Goldbach's conjecture, where we find ζ in another way (Denise Vella-Chemla, 29.5.2019)

One considers here Goldbach's conjecture that asserts that every even number strictly greater than 2 is the sum of two primes.

One recalls that a prime number x lesser than $\dfrac{n}{2}$, that doesn't share any of its division rest with n an even number strictly greater than 2, in all divisions by a prime number lesser than $\sqrt{n}$, is a Goldbach component of n (i.e. $n-x$ is prime too).

Indeed, if x lesser than $\dfrac{n}{2}$ doesn't share any of its division rest with n in any division by a prime lesser than $\sqrt{n}$, then $n-x$ is prime.

The asymptotic probability that an integer x lesser than $\dfrac{n}{2}$ be prime is provided by the prime number theorem ; it equals :

$$\frac{\dfrac{n}{2}}{ln\left(\dfrac{n}{2}\right)}$$

The minoration of $\pi(k)$ (the number of prime numbers lesser than k) by $\dfrac{k}{ln\ k}$ is provided in [1], page 69, for all $x \geq 17$.

Let us suppose now that x is prime. Let us study the probabilities that divisions rests of x and n are equal when one divides them by all the prime numbers lesser than $\sqrt{n}$.

Since we supposed x to be prime, we know at least that x has no rest equal to zero when we divide it by a prime number lesser than $\sqrt{n}$.

n has a certain rest, when we divide it by a prime number lesser than $\sqrt{n}$ and x has to "avoid" the rest in question (it can't have the same).

If we consider a division of n by one of its prime divisors, in which the rest is null, x has only this rest zero (0) to avoid. However x can't have (has yet avoided) the rest 0 since it's prime. It remains $p-1$ possible rests for x when we divide it by p.

Let us consider now a division of n by a prime number which is not an n's divisor, let us call it d. n has, when we divide it by d a rest that is different from 0 that x must avoid. In this case, x has the choice between $p-2$ possible rests in its division by p, that it can have with equal probabilities the one or the other but we are going to use the fact that $\dfrac{1}{p-2} > \dfrac{1}{p-1}$ to minorate each probability modulo a given prime number p by $\dfrac{1}{p-1}$, to homogeneize the different possible cases (if we are considering or not a prime divisor of n).

Let us see examples, to fix ideas : in a division by prime number 3, we minorate the number of possibilities by 2 possibilities for the division rests (1 or 2), and x has one chance among two (i.e. 1/2) to obtain one or the other.

In a division by 5, it remains 4 possibilities for x to have some division rest among 1, 2, 3 or 4, and x has one chance among 4 (i.e. 1/4) to obtain the one or the other.

In a division by 7, it remains 6 possibilities for x to have its division rest among 1, 2, 3, 4, 5 or 6, and x has one chance among 6 (i.e. 1/6) to obtain the one or the other.

More generally, in a division by p, one minorates the probability for x and n to have the same division rest in the following way : there are $p-1$ division rests possibilities at most for x (that are $1, 2, \ldots, p-1$), and x has one chance among $p-1$ (i.e. $\dfrac{1}{p-1}$ to obtain the one or the other of those division rests).

All those events (rests sharings) having independent probabilities, the probability to obtain their conjunction is the product of the probabilities of each event alone (the considered events being "x and n have the same rest in a division by 3", or "x and n have the same rest in a division by 5", etc.).

This product of probabilities can be written :

$$\prod_{p\ premier\ <\sqrt{n}} \frac{1}{p-1}$$

We can transform this in :

$$\prod_{p\ premier\ <\sqrt{n}} \frac{1}{p^{-(-1)}-1}$$

and then in

$$-\prod_{p\ premier\ <\sqrt{n}} \frac{1}{1-p^{-(-1)}}$$

We can extend this product to the set of all primes in infinite number because in fact, it's modulo every prime number that n and x have not to be in the same congruence class (i.e. mustn't share their rest), for the complementary of x to n (i.e. $n-x$) to be prime too. One can recognize then $-\zeta(-1)$ in the calculus of the product for x and n have different rests in a division by whatever prime number. Ramanujan demonstrated that $\zeta(-1) = -\frac{1}{12}$. The note [1] provides a simple demonstration of this fact.

We obtain the cardinal of a set of numbers x that are prime on one side, and that don't have the same division rest than n in a division by any prime number lesser than $\sqrt{n}$ (and in fact by any prime) [2] on the other side :

$$\frac{\frac{n}{2}}{ln\left(\frac{n}{2}\right)} \times (-\zeta(-1))$$

that is :

$$\frac{n}{2\ ln\ n - 2\ ln\ 2} \times \frac{1}{12}.$$

This seems to make Goldbach's conjecture true above $n = 92$ [3].

Attempt to write this reasoning more formally :

We want to demonstrate that $\forall n$ even, $\exists x, 3 \leq x \leq n/2$ odd prime such that $n-x$ is prime too.

$$\begin{array}{llll} (1) & x \text{ prime} & \iff \forall p \text{ prime} \leq \sqrt{x}, & x \not\equiv 0\ (mod\ p). \\ (2) & n-x \text{ prime} & \iff \forall p \text{ prime} \leq \sqrt{n-x}, & n-x \not\equiv 0\ (mod\ p) \\ & & \iff \forall p \text{ prime} \leq \sqrt{n-x}, & x \not\equiv n\ (mod\ p). \end{array}$$

1. Par définition $S = 1+2+3+4+5+....$

One notes than calculating term by term the difference :

$$\begin{array}{rllll} S-B = & 1+2 & +3+4 & +5+6 & ... \\ & -1+2 & -3+4 & -5+6 & ... \\ = & 0+4 & +0+8 & +0+12 & ... = 4(1+2+3+...) = 4S \end{array}$$

So $S-4S = B$, i.e. $-3S = B$, d'où $S = -\frac{B}{3} = -\frac{\frac{1}{4}}{3}$ So one finds the expected result : $S = -\frac{1}{12}$.

2. The fact that x doesn't share any division rest with n in divisions by prime numbers lesser than $\sqrt{n}$ is not the same as the fact to be prime to n (to have no common factor greater than 1 with n). This last condition is necessary (i.e. *implied*) but not sufficient (i.e. *implying*). For instance, 17 and 81, that have a sum equal to 98, are both *prime to* 98, but they are not Goldbach'decomponents of 98 since 17 shares its division rest 2 with 98 when we divide them by 3 (Gauss writes this $17 \equiv 98\ (mod\ 3)$, he is the one who drew attention of everyone on the importance to work in prime fields).

3. $\frac{92}{2\ ln\ 92 - 2\ ln\ 2}.\frac{1}{12} = 1.0012254835$ alors que $\frac{90}{2\ ln\ 90 - 2\ ln\ 2)}.\frac{1}{12} = 0.9851149163$.

One can replace in (1) the condition $\forall p$ prime $\leq \sqrt{x}$ by the strongest condition $\forall p$ prime $\leq \sqrt{n/2}$ since we let $x \leq n/2$.

One can minorate the number of prime numbers lesser than $\dfrac{n}{2}$ by $\dfrac{\frac{n}{2}}{log\left(\frac{n}{2}\right)}$.

It matters then to find how many numbers in this set of prime numbers lesser than $\dfrac{n}{2}$, set whose we know the cardinal, share their division rest with n ; sharing a rest with n, the even number considered, consists in "fixing" the possible rest and so to make decrease by 1 the number of possible rests for each module ; we must multiply the cardinal $\pi\left(\dfrac{n}{2}\right)$ minorated by $\dfrac{\frac{n}{2}}{log\left(\frac{n}{2}\right)}$ (that corresponds to the condition (1) above) by the probability there would be a rest sharing modulo each prime number independently (that corresponds to the condition (2) above) and this probability has as value $-\zeta(-1) = \dfrac{1}{12}$. It's a set cardinal one obtains by this process of multiplying a set cardinal by a probability. Such a calculus seems to make sense and seems to ensure a cardinal equal at least to 1 above 92.

Bibliography

[1] J. B. Rosser et L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, dedicated to Hans Rademacher for his seventieth birthday, Illinois J. Math., Volume 6, Issue 1 (1962), 64-94.