Goldbach's Conjecture, chip-firing game, 2×2 matrices and infinite descent

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Résumé

A modelisation based on a two letters language and on 2×2 counting matrices inspired by chip-firing games allows us to approach a Goldbach's conjecture demonstration.

1 Introduction

We propose here another attempt to demonstrate Goldbach's conjecture.

In the following, we focus on even integers n decompositions of the form $p + (n-p)$ in sum of two odd numbers greater than 3 and for what p is a prime number smaller than or equal to $n/2$.

Our proposition is based on the following elements :

- a two letters language $\mathcal{L}$ based on the alphabet $\mathcal{A} = \{a, b\}$. The letter a codes decompositions of the form $p + (n-p)$ such that $n-p$ is a prime number ; the letter b codes decompositions of the form $p + (n-p)$ such that $n-p$ is compound ;
- transition 2×2 matrices with integer coefficients * ; each matrix integrates in its coefficients some knowledge on n's decompositions as well as some knowledge on $n+2$'s decompositions, we will see how ;
- chip-firing game notion [1] : we won't use here results from this notion theory, but it has been useful for its suggestive potential from one part, and also, it will permit to understand easily how takes place the passage between the matrix associated to even number n to the one associated to even number $n+2$.

2 Examples

To fix ideas, let us begin by studying two examples :

- to the even number 30 is associated matrix $M_{30} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

$*$. See for instance A. Connes, Noncommutative geometry year 2000, http://alainconnes.org/docs/2000.pdf, [2], p.8, to find matrices such that $f_{ab}(n) \neq f_{ba}(n)$ but in this context, matrices are operators. This idea has been of a great suggestive potential for us.

- to the even number 32 is associated matrix $M_{32} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.

To think the transition between M_{30} and M_{32} in terms of chip-firing game consists to tell oneself that the 2 at right top of M_{30} "has given" 1 (chip) to the 1 at left bottom of M_{30}, and this has made this element at left bottom pass from 1 to 2, and has made itself at right top pass from 2 to 1.

Let us explain to what correspond the 4 elements of M_{30} matrix and the 4 elements of M_{32} matrix.

	p	$n-p$	$(n+2)-p$	*doublon*	
$n=30$	3	27	29	*ba*	$\begin{pmatrix} aa\ 1 & ab\ 2 \\ ba\ 1 & bb\ 1 \end{pmatrix}$
	5	25	27	*bb*	
	7	23	25	*ab*	
	11	19	21	*ab*	
	13	17	19	*aa*	

	p	$n-p$	$(n+2)-p$	*doublon*	
$n=32$	3	29	31	*aa*	$\begin{pmatrix} aa\ 1 & ab\ 1 \\ ba\ 2 & bb\ 1 \end{pmatrix}$
	5	27	29	*ba*	
	7	25	27	*bb*	
	11	21	23	*ba*	
	13	19	21	*ab*	

3 Modelisation

As we can see in those examples, each matrix integrates in its coefficients a certain knowledge on n's decompositions (in left letters of couples of letters that its coefficients count) as well as a certain knowledge on $n+2$'s decompositions (in right letters of couples of letters that its coefficient count).

Let us write formally what the 4 elements of each 2×2 matrix count :

$$M_n \begin{pmatrix} f_{aa}(n) & f_{ab}(n) \\ f_{ba}(n) & f_{bb}(n) \end{pmatrix}$$

with :

- $f_{aa}(n) = \#\{(p+q=n) \wedge (p\ prime) \wedge (3 \leq p \leq n/2) \wedge (q\ prime) \wedge (((n+2)-p)\ prime)\}$;
- $f_{ab}(n) = \#\{(p+q=n) \wedge (p\ prime) \wedge (3 \leq p \leq n/2) \wedge (q\ prime) \wedge (((n+2)-p)\ compound)\}$;
- $f_{ba}(n) = \#\{(p+q=n) \wedge (p\ prime) \wedge (3 \leq p \leq n/2) \wedge (q\ compound) \wedge (((n+2)-p)\ prime)\}$;
- $f_{bb}(n) = \#\{(p+q=n) \wedge (p\ prime) \wedge (3 \leq p \leq n/2) \wedge (q\ compound) \wedge (((n+2)-p)\ compound)\}$.

It can be easily understood that the sum $f_{ba}(n) + f_{bb}(n)$ counts the couples $(x, x+2)$ such that $x \geq n/2$ et $n-x$ is prime.

Sum of the 4 elements of matrices is equal to $\pi(n/2) - 1$ (with $\pi(x)$ the usual notation for the number of prime numbers smaller than or equal to x), since we focus only on even numbers'decompositions that are "based" on a small sommant that is a prime number.

This sum is increased by 1 at each even number that is twice a prime.

We provide in annex matrices associated to even numbers between 10 and 100.

4 Infinite descent

The number of n's Goldbach decompositions is equal to $f_{aa}(n) + f_{ab}(n)$. Demonstrating Goldbach's conjecture is the same as demonstrating that $f_{aa}(n) + f_{ab}(n) > 0$ is always true.

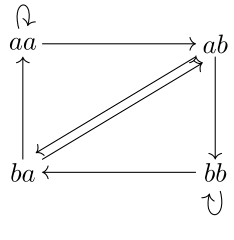
Let us try now to break down the process that is at work during the passing from the even number n to the following even number $n+2$ in terms of each element of matrix M_n associated to n taken separately.

Let us refer to examples matrices M_{30} and M_{32} observing the column containing the couples of letters at the right of decompositions.

If a number would have no Goldbach's decomposition, we should have $f_{aa}(n) = f_{ab}(n) = 0$. Only $f_{ba}(n)$ and $f_{bb}(n)$ would be different from zero. Then, there would be, at right of the matrix M_n, only couples of lettres *ba* and *bb*. But we saw that left letters of couples of letters of $n+2$ and right letters of couples of letters of n are the same (since those letters code the same decompositions).

Let us specify now the chips circulation, in terms of chip-firing game, we have that (*or* are *inclusive or*) :
- *ba* spills in *aa* or in *ab* ;
- *bb* spills in *ba* or in *bb* ;
- *aa* spills in *aa* or in *ab* ;
- *ab* spills in *ba* or in *bb*.

We can represent this chips circulation on the diagram below :



Observing the way "chips circulate" in the diagram, we understand that the only way to obtain a matrix M_n such that $f_{aa}(n)$ and $f_{ab}(n)$ are both equal to zero consists in going from a matrix M_{n-2} such that $f_{aa}(n-2)$ and $f_{ab}(n-2)$

are also both equal to zero.

The contradiction † comes from a kind of reasoning called Fermat infinite descent : we saw that if Goldbach's conjecture wasn't verified by an even number, it wouldn't be verified no more by the even number just before. However, there is no infinitely decreasing sequence of positive integers that verify simultaneously a same property (here for an even number to decompose itself in a sum of two odd prime numbers, such a property for which we recall that it is true for all even numbers between 6 and 100). The set $\mathbb{N}$ of positive natural numbers and all its not empty proper parts have a remarkable property : they admit a smallest element. We made a reductio ad absurdum reasoning : supposing that the matrix associated to n is such that $f_{aa}(n) = f_{ab}(n) = 0$, we saw it was necessary that the matrix associated to $n-2$ should be also such that $f_{aa}(n-2) = f_{ab}(n-2) = 0$. But $n-2 < n$. So we are led to a contradiction by infinite descent and such a reasoning should ensure that Goldbach's conjecture is true.

By the only game of letters manipulations, one has effectively the sensation of something *that turns*, that is to say something that is as *circulating in time*.

Annex 1 : 2×2 matrices associated to even numbers between 10 and 100

10 $\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$	12 $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	14 $\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$	16 $\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$	18 $\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$
20 $\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$	22 $\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$	24 $\begin{pmatrix}1 & 2\\1 & 0\end{pmatrix}$	26 $\begin{pmatrix}0 & 3\\2 & 0\end{pmatrix}$	28 $\begin{pmatrix}1 & 1\\2 & 1\end{pmatrix}$
30 $\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}$	32 $\begin{pmatrix}1 & 1\\2 & 1\end{pmatrix}$	34 $\begin{pmatrix}2 & 2\\2 & 0\end{pmatrix}$	36 $\begin{pmatrix}1 & 3\\0 & 2\end{pmatrix}$	38 $\begin{pmatrix}0 & 2\\3 & 2\end{pmatrix}$
40 $\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}$	42 $\begin{pmatrix}1 & 3\\2 & 1\end{pmatrix}$	44 $\begin{pmatrix}1 & 2\\2 & 2\end{pmatrix}$	46 $\begin{pmatrix}2 & 2\\3 & 1\end{pmatrix}$	48 $\begin{pmatrix}2 & 3\\2 & 1\end{pmatrix}$
50 $\begin{pmatrix}0 & 4\\3 & 1\end{pmatrix}$	52 $\begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}$	54 $\begin{pmatrix}1 & 4\\2 & 1\end{pmatrix}$	56 $\begin{pmatrix}0 & 3\\3 & 2\end{pmatrix}$	58 $\begin{pmatrix}2 & 2\\4 & 1\end{pmatrix}$

†. We would have wished initially to establish the contradiction from the fact that, since the overall sum of the 4 elements of each matrix is equal to $\pi(n/2)-1$, and since we have seen that the sum $f_{ba}(n)+f_{bb}(n)$ counts couples $(x, x+2)$ such that $n-x$ is prime, if only $f_{ba}(n)$ and $f_{bb}(n)$ were equal to zero, it would significate that there are as many primes between 3 and $n/2$ that there are primes between $n/2$ and n, that would be impossible, the number of primes going decreasing on two successive intervals of the same length. But we didn't find in literature a result demonstrating that $\pi(n)-\pi(n/2) < \pi(n/2)$.

60 $\begin{pmatrix}1 & 5\\1 & 2\end{pmatrix}$	62 $\begin{pmatrix}1 & 2\\4 & 3\end{pmatrix}$	64 $\begin{pmatrix}2 & 3\\4 & 1\end{pmatrix}$	66 $\begin{pmatrix}1 & 5\\1 & 3\end{pmatrix}$	68 $\begin{pmatrix}0 & 2\\5 & 3\end{pmatrix}$
70 $\begin{pmatrix}2 & 3\\4 & 1\end{pmatrix}$	72 $\begin{pmatrix}2 & 4\\2 & 2\end{pmatrix}$	74 $\begin{pmatrix}1 & 4\\4 & 2\end{pmatrix}$	76 $\begin{pmatrix}2 & 3\\5 & 1\end{pmatrix}$	78 $\begin{pmatrix}3 & 4\\1 & 3\end{pmatrix}$
80 $\begin{pmatrix}0 & 4\\4 & 3\end{pmatrix}$	82 $\begin{pmatrix}3 & 2\\5 & 2\end{pmatrix}$	84 $\begin{pmatrix}1 & 7\\3 & 1\end{pmatrix}$	86 $\begin{pmatrix}0 & 5\\4 & 4\end{pmatrix}$	88 $\begin{pmatrix}2 & 2\\7 & 2\end{pmatrix}$
90 $\begin{pmatrix}2 & 7\\2 & 2\end{pmatrix}$	92 $\begin{pmatrix}0 & 4\\4 & 5\end{pmatrix}$	94 $\begin{pmatrix}1 & 4\\6 & 3\end{pmatrix}$	96 $\begin{pmatrix}1 & 6\\2 & 5\end{pmatrix}$	98 $\begin{pmatrix}0 & 3\\6 & 5\end{pmatrix}$
		100 $\begin{pmatrix}2 & 4\\6 & 2\end{pmatrix}$		

Annex 2 : Python program for matrices computing

```
import math
from math import *

def prime(atester):
  pastrouve = True
  k = 2
  if (atester == 1): return False
  if (atester == 2): return True
  if (atester == 3): return True
  if (atester == 5): return True
  if (atester == 7): return True
  while (pastrouve):
    if ((k * k) > atester):
        return True
    else:
        if ((atester % k) == 0):
            return False
        else: k=k+1
```

```
for n in range(10,102,2):
  aa = 0 ; ab = 0 ; ba = 0 ; bb = 0 ;
  for x in range(3,n/2+1,2):
    if (prime(x)):
      if ((prime(n-x)) and (not(prime((n+2)-x)))):
        ab=ab+1
      elif ((prime(n-x)) and (prime((n+2)-x))):
        aa=aa+1
      elif ((not(prime(n-x))) and (prime((n+2)-x))):
        ba=ba+1
      elif ((not(prime(n-x))) and (not(prime((n+2)-x)))):
        bb=bb+1
  s=str(n)+'--->'
  print(s)
  s=str(aa)+' '+str(ab)
  print(s)
  s=str(ba)+' '+str(bb)
  print(s)
```

Bibliography

[1] : A. Björner, L. Lovász, P. W. Shor : Chip-firing games on graphs. European Journal of Combinatorics archive, Volume 12 Issue 4, July 1991, Pages 283–291.

[2] : A. Connes, Noncommutative geometry year 2000, In : Alon N., Bourgain J., Connes A., Gromov M., Milman V. (eds) Visions in Mathematics. Modern Birkhäuser Classics. Birkhäuser Basel, 2000, p.481-559.