

A SUMMATION FORMULA IN THE THEORY OF PRIME NUMBERS

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1. Introduction

It has long been known that there is a close connexion between the prime numbers and the non-trivial zeros of the Riemann zeta-function,* but the precise nature of this connexion has not been elucidated. However, relations are known† between certain sums involving the prime numbers and other sums involving the zeros of $\zeta(s)$.

In this paper it is shown that, if the Riemann hypothesis is true, then there exists a general summation formula connecting sums of the above types. With appropriate conditions on $f(x)$ the main result is

$$\begin{aligned} \lim_{T \rightarrow \infty} \left\{ \sum_{0 < m \log p < T} \frac{\log p}{p^{im}} f(m \log p) - \int_0^T f(t) e^{it} dt \right\} - \frac{1}{2} \int_0^\infty f(t) \left(\frac{1}{t} - \frac{e^{-it}}{\sinh t} \right) dt \\ = -(2\pi)^{\frac{1}{2}} \lim_{T \rightarrow \infty} \left\{ \sum_{0 < \gamma < T} g(\gamma) - \frac{1}{2\pi} \int_0^T g(t) \log \frac{t}{2\pi} dt \right\}, \quad (1.1) \end{aligned}$$

where‡
$$g(x) = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\infty f(t) \cos xt dt,$$

p runs through the prime numbers, m through the positive integers, and $\frac{1}{2} + i\gamma$ through the non-trivial zeros§ of $\zeta(s)$.

* E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 1 (Leipzig, 1909), 367–368.

† E. Landau, *loc. cit.* 348–368, and A. E. Ingham, *The Distribution of Prime Numbers* (Cambridge, 1932), chapter iv.

‡ We use the notation $\int_0^\infty$ for $\lim_{X \rightarrow \infty} \int_0^X$.

§ If multiple zeros of $\zeta(s)$ exist then the corresponding terms $g(\gamma)$ in (1.1) are to be multiplied by the orders of the zeros. This convention will be used throughout the paper.

This result resembles Poisson's summation formula, and the method of proof used here was originally applied to the latter formula.*

An expression is derived for the remainder in the Riemann-von Mangoldt formula† for $N(T)$, the number of zeros of $\zeta(s)$ in $0 < I(s) < T$.

Some examples and applications of the summation formula (1.1) are given in the last section.

The truth of the Riemann hypothesis is assumed throughout the paper.

2. A pair of Hankel transforms

Consider the integral

$$\int_0^{\rightarrow\infty} (xt)^{\frac{1}{2}} J_{\frac{1}{2}}(xt) \left\{ N(t) - \left(\frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} \right) \right\} \frac{dt}{t}. \quad (2.1)$$

Now
$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z} \right)^{\frac{1}{2}} \left(\frac{\sin z}{z} - \cos z \right),$$

and hence
$$\left(\frac{x}{t} \right)^{\frac{1}{2}} J_{\frac{1}{2}}(xt) = - \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{d}{dt} \left(\frac{\sin xt}{xt} \right).$$

Also, by definition,
$$N(t) = \sum_{0 < \gamma < t} 1,$$

using the convention explained earlier if multiple zeros of $\zeta(s)$ occur. Hence the integral (2.1) may be written

$$- \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^{\rightarrow\infty} \left\{ \sum_{0 < \gamma < t} 1 - \left(\frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} \right) \right\} \frac{d}{dt} \left(\frac{\sin xt}{xt} \right) dt.$$

If we now write γ_r to distinguish the r th value of γ , this becomes

$$\begin{aligned} & - \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \lim_{N \rightarrow \infty} \int_0^{\gamma_{N+1}} \left\{ \sum_{0 < \gamma_r < t} 1 - \left(\frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} \right) \right\} \frac{d}{dt} \left(\frac{\sin xt}{xt} \right) dt \\ &= \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \lim_{N \rightarrow \infty} \left\{ - \sum_{r=1}^N r \int_{\gamma_r}^{\gamma_{r+1}} \frac{d}{dt} \left(\frac{\sin xt}{xt} \right) dt + \left[\frac{\sin xt}{xt} \left(\frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} \right) \right]_0^{\gamma_{N+1}} \right. \\ &\quad \left. - \frac{1}{2\pi} \int_0^{\gamma_{N+1}} \frac{\sin xt}{xt} \log \frac{t}{2\pi} dt \right\} \\ &= \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \lim_{N \rightarrow \infty} \left\{ \frac{1}{x} \sum_{r=1}^N r \left(\frac{\sin x\gamma_r}{\gamma_r} - \frac{\sin x\gamma_{r+1}}{\gamma_{r+1}} \right) \right. \\ &\quad \left. + \frac{\sin x\gamma_{N+1}}{x\gamma_{N+1}} \left(\frac{\gamma_{N+1}}{2\pi} \log \frac{\gamma_{N+1}}{2\pi} - \frac{\gamma_{N+1}}{2\pi} \right) \right\} \\ &\quad - \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \right) \int_0^{\rightarrow\infty} \frac{\sin xt}{xt} \log \frac{t}{2\pi} dt. \quad (2.2) \end{aligned}$$

* A. P. Guinand, *Annals of Math.* 42 (1941), 591–603.

† E. Landau, *loc. cit.* chapter xx.

$$\text{Now} \quad \int_0^{\rightarrow\infty} \sin u \frac{du}{u} = \frac{1}{2}\pi, \quad \int_0^{\rightarrow\infty} \sin u \log u \frac{du}{u} = -\frac{1}{2}\pi C,$$

where C is Euler's constant. Hence the integral term in (2.2) is equal to

$$-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi x}\right) \int_0^{\rightarrow\infty} \sin u (\log u - \log 2\pi x) \frac{du}{u} = \frac{1}{2x(2\pi)^{\frac{1}{2}}} (C + \log 2\pi x),$$

and (2.2) becomes

$$\begin{aligned} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{N \rightarrow \infty} \left\{ \frac{1}{x} \sum_{r=1}^N \frac{\sin x\gamma_r}{\gamma_r} - \frac{\sin x\gamma_{N+1}}{x\gamma_{N+1}} \left(N - \frac{\gamma_{N+1}}{2\pi} \log \frac{\gamma_{N+1}}{2\pi} + \frac{\gamma_{N+1}}{2\pi} \right) \right\} \\ + \frac{1}{2x(2\pi)^{\frac{1}{2}}} (C + \log 2\pi x). \end{aligned} \quad (2.3)$$

$$\text{Now, } * \text{ as } t \rightarrow \infty \quad N(t) - \left(\frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} \right) = O(\log t). \quad (2.4)$$

Hence, as $N \rightarrow \infty$

$$\frac{\sin x\gamma_{N+1}}{\gamma_{N+1}} \left(N - \frac{\gamma_{N+1}}{2\pi} \log \frac{\gamma_{N+1}}{2\pi} + \frac{\gamma_{N+1}}{2\pi} \right) = O\left(\frac{\log \gamma_{N+1}}{\gamma_{N+1}}\right)$$

and (2.3) becomes

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{1}{x}\right) \left\{ \sum_{r=1}^{\infty} \frac{\sin x\gamma_r}{\gamma_r} + \frac{1}{2} (C + \log 2\pi x) \right\}, \quad (2.5)$$

if this series converges. Now it is known that†

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum'_{n \leq x} \Lambda(n) n^{-s} - \frac{x^{1-s}}{1-s} - \sum_{q=1}^{\infty} \frac{x^{-2q-s}}{2q+s} + \sum_{\rho} \frac{x^{\rho-s}}{\rho-s} \quad (x > 1, s \neq 1, \rho, -2n), \quad (2.6)$$

where the dash indicates that the term $n = x$, if it occurs, is to be halved, and ρ runs through the non-trivial zeros of $\zeta(s)$. The series $\sum_{\rho}$ is to be interpreted as

$$\lim_{T \rightarrow \infty} \left\{ \sum_{-T < I(\rho) < T} \right\}.$$

Putting $s = \frac{1}{2}$, $x = e^z$, and $\rho = \frac{1}{2} + i\gamma$, this becomes

$$\begin{aligned} -\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} &= \sum'_{n \leq e^z} \Lambda(n) n^{-\frac{1}{2}} - 2e^{\frac{1}{2}z} - \sum_{q=1}^{\infty} \frac{e^{-(2q+\frac{1}{2})z}}{2q+\frac{1}{2}} + \sum_{\gamma} \frac{e^{i\gamma z}}{i\gamma} \\ &= \sum'_{\log n \leq z} \Lambda(n) n^{-\frac{1}{2}} - 2e^{\frac{1}{2}z} + 2e^{-\frac{1}{2}z} - \frac{1}{2} \log \coth \frac{1}{2}z \\ &\quad - \arctan e^{-\frac{1}{2}z} + 2 \sum_{\gamma > 0} \frac{\sin \gamma z}{\gamma}. \end{aligned}$$

* E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 4. Referred to in the sequel as *Tract*.

† E. C. Titchmarsh, *Tract*, 81.

Hence the series in (2.5) does converge, and it follows that the integral (2.1) exists. Using the definition* of $\Lambda(n)$ we obtain

$$\sum_{\gamma > 0} \frac{\sin \gamma z}{\gamma} = -\frac{1}{2} \sum'_{m \log p \leq s} \frac{\log p}{p^{im}} + 2 \sinh \frac{1}{2} z \\ + \frac{1}{4} \log \coth \frac{1}{4} z + \frac{1}{2} \arctan e^{-iz} - \frac{1}{4} C - \frac{1}{8} \pi - \frac{1}{4} \log 8\pi, \quad (2.7)$$

since†
$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \frac{1}{2} C + \frac{1}{4} \pi + \frac{1}{2} \log 8\pi.$$

Substituting (2.7) in (2.5) we obtain

THEOREM 1. *If the Riemann hypothesis is true and $x > 0$, then*

$$\int_0^{\infty} (xt)^{\frac{1}{2}} J_{\frac{1}{2}}(xt) \left\{ N(t) - \left(\frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} \right) \right\} \frac{dt}{t} \\ = -\frac{1}{x(2\pi)^{\frac{1}{2}}} \left\{ \sum'_{0 < m \log p \leq x} \frac{\log p}{p^{im}} - 4 \sinh \frac{1}{2} x \right\} \\ + \frac{1}{x(2\pi)^{\frac{1}{2}}} \left\{ \frac{1}{2} \log \left(\frac{1}{4} x \coth \frac{1}{4} x \right) + \arctan e^{-ix} - \frac{1}{4} \pi \right\}.$$

If we put
$$F(x) = \frac{1}{x} \left\{ N(x) - \left(\frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} \right) \right\}, \quad (2.8)$$

and
$$G(x) = -\frac{1}{x(2\pi)^{\frac{1}{2}}} \left\{ \sum'_{0 < m \log p \leq x} \frac{\log p}{p^{im}} - 4 \sinh \frac{1}{2} x \right\} \\ + \frac{1}{x(2\pi)^{\frac{1}{2}}} \left\{ \frac{1}{2} \log \left(\frac{1}{4} x \coth \frac{1}{4} x \right) + \arctan e^{-ix} - \frac{1}{4} \pi \right\}, \quad (2.9)$$

then theorem 1 implies that $G(x)$ is the Hankel transform‡ of order $\frac{3}{2}$ of $F(x)$.

We can deduce the inverse transformation from theorem 1 by using the following lemmas:

LEMMA § α . *If $F(x)$ belongs to $L^p(0, \infty)$ for some p in $1 < p \leq 2$, then there exists a function $G(x)$ belonging to $L^{p'}(0, \infty)$ for $p' = p/(p-1)$, defined almost everywhere by*

$$G(x) = \text{l.i.m.}_{T \rightarrow \infty} \int_0^T (xt)^{\frac{1}{2}} J_{\frac{1}{2}}(xt) F(t) dt. \quad (2.10)$$

Further
$$F(x) = \text{l.i.m.}_{T \rightarrow \infty} \int_0^T (xt)^{\frac{1}{2}} J_{\frac{1}{2}}(xt) G(t) dt,$$

* $\Lambda(n) = \log p$ when $n = p^m$ and vanishes elsewhere.

† This follows on putting $s = \frac{1}{2}$ in the functional equation for $\zeta'(s)/\zeta(s)$. See A. E. Ingham, *loc. cit.* 73.

‡ E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 214–215. Referred to in the sequel as *F.I.*

§ H. Kober, *Quart. J. of Math.* 8 (1937), 186–199, and I. W. Busbridge, *ibid.* 9 (1938), 148–160.

and if $f(x), g(x)$ is another pair of such transforms for the same value of p , then

$$\int_0^\infty f(x) G(x) dx = \int_0^\infty F(x) g(x) dx. \quad (2.11)$$

LEMMA* β . If $F(x)$ belongs to $L^2(0, \infty)$ and is of bounded variation in some neighbourhood of $x = y$, and $G(x)$ is defined by (2.10), then

$$\frac{1}{2}\{F(y+0) + F(y-0)\} = \int_0^{\rightarrow\infty} (yt)^{\frac{1}{2}} J_{\frac{1}{2}}(yt) G(t) dt.$$

Now if $F(x)$ is defined by (2.8), then it follows from (2.4) that $F(x)$ belongs to $L^p(0, \infty)$ for any $p > 1$, and the conditions of lemmas α and β are fulfilled for any $y > 0$. Hence

$$\begin{aligned} & \frac{1}{y} \left[\frac{1}{2}\{N(y+0) + N(y-0)\} - \left(\frac{y}{2\pi} \log \frac{y}{2\pi} - \frac{y}{2\pi} \right) \right] \\ &= \int_0^{\rightarrow\infty} (yt)^{\frac{1}{2}} J_{\frac{1}{2}}(yt) \left[-\frac{1}{t(2\pi)^{\frac{1}{2}}} \left\{ \sum'_{0 < m \log p \leq t} \frac{\log p}{p^{\frac{1}{2}m}} - 4 \sinh \frac{1}{2}t \right\} \right. \\ & \quad \left. + \frac{1}{t(2\pi)^{\frac{1}{2}}} \left\{ \frac{1}{2} \log \left(\frac{1}{4}t \coth \frac{1}{4}t \right) + \arctan e^{-t} - \frac{1}{4}\pi \right\} \right] dt. \end{aligned}$$

We can deal with the first part of this integral as we dealt with the integral (2.1). The second part of the integral can be evaluated by partial integration and the use of known integrals. We obtain:

THEOREM 2. If the Riemann hypothesis is true and $T \geq 0$, then

$$\begin{aligned} & \frac{1}{2}\{N(T+0) + N(T-0)\} - \left(\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} \right) \\ &= -\frac{1}{\pi} \lim_{N \rightarrow \infty} \left[\sum_{n=1}^N \Lambda(n) \frac{\sin(T \log n)}{n^{\frac{1}{2}} \log n} - \int_1^N \frac{\sin(T \log t)}{t^{\frac{1}{2}} \log t} dt \right. \\ & \quad \left. - \frac{\sin(T \log N)}{\log N} \left\{ \sum_{n=1}^N \Lambda(n) n^{-\frac{1}{2}} - 2N^{\frac{1}{2}} \right\} \right] + \frac{1}{2\pi} \{ \text{am } \Gamma(\tfrac{1}{2} + iT) - T \log T + T \} \\ & \quad + \frac{1}{\pi} \arctan 2T - \frac{1}{4\pi} \arctan(\sinh \pi T), \quad (2.12) \end{aligned}$$

where $\text{am } \Gamma(\tfrac{1}{2} + iT)$ is defined by making $\text{am } \Gamma(\tfrac{1}{2}) = 0$ and continuing analytically along any path not meeting the real axis.

Other writers have remarked† previously that a connexion exists between $N(T)$ and the first series in (2.12), but as far as I am aware no exact formula of this type for $N(T)$ has been given before.

* E. C. Titchmarsh, *F.I.* 83 and 266.

† E. C. Titchmarsh, *Tract*, 63; A. Wintner, *Duke Math. J.* 10 (1943), 99–105; and A. Selberg, *Avhandlingar utgitt av det Norske Videnskaps-Akademi i Oslo*, no. 1 (1944). 1–27.

Now it has been shown* that, if the Riemann hypothesis is true, then as $N \rightarrow \infty$

$$\sum_{n=1}^N \Lambda(n) - N = O(N^{\frac{1}{2}} \log^{\alpha} N) \quad (2.13)$$

holds for $\alpha = 2$, and also that the expression on the left is

$$\Omega \pm (N^{\frac{1}{2}} \log \log \log N).$$

These results are the best of their kind yet proved, so it is still possible that (2.13) may hold for some α in $0 < \alpha < 2$.

If we put $s = 0$ and $s = \frac{1}{2}$ successively in (2.6), then we can show that

$$\begin{aligned} & \left\{ \sum_{n=1}^N \Lambda(n) n^{-\frac{1}{2}} - 2N^{\frac{1}{2}} \right\} - N^{-\frac{1}{2}} \left\{ \sum_{n=1}^N \Lambda(n) - N \right\} \\ &= - \sum_{\rho} N^{\rho-\frac{1}{2}} \left\{ \frac{1}{\rho-\frac{1}{2}} - \frac{1}{\rho} \right\} + O(1) \\ &= - \frac{1}{2} \sum_{\gamma} \frac{N^{i\gamma}}{i\gamma(\frac{1}{2} + i\gamma)} + O(1) \\ &= O(1), \end{aligned}$$

since $\sum_{\gamma} \gamma^{-2}$ is absolutely convergent.

$$\text{Hence} \quad \sum_{n=1}^N \Lambda(n) n^{-\frac{1}{2}} - 2N^{\frac{1}{2}} = O(\log^{\alpha} N). \quad (2.14)$$

If this holds for some $\alpha < 1$, then in theorem 2

$$\lim_{T \rightarrow \infty} \frac{\sin(T \log N)}{\log N} \left\{ \sum_{n=1}^N \Lambda(n) n^{-\frac{1}{2}} - 2N^{\frac{1}{2}} \right\} = 0,$$

so this term could be omitted from (2.11).

3. The summation formula

We now need the following lemmas:†

LEMMA γ . *If $f(x)$ is an integral, tends to zero as x tends to infinity, and $xf'(x)$ belongs to $L^p(0, \infty)$ for some $p > 1$, then $f(x)$ belongs to $L^p(0, \infty)$ and $x^{1/p}f(x)$ tends to zero as x tends to $+\infty$ or to infinity.*

LEMMA δ . *If $f(x)$ satisfies the conditions of lemma γ for some p in $1 < p \leq 2$, then it has a Fourier cosine transform $g(x)$ given for all positive x by*

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{p}} \int_0^{\infty} f(t) \cos xt dt.$$

* A. E. Ingham, *loc. cit.* theorems 30 and 34.

† A. P. Guinand, *loc. cit.* lemmas 2 and 4.

Further
$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\rightarrow \infty} g(t) \cos xt dt,$$

$g(x)$ is an integral, both $g(x)$ and $xg'(x)$ belong to $L^{p'}(0, \infty)$, $x^{1/p'}g(x)$ tends to zero as x tends to $+0$ or to infinity, and $xf'(x)$, $xg'(x)$ is a pair of Hankel transforms of order $\frac{3}{2}$ in the sense of lemma α .

Now if $f(x)$ and $g(x)$ satisfy the conditions of lemma δ and $F(x)$ and $G(x)$ are defined by (2.7) and (2.8), then by (2.10)

$$\int_0^{\infty} xf'(x) G(x) dx = \int_0^{\infty} xg'(x) F(x) dx. \quad (3.1)$$

The left-hand side of (3.1) is

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} f'(x) \left\{ - \sum'_{0 < m \log p \leq x} \frac{\log p}{p^{\frac{1}{2}m}} + 4 \sinh \frac{1}{2}x + \frac{1}{2} \log \left(\frac{1}{2}x \coth \frac{1}{2}x \right) + \arctan e^{-\frac{1}{2}x} - \frac{1}{2}\pi \right\} dx. \quad (3.2)$$

Now
$$\frac{1}{x} \left\{ \frac{1}{2} \log \left(\frac{1}{2}x \coth \frac{1}{2}x \right) + \arctan e^{-\frac{1}{2}x} - \frac{1}{2}\pi \right\}$$

belongs to $L^p(0, \infty)$ for any $p > 1$. Hence the integral

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} f'(x) \left\{ \frac{1}{2} \log \left(\frac{1}{2}x \coth \frac{1}{2}x \right) + \arctan e^{-\frac{1}{2}x} - \frac{1}{2}\pi \right\} dx \quad (3.3)$$

converges absolutely. Integrating by parts, the integrated terms vanish by lemma γ , and (3.3) becomes

$$\begin{aligned} & - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} f(x) \left\{ \frac{1}{2x} - \frac{\operatorname{cosech}^2 \frac{1}{2}x}{8 \coth \frac{1}{2}x} - \frac{\frac{1}{2}e^{-\frac{1}{2}x}}{1 + e^{-x}} \right\} dx \\ & = \frac{1}{2(2\pi)^{\frac{1}{2}}} \int_0^{\infty} f(x) \left(\frac{e^{\frac{1}{2}x}}{\sinh x} - \frac{1}{x} \right) dx. \end{aligned} \quad (3.4)$$

The other terms in (3.2) are

$$\begin{aligned} & - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} f'(x) \left\{ \sum'_{0 < m \log p \leq x} \frac{\log p}{p^{\frac{1}{2}m}} - 4 \sinh \frac{1}{2}x \right\} dx \\ & = - \frac{1}{(2\pi)^{\frac{1}{2}}} \lim_{T \rightarrow \infty} \left\{ \left[f(x) \left(\sum'_{0 < m \log p \leq x} \frac{\log p}{p^{\frac{1}{2}m}} - 4 \sinh \frac{1}{2}x \right) \right]_0^T \right. \\ & \quad \left. - \int_0^T f(x) d \left(\sum'_{0 < m \log p \leq x} \frac{\log p}{p^{\frac{1}{2}m}} \right) + 2 \int_0^T f(x) \cosh \frac{1}{2}x dx \right\} \\ & = - \frac{1}{(2\pi)^{\frac{1}{2}}} \lim_{T \rightarrow \infty} \left\{ f(T) \left(\sum'_{0 < m \log p \leq T} \frac{\log p}{p^{\frac{1}{2}m}} - 2e^{\frac{1}{2}T} \right) \right. \\ & \quad \left. - \sum_{0 < m \log p < T} \frac{\log p}{p^{\frac{1}{2}m}} f(m \log p) + \int_0^T f(x) e^{\frac{1}{2}x} dx \right\} \\ & \quad - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} f(x) e^{-\frac{1}{2}x} dx, \end{aligned} \quad (3.5)$$

since the last integral converges absolutely. Adding (3.4) and (3.5) we find that (3.2) is equal to

$$\begin{aligned} \frac{1}{(2\pi)^{\frac{1}{2}}} \lim_{T \rightarrow \infty} \left\{ \sum_{0 < m \log p < T} \frac{\log p}{p^{\frac{1}{2}m}} f(m \log p) - \int_0^T f(x) e^{\frac{1}{2}ix} dx \right. \\ \left. - f(T) \left(\sum_{0 < m \log p < T} \frac{\log p}{p^{\frac{1}{2}m}} - 2e^{\frac{1}{2}iT} \right) \right\} \\ - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty f(x) e^{-\frac{1}{2}ix} dx + \frac{1}{2(2\pi)^{\frac{1}{2}}} \int_0^\infty f(x) \left(\frac{e^{\frac{1}{2}ix}}{\sinh x} - \frac{1}{x} \right) dx. \end{aligned}$$

The sum of the last two integrals is equal to

$$\frac{1}{2(2\pi)^{\frac{1}{2}}} \int_0^\infty f(x) \left(\frac{e^{-\frac{1}{2}ix}}{\sinh x} - \frac{1}{x} \right) dx.$$

Now consider the right-hand side of (3.1). It is equal to

$$\begin{aligned} \int_0^\infty g'(x) \left\{ N(x) - \left(\frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} \right) \right\} dx \\ = \lim_{T \rightarrow \infty} \left\{ \left[g(x) \left(N(x) - \frac{x}{2\pi} \log \frac{x}{2\pi} + \frac{x}{2\pi} \right) \right]_0^T - \int_0^T g(x) dN(x) \right. \\ \left. + \frac{1}{2\pi} \int_0^T g(x) \log \frac{x}{2\pi} dx \right\} \\ = - \lim_{T \rightarrow \infty} \left\{ \sum_{0 < \gamma < T} g(\gamma) - \frac{1}{2\pi} \int_0^T g(x) \log \frac{x}{2\pi} dx \right\}, \end{aligned}$$

since the integrated terms vanish by (2.4) and lemma δ . Hence we have, altogether:

THEOREM 3. *If (i) the Riemann hypothesis is true,*

(ii) $f(x)$ is an integral, tends to zero as x tends to infinity, and $xf'(x)$ belongs to $L^p(0, \infty)$ for some p in $1 < p \leq 2$, and

$$(iii) \quad g(x) = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^\infty f(t) \cos xt dt,$$

$$\begin{aligned} \text{then} \quad \lim_{T \rightarrow \infty} \left\{ \sum_{0 < m \log p < T} \frac{\log p}{p^{\frac{1}{2}m}} f(m \log p) - \int_0^T f(x) e^{\frac{1}{2}ix} dx \right. \\ \left. - f(T) \left(\sum_{0 < m \log p < T} \frac{\log p}{p^{\frac{1}{2}m}} - 2e^{\frac{1}{2}iT} \right) \right\} - \frac{1}{2} \int_0^\infty f(x) \left(\frac{1}{x} - \frac{e^{-\frac{1}{2}ix}}{\sinh x} \right) dx \\ = - (2\pi)^{\frac{1}{2}} \lim_{T \rightarrow \infty} \left\{ \sum_{0 < \gamma < T} g(\gamma) - \frac{1}{2\pi} \int_0^T g(x) \log \frac{x}{2\pi} dx \right\}. \quad (3.6) \end{aligned}$$

We can simplify this result slightly by making further assumptions. By lemma γ and (2.14)

$$f(T) \left(\sum_{0 < m \log p < T} \frac{\log p}{p^{im}} - 2e^{ix} \right) = O\{T^\alpha f(T)\} \\ = o(T^{\alpha-(1/p)}). \quad (3.7)$$

Hence:

THEOREM 3'. *If, in addition to the assumptions of theorem 3, either (2.13) holds for $\alpha = 1/p$ or $T^2 f(T)$ tends to zero as T tends to infinity, then the term (3.7) can be omitted from (3.6).*

Further, if it is desired, the left-hand side of (3.6) can be expressed in terms of $\Lambda(n)$. Putting $x = \log t$, $U = e^T$ we have

$$\lim_{T \rightarrow \infty} \left\{ \sum_{0 < m \log p < T} \frac{\log p}{p^{im}} f(m \log p) - \int_0^T f(x) e^{ix} dx \right\} \\ = \lim_{U \rightarrow \infty} \left\{ \sum_{n < U} \frac{\Lambda(n)}{n^{\frac{1}{p}}} f(\log n) - \int_1^U f(\log t) \frac{dt}{t^{\frac{1}{p}}} \right\}.$$

4. Examples and applications

Numerous examples of functions to which the formula (3.6) is applicable can easily be found, corresponding to the many known applications of Poisson's summation formula.* Two examples chosen for their special interest are given here.

(A) Suppose† $\frac{1}{2} < R(s) < 1$, and

$$f_1(x) = \begin{cases} x^{-s} & (x \leq 1), \\ e^{1-x} & (x \geq 1), \end{cases} \\ f_2(x) = \begin{cases} 0 & (x \leq 1), \\ x^{-s} - e^{1-x} & (x \geq 1). \end{cases}$$

Then $f_1(x)$ satisfies the conditions of theorem 3 for $1 < p < 1/R(s)$, and $f_2(x)$ satisfies them for $p = 2$. Hence the result (3.6) holds for both functions, and by addition it also holds for the function

$$f(x) = f_1(x) + f_2(x) = x^{-s}.$$

This gives

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} t^{-s} \cos xt dt \\ = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{\Gamma(1-s) \sin \tfrac{1}{2}s\pi\} x^{1-s}.$$

* E. C. Titchmarsh, *F.I.* 64-65.

† Cf. A. P. Guinand, *loc. cit.* 597.

Also if we write for the infinite integral in (3.6)

$$I(s) = \frac{1}{2} \int_0^\infty x^{-s} \left(\frac{1}{x} - \frac{e^{-ix}}{\sinh x} \right) dx,$$

then this defines an analytic function of s for $0 < R(s) < 1$, and in this strip

$$\begin{aligned} I(s) &= \frac{1}{2} \int_0^1 x^{-s} \left(\frac{1}{x} - \frac{e^{-ix}}{\sinh x} \right) dx + \frac{1}{2} \int_1^\infty x^{-1-s} dx - \frac{1}{2} \int_1^\infty \frac{x^{-s} e^{-ix}}{\sinh x} dx \\ &= \frac{1}{2} \int_0^1 x^{-s} \left(\frac{1}{x} - \frac{e^{-ix}}{\sinh x} \right) dx + \frac{1}{2s} - \frac{1}{2} \int_1^\infty \frac{x^{-s} e^{-ix}}{\sinh x} dx. \end{aligned}$$

This defines the analytic continuation of $I(s)$ for $R(s) < 0$, where it is

$$\begin{aligned} I(s) &= \frac{1}{2} \int_0^1 x^{-s-1} dx + \frac{1}{2s} - \frac{1}{2} \int_0^\infty \frac{x^{-s} e^{-ix}}{\sinh x} dx \\ &= - \int_0^\infty \frac{x^{-s} e^{-ix}}{e^x - e^{-x}} dx \\ &= - \sum_{n=0}^\infty \int_0^\infty x^{-s} e^{-(2n+1)x} dx \\ &= - \Gamma(1-s) \sum_{n=0}^\infty (2n + \frac{1}{2})^{s-1} \\ &= - 2^{1-s} \Gamma(1-s) \sum_{n=1}^\infty (4n+1)^{s-1}. \end{aligned}$$

Now, if we write

$$\begin{aligned} \eta(s) &= 1^{-s} - 3^{-s} + 5^{-s} - \dots \quad \{R(s) > 1\} \\ &= \sum_{n=0}^\infty (4n+1)^{-s} - \sum_{n=0}^\infty (4n+3)^{-s}, \end{aligned}$$

$$\begin{aligned} \text{then} \quad \sum_{n=1}^\infty (4n+1)^{-s} &= \frac{1}{2} \eta(s) + \frac{1}{2} \sum_{n=0}^\infty (4n+1)^{-s} + \frac{1}{2} \sum_{n=0}^\infty (4n+3)^{-s} - 1 \\ &= \frac{1}{2} \eta(s) + \frac{1}{2} \sum_{n=0}^\infty (2n+1)^{-s} - 1 \\ &= \frac{1}{2} \eta(s) + \frac{1}{2} (1 - 2^{-s}) \zeta(s) - 1. \end{aligned}$$

$$\text{Hence} \quad I(s) = \Gamma(1-s) \{2^{1-s} - 2^{-s} \eta(1-s) - (2^{-s} - \frac{1}{2}) \zeta(1-s)\}.$$

If (2.13) holds for some $\alpha < R(s)$, then (3.6) gives

$$\begin{aligned} &\lim_{T \rightarrow \infty} \left\{ \sum_{0 < m \log p < T} \frac{\log p}{p^{im}} (m \log p)^{-s} - \int_0^T x^{-s} e^{ix} dx \right\} \\ &\quad - \Gamma(1-s) \{2^{1-s} - 2^{-s} \eta(1-s) - (2^{-s} - \frac{1}{2}) \zeta(1-s)\} \\ &= -2\Gamma(1-s) \sin \frac{1}{2} \pi s \lim_{T \rightarrow \infty} \left\{ \sum_{0 < \gamma < T} \gamma^{s-1} - \frac{T^s}{2\pi s} \log \frac{T}{2\pi} + \frac{T^s}{2\pi s^2} \right\}. \end{aligned}$$

(B) Put $y > 0$ and

$$f(x) = \begin{cases} \cos yx - \cos yX & (0 \leq x \leq X), \\ 0 & (x \geq X). \end{cases}$$

The conditions of theorem 3' are satisfied, and

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{y}{x(y^2 - x^2)} (x \sin yX \cos xX - y \cos yX \sin xX)$$

if $x \neq y$. If $x = y$ then this becomes

$$g(y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \left(X - \frac{\sin 2yX}{2y} \right). \quad (4.1)$$

Substituting this in (3.6) we obtain

$$\begin{aligned} & \sum_{0 < m \log p < X} \frac{\log p}{p^{\frac{1}{2}m}} \{ \cos (ym \log p) - \cos yX \} - \int_0^X (\cos yx - \cos yX) e^{\frac{1}{2}ix} dx \\ & \quad - \frac{1}{2} \int_0^X (\cos yx - \cos yX) \left(\frac{1}{x} - \frac{e^{-\frac{1}{2}ix}}{\sinh x} \right) dx \\ & = -2 \sum_{\gamma > 0} \frac{y}{\gamma(y^2 - \gamma^2)} (\gamma \sin yX \cos \gamma X - y \cos yX \sin \gamma X) \\ & \quad + \frac{1}{\pi} \int_0^\infty \frac{y}{x(y^2 - x^2)} (x \sin yX \cos xX - y \cos yX \sin xX) \log \frac{x}{2\pi} dx, \quad (4.2) \end{aligned}$$

where the term $\gamma = y$, if it occurs, is to be interpreted as in (4.1).

Now let X tend to infinity. For fixed y

$$\begin{aligned} \frac{1}{2} \int_0^X (\cos yx - \cos yX) \left(\frac{1}{x} - \frac{e^{-\frac{1}{2}ix}}{\sinh x} \right) dx &= O(1) - \frac{1}{2} \cos yX \int_1^X \frac{dx}{x} \\ &= O(1) - \frac{1}{2} \cos yX \log X. \end{aligned}$$

$$\begin{aligned} \text{Also } & \left| \sum_{\gamma > 0, \gamma \neq y} \frac{y}{\gamma(y^2 - \gamma^2)} (\gamma \sin yX \cos \gamma X - y \cos yX \sin \gamma X) \right| \\ & \leq \sum_{\gamma > 0, \gamma \neq y} \frac{y(\gamma + y)}{\gamma |y^2 - \gamma^2|} = y \sum_{\gamma > 0, \gamma \neq y} \frac{1}{\gamma |y - \gamma|} \\ & = O(1), \end{aligned}$$

since the latter series converges.* The term $\gamma = y$, if it occurs, gives a term

$$-X + \frac{\sin 2yX}{2y} = -X + O(1).$$

* A. E. Ingham, *loc. cit.* theorem 25b.

Further

$$\begin{aligned}
 & \frac{1}{\pi} \int_0^\infty \frac{y}{x(y^2 - x^2)} (x \sin yX \cos xX - y \cos yX \sin xX) \log \frac{x}{2\pi} dx \\
 &= \frac{1}{\pi} \int_0^\infty \frac{y}{x(y^2 - x^2)} \{(x - y) \cos yX \sin xX - x \sin (x - y) X\} \log \frac{x}{2\pi} dx \\
 &= -\frac{1}{\pi} y \cos yX \int_0^\infty \frac{\sin xX}{x(y + x)} \log \frac{x}{2\pi} dx + \frac{1}{\pi} y \int_0^\infty \frac{\sin (x - y) X}{(x^2 - y^2)} \log \frac{x}{2\pi} dx \\
 &= \frac{1}{\pi} \cos yX \int_0^\infty \frac{\sin xX}{y + x} \log \frac{x}{2\pi} dx - \frac{1}{\pi} \cos yX \int_0^\infty \frac{\sin xX}{x} \log \frac{x}{2\pi} dx \\
 &\quad + \frac{1}{\pi} y \int_0^\infty \frac{\sin (x - y) X}{x^2 - y^2} \log \frac{x}{2\pi} dx. \quad (4.3)
 \end{aligned}$$

By Fourier's single integral theorem* the first of these integrals tends to zero as X tends to infinity, and the last tends to

$$\frac{1}{2} \log (y/2\pi) = O(1).$$

There remains in (4.3)

$$\begin{aligned}
 -\frac{1}{\pi} \cos yX \int_0^\infty \frac{\sin xX}{x} \log \frac{x}{2\pi} dx &= -\frac{1}{\pi} \cos yX \int_0^\infty \frac{\sin t}{t} \log \left(\frac{t}{2\pi X} \right) dt \\
 &= -\frac{1}{2} \cos yX (C + \log 2\pi X) \\
 &= O(1) + \frac{1}{2} \cos yX \log X.
 \end{aligned}$$

Hence (4.2) becomes

$$\begin{aligned}
 \sum_{0 < m \log p < X} \frac{\log p}{p^{im}} \{\cos (ym \log p) - \cos yX\} - \int_0^X (\cos yx - \cos yX) e^{ix} dx \\
 = \begin{cases} -X + O(1) & (y = \gamma), \\ O(1) & (y \neq \gamma). \end{cases}
 \end{aligned}$$

Hence we have:

THEOREM 4. *If the Riemann hypothesis is true, then as X tends to infinity*

$$\begin{aligned}
 \sum_{0 < m \log p < X} \frac{\log p}{p^{im}} \cos (ym \log p) - \frac{e^{iX}}{\frac{1}{4} + y^2} \left(\frac{1}{2} \cos yX + y \sin yX \right) \\
 - \cos yX \left\{ \sum_{0 < m \log p < X} \frac{\log p}{p^{im}} - 2e^{iX} \right\} \\
 = \begin{cases} -X + O(1) & (y = \gamma), \\ O(1) & (y \neq \gamma). \end{cases}
 \end{aligned}$$

* E. C. Titchmarsh, *F.I.* theorem 12.

Inversely, if we interchange $f(x)$ and $g(x)$ we can prove:

THEOREM 5. *If the Riemann hypothesis is true, then as X tends to infinity*

$$\sum_{0 < \gamma < X} \cos y\gamma = \begin{cases} -\frac{X \log p}{2\pi p^{im}} + O(\log X) & (y = m \log p), \\ O(\log X) & -(y \neq m \log p). \end{cases}$$

However, this result is a simple consequence of the known result* for $x > 1$

$$\sum_{0 < I(\rho) < T} x^\rho = \begin{cases} -\frac{T}{2\pi} \log p + O(\log T) & (x = p^m), \\ O(\log T) & (x \neq p^m). \end{cases}$$

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* E. C. Titchmarsh, *Tract*, theorem 42.