Goldbach's conjecture and propositional logic (one-variable propositions)

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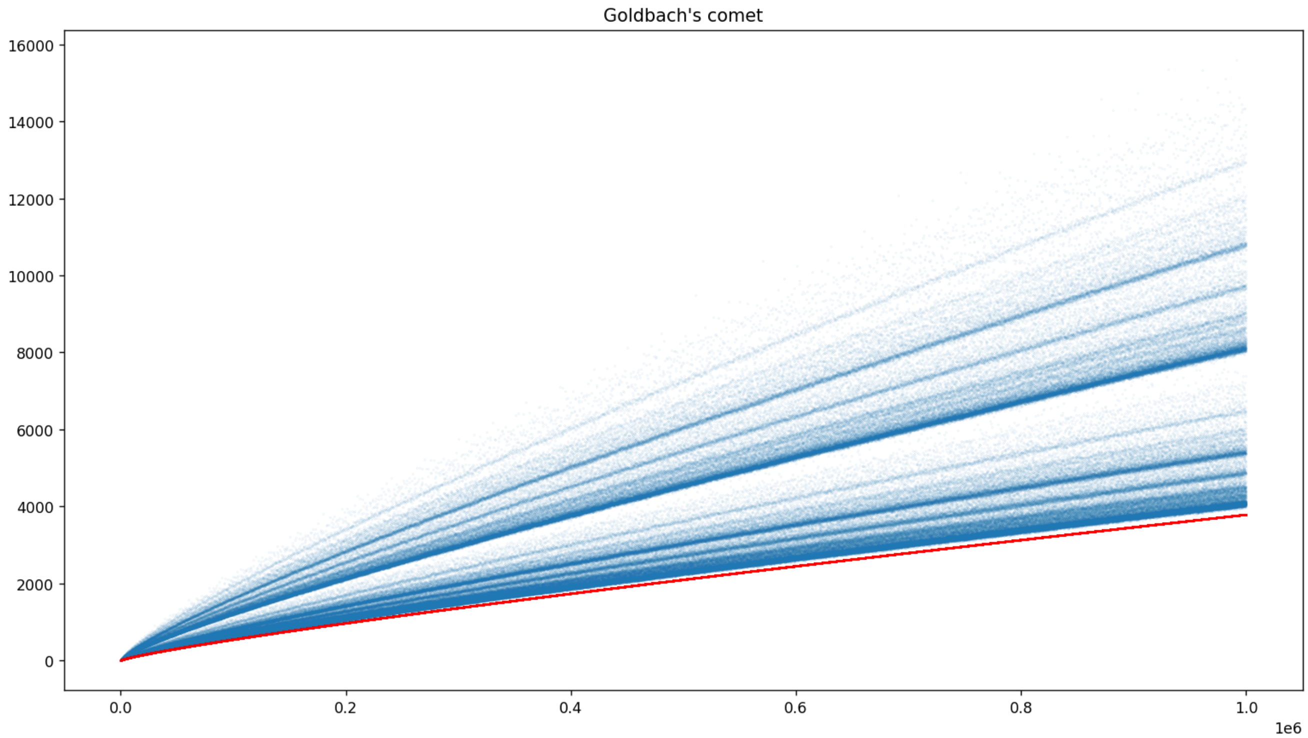
This note situates itself in propositional logic of propositions with one variable. We have true statements and false statements. If the variable z represents the statement '3+9=10', it takes the truth value $False$. If the variable z represents the statement '3+35=38', it takes the truth value $True$. In what follows, we will only be interested in statements that are true, of the form $n = p + q$ with n an even number, and p and q two odd numbers.

Among these statements, a certain number involve two prime numbers in place of p and q. Let us represent these statements in Figure 1. We have in the figure ordered the statements in the plan to fix the ideas, but we can quite imagine them "in bulk".

6= $3+3$	6= $5+1$	6= $7+(-1)$	6= $9+(-3)$	6= $11+(-5)$	6= $13+(-7)$	6= $15+(-9)$	6= $17+(-11)$
8= $3+5$	8= $5+3$	8= $7+1$	8= $9+(-1)$	8= $11+(-3)$	8= $13+(-5)$	8= $15+(-7)$	8= $17+(-9)$
10= $3+7$	10= $5+5$	10= $7+3$	10= $9+1$	10= $11+(-1)$	10= $13+(-3)$	10= $15+(-5)$	10= $17+(-7)$
12= $3+9$	12= $5+7$	12= $7+5$	12= $9+3$	12= $11+1$	12= $13+(-1)$	12= $15+(-3)$	12= $17+(-5)$
14= $3+11$	14= $5+9$	14= $7+7$	14= $9+5$	14= $11+3$	14= $13+1$	14= $15+(-1)$	14= $17+(-3)$
16= $3+13$	16= $5+11$	16= $7+9$	16= $9+7$	16= $11+5$	16= $13+3$	16= $15+1$	16= $17+(-1)$
18= $3+15$	18= $5+13$	18= $7+11$	18= $9+9$	18= $11+7$	18= $13+5$	18= $15+3$	18= $17+1$
20= $3+17$	20= $5+15$	20= $7+13$	20= $9+11$	20= $11+9$	20= $13+7$	20= $15+5$	20= $17+3$
22= $3+19$	22= $5+17$	22= $7+15$	22= $9+13$	22= $11+11$	22= $13+9$	22= $15+7$	22= $17+5$
24= $3+21$	24= $5+19$	24= $7+17$	24= $9+15$	24= $11+13$	24= $13+11$	24= $15+9$	24= $17+7$

FIGURE 1 : True statements of the form $even = odd + odd$

Let us now consider the Goldbach's conjecture. When we represent the comet of decompositions, what do the points of the comet represent? (assume that they are positioned at the integer points of an $\mathbb{N} \times \mathbb{N}$ mesh).



A blue point (x, y) of the comet represents that we have stated y true statements of the form '$x = p + q$' with x even, p and q two prime numbers (and $p \leqslant q$ as it happens). For example, for the number 98, which has 3 Goldbach decompositions $19 + 79$, $31 + 67$ and $37 + 61$, the point of the comet $(98, 3)$ represents the fact that among the multiple possibilities of statements in propositional logic representing the decompositions of 98 into a sum of two odd numbers, only 3 of them involve two prime numbers.

It should be borne in mind that the point $(98, 3)$ represents the Goldbach's decompositions of 98 among a multitude of other points of the vertical $x = 98$, on which one could position a multitude of other points which could mean for example '98 has 10 Goldbach decompositions' or even '98 has $\sqrt{2}$ Goldbach decompositions' (if we place ourselves in $\mathbb{R}^2$ rather than in $\mathbb{N}^2$).

The paper [1] provides an asymptotic limit to the proportion of univariate tautologies among univariate propositions. A reformulation of an excerpt from the abstract of this article is :

> "*For types with one propositional variable ground type formulae, we prove that the limit of the density (or asymptotical probability) of provable intuitionistic propositional formulae among all formulae exists and is equal to* $1/2 + \sqrt{5}/10$*, which is approximately equal to 72%. This means. that a large size random type formula is as likely as about 72% to be an inhabited formula (a tautology).*"

This ratio is equal to $\dfrac{1}{2}+\dfrac{\sqrt{5}}{10}=0.72360679775=\dfrac{1+\sqrt{5}}{2\sqrt{5}}$, the golden ratio divided by $\sqrt{5}$.

If we say that the points of the comet represent a certain number of tautologies among multiple possible one variable propositions, the ratio in question should understate the comet.

This is shown by the red curve, under the comet, which is the graph of the function $f(x)=\dfrac{1+\sqrt{5}}{2\sqrt{5}}\,\dfrac{x}{\ln x\cdot\ln x}$.

We have a problem with respect to the interpretation of the terms of the summary "*a random type (in terms of formula) of* a large size" : all our statements of the form *even* $=$ *odd* $+$ *odd* are of the same size (there are no statements involving multiple implications, even if they relate to one and only one variable). One could consider that the statement '100=3+97' is of size bigger than the statement '6=3+3' because, computationnaly speaking, the execution of the computation takes more time to perform (proving by computer that 97 is prime requires more time than proving that 3 is prime) but we rather thought that we had to be there in a kind of ideal world, where all statements of the form 'this number is prime' are given "at once". It is therefore difficult to know if (and via) which logical transposition of Goldbach's conjecture in the one single variable propositional logic the result of [1] is applicable.

However, if the interpretation of the comet and the transposition that we have proposed were valid, we would have established a nice "bridge", in the sense of Olivia Caramello, between arithmetic and one-variable propositional logic.

Note : we have deposited in a spreadsheet file of 25 mega the numbers of Goldbach decompositions of the even numbers from 6 to 10^6 as well as the value of the red minoring function here http://denise.vella.chemla.free.fr/dg-calc-1000000-dvc.pdf[1].

Références

[1] Malgorzata Moczurad, Jerzy Tyszkiewicz, Marek Zaionc, *Statistical properties of simple types.* Math. Struct. Comput. Sci. 10(5): 575-594 (2000).
French translation : http://denise.vella.chemla.free.fr/trad-Prop-stat-tautologies.pdf.
pdf article(postscript file is downloadable on M. Zaionk's personal page).
http://denise.vella.chemla.free.fr/Statistical$_{P}roperties_{o}f_{S}imple_{T}ypes.pdf$.

[2] Olivia Caramello, *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'*, Oxford University Press, 2017.

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[1] There are 32 exceptions to the lower bound for even numbers $\leqslant 10^6$. These exceptions concern numbers : $6=2\cdot3, 8=2^3, 12=2^2\cdot3, 38=2\cdot19, 68=2^2\cdot17, 98=2\cdot7^2, 128=2^7, 152=2^3\cdot19, 326=2\cdot163, 332=2^2\cdot183, 398=2\cdot199, 488=2^3\cdot61, 632=2^3\cdot79, 668=2^2\cdot167, 692=2^2\cdot173, 992=2^5\cdot31, 1112=2^3\cdot139, 1412=2^2\cdot353, 1718=2\cdot859, 2048=2^{11}, 2252=2^2\cdot563, 2642=2\cdot1321, 2672=2^4\cdot167, 2936=2^3\cdot367, 4412=2^2\cdot1103, 5468=2^2\cdot1367, 5948=2^2\cdot1487, 7508=2^2\cdot1877, 8042=2\cdot4021, 8048=2^4\cdot503, 8552=2^3\cdot1069, 9602=2\cdot4801$.