, i.e. so it is to multiply by 3 / 2, that agrees.

Well, that's the music, well known now, to which the ear is sensitive, etc. Okay. But... there is an obvious question ! It is "is there a geometrical object which range gives us the range we use in music ?". This is an absolutely obvious question.

If you look at what is going on, like these are the powers of q, you notice that the dimension of the space in question is necessarily equal to 0. Why ? Because earlier, I had shown you its limits. *(pauses to tell someone "I'm coming.")*. So I had shown you *(pauses to tell the person "I still have 5 minutes.")*. I had shown you earlier that the objects had a range that looked like a parabola when they were of dimension 2. When an object is larger, it will be a little more complicated than a parabola.

For example, if it is in dimension 3, it will be $y = x^{\frac{1}{3}}$, okay, but here, it's not at all a thing that is round like a parabola like that *(AC draws a parabola in the air)*. This is something that pffuiittt ! *(AC makes the gesture of an exponential in air.)* that gets up in the air like that. And what it tells you is that the object in question must be of dimension 0. So you say to yourself, "an object of dimension 0, What does it mean ? etc. Well.".

Well, when you develop geometry, which I have done for years and years, from a spectral point of view and for the development of what is called non-commutative geometry, etc., well, you actually see that these objects exist with a small nuance, it is that the algebra which is going to intervene is not going to be necessarily commutative.

And the wonder of wonders is that I realized, while preparing my talk, that there was an object, well known to mathematicians, that study non-commutative geometry or quantum things, which works for that thing and which gives you the right range.

And what is this object ? It is none other than what is called the quantum sphere S^2 index q. So this object is therefore a more delicate object. It was considered in particular by these three names *(on the transparency are noted the names Poddles, Brain and Landi)*. It has a spectrum, it has a spectrum. And this spectrum ? If you choose the number q carefully, it will correspond exactly to the musical spectrum. So, I return now to my experiments and I finish on it. So I will try. I hope that it will work. So we're going to go back to the experimentation. So we made the sphere and now we're looking for this musical shape which is going to be dimension 0. Okay, so we're going to try to play *Standing in the moonlight.* As I am tired, I will surely be wrong. But it's not a big deal. So see. *(AC returns to a rainbow spectrum and plays* Standing in the moonlight *on a keyboard with buttons indicated by integers, 25-25-25-27-29-27-25-29-27-27-25. 27-27-27-27-24, he makes a mistake, plays a B instead of an A, laughs, takes it up, etc... applause and ecstasy !).*

So now what is absolutely extraordinary with this range is... Can someone give me a number, above 10 anyway ? *(We see Jean-Pierre Changeux who waits positioned at the office to give his own Classes.)* 13, very good. Well I will see, I will try again but I promise nothing *Standing in the moonlight* from 13 *(AC plays 13-13-13-15-17-15-13-17- 15-15-13...)*

(When looking for the lowest score discarded from the others, AC says "So it's here I must not be mistaken, otherwise you will yell at me...". Re-applause.)

I will finish by saying the following thing, if you will, is that "nothing is too much beautiful to be realized in Nature". Recently, the Nobel Prize in Chemistry has been awarded to a chemist who discovered quasicrystals, which have a wonderful mathematical history, in Nature. What I hope is that one day, we will find the non-commutative sphere S_2^q in nature and one will be able to use it as a musical instrument, and it will be a wonderful instrument because it will never detune. Here. Thank you.

(Applause).