4. Typical Subobject Classifiers

Each of our typical categories (i)–(xiv) has a subobject classifier. We will now explicitly construct these classifiers in order to exemplify the general notion.

The classifier true: $1 \to 2$ for **Sets** is evidently also a subobject classifier for **FinSets**; indeed, the usual characteristic functions are still effective for finite sets.

In **Sets** $\times$ **Sets**, an arrow is a pair of functions $f\colon Y \to X$, $f'\colon Y' \to X'$. The pair of subsets $(1 \subset 2, 1 \subset 2)$ is a subobject classifier, and the characteristic arrow of any subobject $(S \subset X, S' \subset X')$ is evidently just the pair of characteristic functions $(\phi_S\colon X \to 2, \phi_{S'}\colon X' \to 2)$ from the category **Sets**. Thus, there are, in 2×2, four "truth-values". The corresponding subobject classifier for $\mathbf{Sets}^n$ has 2^n truth-values; as we shall see, it is the Boolean algebra of all 2^n subsets of n.

In the category $\mathbf{B}G = G$-**Sets** of representations of G [Example (iv) of §1], an object is an action $X \times G \to X$ of the fixed group G on some set X, and a subobject is just a subset $S \subset X$ closed under this action (i.e., $s \cdot g \in S$ whenever $s \in S$ and $g \in G$). The complement of S in X is thus also invariant under this action, so we can still use the ordinary characteristic function $\phi_S : X \to 2$ of S, where the subobject classifier is the usual map true: $1 \to 2$, with G acting trivially on both sets 1 and 2. Exactly the same argument applies in the case where G is a topological group [Example (xi) of §1].

For $\mathbf{B}M$ [Example (v) of §1], an object is again a right action $X \times M \to X$ of the fixed monoid M on some set X, and a subobject is again just a subset $S \subset X$ closed under this action, but the previous characteristic function will not do because the complement of S need not be closed under this action. Instead, we may define a function ϕ_S sending each $x \in X$ to the set L of all those $\ell \in M$ with $x \cdot \ell \in S$. This set L is a "right ideal" of M (a subset of M mapped into itself by the right action of M on itself, via right multiplication). Therefore, take $\Omega = \Omega_M$ to be the set of all right ideals L of M with action $\Omega \times M \to \Omega$ defined by $L{\cdot}m = \{\, k \in M \mid m{\cdot}k \in L \,\}$ for $L \in \Omega$ and $m \in M$. Then, the function ϕ_S above is an arrow $\phi_S\colon X \to \Omega$; in particular, it determines the given S as the inverse image of the right ideal M. Therefore, the subobject classifier is the function $\mathrm{true}_M\colon 1 \rightarrowtail \Omega_M$ which sends the one point of the object 1 to the "maximal" right ideal $M \in \Omega_M$.

In case M is a group G the only right ideals are G and $\emptyset$, so this Ω_G reduces to the previous set 2 with trivial G-action. In case M is the additive monoid of natural numbers, the right ideals are the empty set and the sets of numbers larger than some fixed number n.

For the arrow category **2** and $\mathbf{Sets}^{\mathbf{2}}$, a subset $(S_0 \xrightarrow{\sigma} S_1) \rightarrowtail (X_0 \xrightarrow{\sigma} X_1)$ is a pair of subsets $S_0 \subset X_0$, $S_1 \subset X_1$ with $\sigma S_0 \subset S_1$. Relative to this subset S there are three sorts of elements x of X_0: Those x in S_0, those $x \notin S_0$ with $\sigma x \in S_1$, and those x with $\sigma x \notin S_1$. Define $\phi_0 x = 0$, 1, or 2 accordingly. Then, ϕ_0 on S_0, with the usual characteristic function ϕ_1 of $S_1 \subset X_1$, is an arrow $\phi = (\phi_0, \phi_1)$ to the object Ω displayed below,

$$\begin{array}{llcl} X\colon & X_0 & \xrightarrow{\ \sigma\ } & X_1 \\ \phi\downarrow & \phi_0\downarrow & & \downarrow\phi_1 \\ \Omega\colon & \{0,1,2\} & \xrightarrow[\sigma]{} & \{0,1\}, \end{array} \qquad \sigma 0 = 0,\ \sigma 1 = 0,\ \sigma 2 = 1,$$

in $\mathbf{Sets}^{\mathbf{2}}$, and $S_0 \to S_1$ is the inverse image of $(\{0\} \xrightarrow{1} \{0\}) = 1 \rightarrowtail \Omega$.

In brief, this characteristic function $\phi = \langle \phi_0, \phi_1 \rangle$ is that arrow which specifies whether "x is in S" is "true" always, only at 1, or never. One may say that ϕ gives the "time till truth".

For $\mathbf{Sets}^{\mathbf{N}}$, a subobject of X has the form of a sequence S of subsets

$$\begin{array}{lccccccc} S\colon & S_0 & \longrightarrow & S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 \cdots \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & X_0 & \xrightarrow[\sigma]{} & X_1 & \xrightarrow[\sigma]{} & X_2 & \xrightarrow[\sigma]{} & X_3 \cdots \end{array}$$

with $\sigma S_k \subset S_{k+1}$; for example, if X_k is constant and each $\sigma = 1$, this S is a monotone increasing sequence of subsets. For any $x \in X_k$ we can then measure the "time till truth" (the time till inclusion in S) by the function ϕ_k on X_k defined as

$$\begin{aligned} \phi_k x = {} & \text{ the least } n \text{ with } \sigma^n x \in S_{k+n}, \text{ if such exists,} \\ & = \infty \text{ otherwise.} \end{aligned}$$

Then $\phi_k\colon X_k \to \mathbf{N} + \{\infty\}$, so the sequence of these maps ϕ_k is an arrow to the sequence of sets

$$\Omega\colon \mathbf{N} + \{\infty\} \xrightarrow{\tau} \mathbf{N} + \{\infty\} \xrightarrow{\tau} \mathbf{N} + \{\infty\} \longrightarrow \cdots \tag{1}$$

where each τ has $\tau(0) = 0$, $\tau(n+1) = n$ for $n \neq 0$ and $\tau(\infty) = \infty$. Then, $\Omega \in \mathbf{Sets}^{\mathbf{N}}$ has $1\colon \{0\} \to \{0\} \to \{0\} \to \cdots$ as subobject, and the given S is the pullback of 1 along ϕ. In brief, "time till truth" provides a subobject classifier Ω.

For an arbitrary small category $\mathbf{C}$, a *subfunctor* of $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ is defined to be another functor $Q\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ with each QC a subset of PC and each $Qf\colon SD \to SC$ a restriction of Pf, for all arrows $f\colon C \to D$ of $\mathbf{C}$. The inclusion $Q \to P$ is then a monic arrow in the functor category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, so that each subfunctor Q is a subobject. Conversely, all subobjects are given by subfunctors; if a natural transformation $\theta\colon R \rightarrowtail P$ is monic in the functor category, then each function $\theta C\colon RC \to PC$ is an injection (monics, like limits in the functor category, are taken pointwise). For each C let QC be the image of $RC \rightarrowtail PC$; thus Q is manifestly a subfunctor of P, and the given R is equivalent (as a subobject) to Q.

For an arbitrary presheaf category $\widehat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, if there is a subobject classifier Ω, it must, in particular, classify the subobjects of each representable presheaf $\mathbf{y}C = \mathrm{Hom}_{\mathbf{C}}(-,C)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$. Therefore,

$$\mathrm{Sub}_{\widehat{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(-,C)) \cong \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(-,C),\Omega) = \mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-,C),\Omega$$

By the Yoneda lemma [see §1(6) above], the set on the right is (up to isomorphism) $\Omega(C)$. Thus the subobject classifier Ω, if it exists, must be the functor $\Omega\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ with object function

$$\begin{aligned}\Omega(C) &= \mathrm{Sub}_{\widehat{\mathbf{C}}}(\mathrm{Hom}_{\mathbf{C}}(-,C)) \\ &= \{\, S \mid S \text{ a subfunctor of } \mathrm{Hom}_{\mathbf{C}}(-,C)\,\},\end{aligned} \tag{2}$$

and with a suitable mapping function.

To understand this it is customary and useful to use an alternative terminology for subfunctors of a representable functor $\mathrm{Hom}(-,C)$. Given an object C in the category $\mathbf{C}$, a *sieve* on C (in French, a "crible" on C) is a set S of arrows with codomain C such that

$$f \in S \text{ and the composite } fh \text{ is defined implies } fh \in S.$$

If we think of the arrows $f \in S$ as those paths which are "allowed to get through" to C, this definition means that any path to some other B followed by an allowed path from B to C is allowed. For example, if the category $\mathbf{C}$ is a monoid M, a sieve is just a right ideal in M; if the category $\mathbf{C}$ is a partially ordered set regarded as a category, a sieve on $C \in \mathbf{C}$ is a set S of elements $B \leq C$ such that $A \leq B \in S$ implies $A \in S$: If B "goes through" the sieve, so does anything smaller: a sieve is a "downwards closed" subset.

Now if $Q \subset \mathrm{Hom}_{\mathbf{C}}(-,C)$ is a subfunctor, the set

$$S = \{\, f \mid \text{ for some object } A,\ f\colon A \to C \text{ and } f \in Q(A)\,\}$$

is clearly a sieve on C. Conversely, given a sieve S on C, the definition

$$Q(A) = \{\, f \mid f\colon A \to C \text{ and } f \in S\,\} \subseteq \mathrm{Hom}_{\mathbf{C}}(A,C)$$

yields a functor $Q\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ which is a subfunctor of the Hom-functor $\mathrm{Hom}_{\mathbf{C}}(-,C)$. The passages S to Q and Q to S are reciprocal;

hence, we can identify sieves and subfunctors in any locally small category $\mathbf{C}$. Thus,

$$\text{Sieve on } C = \text{Subfunctor of } \mathrm{Hom}_{\mathbf{C}}(-,C). \tag{3}$$

Moreover, for any arrow $g\colon B \to C$, a subobject Q of the functor $\mathrm{Hom}_{\mathbf{C}}(-,C)$ determines a subobject of $\mathrm{Hom}_{\mathbf{C}}(-,B)$ by pullback along g, and similarly each sieve S on C determines the following sieve on B:

$$S \cdot g = \{\, h \mid g \circ h \in S \,\}.$$

With this motivation, the proposed subobject classifier Ω for the functor category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is defined on objects by

$$\Omega(C) = \{\, S \mid S \text{ is a sieve on } C \text{ in } \mathbf{C} \,\} \tag{4}$$

and on arrows $g\colon C' \to C$ by

$$(-)\cdot g\colon \Omega(C) \to \Omega(C'), \qquad S \cdot g = \{\, h \mid g \circ h \in S \,\}. \tag{5}$$

For an object C of $\mathbf{C}$, the set $t(C)$ of *all* arrows into C is a sieve, called the *maximal sieve* on C. These maximal sieves patch together to give a morphism (natural transformation)

$$\text{true}\colon 1 \to \Omega \tag{6}$$

in the presheaf category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

To see that (6) defines a subobject classifier in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, consider any subfunctor Q of a given functor $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$. Then each morphism $f\colon A \to C$ in $\mathbf{C}$ determines a function $P(f)\colon P(C) \to P(A)$ in $\mathbf{Sets}$ which may or may not take a given $x \in P(C)$ into $Q(A) \subseteq P(A)$. For a given $x \in P(C)$ set

$$\phi_C(x) = \{\, f \mid x \cdot f \in Q(\mathrm{dom}(f)) \,\}, \tag{7}$$

where f ranges over all morphisms in $\mathbf{C}$ with codomain C. Then $\phi_C(x)$ is a sieve on C, and $\phi\colon P \to \Omega$ is natural. Moreover, $\phi_C(x)$ is the maximal sieve $t(C)$ iff $x \in Q(C)$, so the given subfunctor $Q \subseteq P$ is the pullback along ϕ of the map "true" defined in (6) above.

$$\begin{array}{ccc} Q & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \text{true}} \\ P & \xrightarrow[\phi]{} & \Omega. \end{array} \tag{8}$$

This shows that ϕ is indeed a possible characteristic map for the subfunctor Q. But this ϕ is also the unique natural transformation $\theta\colon P \to \Omega$ making this diagram into a pullback. Indeed, given $x \in P(C)$ and $f\colon A \to C$, the pullback condition means that $x \cdot f \in Q(A)$ iff $\theta_A(x{\cdot}f) = \mathrm{true}_A$; by naturality of θ, this is equivalent to $\theta_C(x){\cdot}f = \mathrm{true}_A$ and this, in turn, by the definition (5), means that $f \in \theta_C(x)$. The elements f of $\theta_C(x)$ are thus exactly those f with $x \cdot f \in Q(A)$, i.e., those $f \in \phi_C(x)$ as defined in (7). Thus, the definition (7) of ϕ is forced upon us if (8) is to be a pullback. Hence, we have shown that the mono $\mathrm{true}\colon 1 \rightarrowtail \Omega$ defined in (6) provides a subobject classifier for the presheaf category $\widehat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

Intuitively, the sieve $\phi_C(x)$ considered in (7) is the set of all those paths f to C which translate the element x of $P(C)$ into the subfunctor Q. As the set of "paths to truth", it clearly agrees with the characteristic arrows we have already constructed for the special functor categories $\mathbf{Sets}^{\mathbf{2}}$, $\mathbf{Sets}^{M}$, and $\mathbf{Sets}^{\mathbf{N}}$.

We have assumed $\mathbf{C}$ small because we must. Were $\mathbf{C}$ large—say the ordered set of all small ordinal numbers—the number of paths to truth would not in general be small, hence not an object of **Sets**.

The exhibition of subobject classifiers for our typical categories is completed by noting, for any set J, that the projection $J \times 2 \to J$ is a classifier for the category $\mathbf{Sets}/J$, while in $\mathbf{FinSets}^{\mathbf{C}^{\mathrm{op}}}$ with $\mathbf{C}$ finite, the set $\Omega(C)$ of sieves on C is again finite so provides a suitable subobject classifier Ω.

Observe, however, that there are many "reasonable" categories with no such subobject classifier. The category $(\mathbf{FinSets})^{\mathbf{N}}$ provides an immediate such example, because in the linearly ordered set $\mathbf{N}^{\mathrm{op}}$, the number of sieves on each object n is infinite. Another example is the category **Ab** of all (small) abelian groups. For, the terminal object 1 in **Ab** is the zero-group, so the group homomorphism $\mathrm{true}\colon 1 \rightarrowtail \Omega$ must send 0 to $0 \in \Omega$, and thus its pullback along any $\phi\colon A \to \Omega$ is the subgroup $S = \operatorname{Ker}\phi = \phi^{-1}(0)$ of A. This implies that the proposed subobject classifier must be an abelian group which contains a copy of every quotient group A/S of every group A, an absurdity.

5. Colimits

Each of our typical categories has all finite colimits. To show this it suffices (as in the dual case of finite limits discusssed in §2) to observe that each has an initial object 0 and pushouts (or cocartesian squares, as they are sometimes called). In **Sets**, the empty set $\emptyset$ is an initial object because there is for each set X exactly one function $\emptyset \to X$; in a functor category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ the constantly empty functor is initial. And for a topological group G, the empty set (with its unique action by G)