Mathematics and thinking in motion

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The purpose of my talk is to make you feel two things : the first one is that I am going to tell you a certain number of stories, on mathematicians, and the second is to make you understand that mathematics are a factory of concepts, but of concepts absolutely fundamentals and concepts that relate, if you will, to life, and that are not at all confined to calculations with numbers, or things like it ; too often we get the impression that the mathematician is someone who does calculations ; of course, he happens to do calculations, but what I'm going to try to make you understand, precisely, in this talk, is that the mathematical technique emerges from time to time on fundamental concepts, on fundamental ideas, and these are ideas that can be explained simply and that relate to life, that is to say that they are also important, I think, to people who deal with humanities and not only for people who are going to do hard sciences. So, there will be a portrait gallery. We will start with Galois.

And if you want, Galois, it's the prototype of the mathematician who had an absolutely incredible life : he was born in 1811, and he was 17 years old when he found its most important things. And what happened then, there was a succession of misunderstandings, in fact. If you like, in 1829, Abel died. And basically, it's Galois who takes up the torch of Abel's ideas. But in fact, I learned this from specialists of Abel. Abel had come in Paris, but it is absolutely impossible that he met Galois : Galois was too young when Abel came to Paris ; I always imagined that they had met in a Parisian cafe, where they had both discussed. Apparently this is not possible. So when he was 17, Cauchy who was an academician, had already done in 1829 two talks on the works de Galois, at the Academy, in May and June. That was so in 1829. And in July 1829, Galois' father committed suicide because he had been the victim of a slander campaign which had been carried out against him, and in addition, Galois fails for the second time at the Polytechnic School. So

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it was the second time that he appeared at the Polytechnic School. At that time, the Polytechnic School was at the top of the Grandes Écoles, it was the second time he failed. This is where there was the scene apparently where he threw the chalk cloth over the head of the math examiner, because the examiner did not understand Galois'explanations for the logarithm.

Fortunately, Galois was received at the teacher training college. And in January 1830, there is a letter from Cauchy to the Academy which says that he will speak to Galois. So it would be the third time, and then finally Cauchy gives up, and I think, well, we think, and historians think, if you like, that it was agreed with Galois because there was the Big Price of the Academy which was to be given in 1830 and Cauchy had convinced Galois to rewrite his article, and to run for this grand prize. Then, what happened was absolutely dramatic because the academician who was to report on Galois' article, it was Joseph Fourier. It is a very very great mathematician and Fourier apparently he was at the top of his stairs at home, he got his feet in his dressing gown and he tumbled the stairs, he died. So big problem, big problem, and if you want, there was such a mess at that time that the Galois manuscript was lost. So not only Galois did not get the price he might have deserved, the prize was given to Jacobi and Abel, of course two immense mathematicians. Jacobi was a German mathematician, Abel was dead, he died in 1829, the prize was given to Abel posthumously.

But if you want, Galois couldn't complain about not having had its great price ; on the other hand, he could complain, at the time, there was no photocopier. So he had written his manuscript ; at the time, you were writing the manuscript and then it was over. He gave it to the Academy but handwritten lost. So he had complained several times to the Academy, but manuscript lost. And so in 1830, the grand prize was given in June 1830 and in July 1830, there were the Three Glorious Years. It's Les Trois Glorieuses, and Galois was at Normal School and there he was bitching because, in Normal School, the students were confined, they could not go on the barricades. On the other hand, the students of the Polytechnic School, them, they could, so there, Galois started to really revolt. It's very, very weird, if you will, well, he had barely 18 years old. So he started to rebel and he rebelled against the director of the teacher training college. And after the summer, therefore, he started to campaign more or less, and little by little, he managed to get kicked out of Normal School. So he was expelled from the Normal School in January 1831, and there are something incredibly ironic, which is that Galois was on the street, if you want, he didn't have a salary anymore because at the time, and that's still the case now, the students of the Normal School received a small salary. So he was on the street and then to earn some money, he had created an algebra course, a course which brought together a number of people who came to listen to him because he was a magnificent mathematician despite his very young age. And the total irony is that its course of algebra, it gave it in the street which now is a street adjoining the Sorbonne, which is called rue Victor Cousin. Why is it ironic ? It is ironic because the person who signed the dismissal from the Normal School of Galois is called Victor Cousin. So a few years ago, for the 200 years from the birth of Galois, I had to give the talk at the Academy of Sciences on Galois. And at that time, I wanted everyone to agree to rename rue Victor Cousin rue Galois. Well, it was not possible, but it should still be incredible.

So this is what happened. So after, therefore, it must be said that Galois died at 20 years old. And the last two years of his life, he didn't a lot of math. It's incredible, it's absolutely incredible. And what happened is that once he was expelled from the Normal School, there still had been another academician who wished him well, his name was Poisson. And in mathematics, there is a well-known formula called the Poisson formula. And if you want, Poisson had convinced him to rewrite his manuscript and to present it to the Academy. So Galois had executed. He had rewritten his manuscript. He had worked, etc. And in the meantime, of course, after The Three Glorious, everyone started to be extremely disappointed with the new power. And Galois was one of those people. So the first thing he did was not very, not very clever, well. It was at a banquet celebrating the release of opponents of power. So what, he was at this banquet, and he had raised his glass to Louis-Philippe. So, all the people were saying "He's completely crazy !" : he was at a banquet against Louis-Philippe and he raised his glass to Louis-Philippe. And in the hand, he had a knife. First, people didn't understand why he was raising his glass at Louis-Philippe ; secondly, there was a spy who was there and who saw that he had a knife in his hand. He had been arrested, that was in the month of May 1831 he had been arrested, and had been judged fairly quickly. He had been judged by a popular jury. But as he had been judged by a popular jury, the people had seen it was a little weird, well, well, I mean, it didn't defend himself, basically, he said... So they acquitted him. I believe he had been acquitted in June 1831. And a month later, he received

Poisson's report on his article. So there, disaster because Poisson said it was a very very nice safe-lying theory, but that there weren't enough details in demonstrations, etc. So he couldn't accept the article. And Galois, when he received this report, he wrote by hand, in the margin of the report, he writes "Oh, cherubs !". It means he saw that people didn't understand nothing he did. At that time, he defended himself a little, that is to say, there, he got arrested. It was on July 4 that he received Poisson's report, he was arrested on July 14 at the head of a demonstration against Louis-Philippe. And there he was put in prison for good ; he was put in a prison which is called Sainte-Pélagie ; and there are a lot of you probably, who imagine that if they were in prison, they could at least think quietly with books ; in fact, it was not like that at all, because Galois, he was among the condemned and it was absolutely terrible because that the other condemned forced him to drink very strong liquor, etc. ; I mean it was absolutely orthogonal to his... what he was doing and in fact there, he met Nerval. Nerval met him while he was in prison.

And then, it's terrible, it's terrible, because if you want, Galois remained in prison until March of the following year, 1832. He was not 20 years old. And in March 1832, the reason for which he was released is that there was cholera in Paris. And they emptied prisons so that there's not too much damage. So he was put in a nursing home and in this nursing home, he more or less fell in love with a girl who was there, without realizing that she was already with someone else.

Well, it all ended in a duel, okay. And then there, it's the same, if you want, I guess each of you would imagine that if he were on duty to fight in a duel, he would have more skill than the opponent, so it would be fine, he would get out of it. Unfortunately, the duel in which Galois was caught, he tried to get out of it before. He tried to say that... But unfortunately, it was an absolutely terrible duel, it was like russian roulette, it was a duel in which there were two revolvers of which only one of the two was charge. And they had to put them on their stomachs. So he had, of course, a bullet in the stomach. Back then, and even now, it was deadly, and the others left him there.

He was found by a peasant on the spot, who took him to the hospital and he died the day after. Well. And he left a wad of papers, that's what he says, that's what he says in his stuff, so it was... So there are people who will make you believe that he found all his results the day before his duel. It is absolutely not true, I mean, obviously he had kept thinking and it was to the point... he had to force himself so much to keep doing math while he was in awful circumstances, that people who saw him when he got out of prison said he looked 50 years old then that he was 20 years old. Okay, so that's to tell you a little bit what the passion that inhabits him was, and it's a miracle, finally, it's a miracle that we had his work.

It is an absolute miracle that we had his work. So that's what he wrote and that he left in his letter-will. It's his testament that he had left to his brother, and to his friend, he had a friend too.

And what has happened, then, is that 10 years have passed. And by an extraordinary hazard, Liouville, who was a contemporary of Galois, who was just two years older than Galois, found the Galois papers. And he understood that these were absolutely great things. And he talked to the Academy. So if you want, 10 years after Galois'death, it's Liouville who is here. Well there, obviously, he is much older but he was a contemporary of Galois, he was someone who was born in 1809, therefore two years before Galois. And so, Liouville understood the extraordinary strength of the works of Galois if you want.

So he wrote that, but I show you that he wrote it correctly, so it's like that.

He talked about it at the Academy. And then, gradually, the work of Galois have been understood. And then what I'm going to do, I don't want to bother you with too complicated math, I'm just going to give you the essence of Galois theory. I'll give you the essence by giving you an example. What Galois says in his testament is something incredibly visionary, if you will, what he says is :

"You know my dear Auguste, (he had a friend named Auguste) that these are not the only subjects I have explored. My main meditations for some time have been directed on the application to transcendantal analysis of the ambiguity theory."

So Galois discovered this theory of ambiguity. And in this letter, at the end of his life he says that not only he applied it to polynomial equations.

But in fact, he applied it to the theory of transcendantal functions. No one knows exactly what he had in mind. It's a fact that nobody can say that we now know what Galois had in mind.

On the other hand, we know very well what he had in mind for the polynomial equations. And so, for polynomial equations, I'm going to explain what ambiguity theory. So what Galois understood, if you like, is something quite extraordinary ; it is that when you give yourself an algebraic equation, for example, I gave you an equation so, we knows how to solve it. You know now, I mean with the computer, you can control all that, you can plot the graph of a function, you can solve a polynomial equation, and all that. But the computer will never give you zeros unless with some precision, it will only give the roots except with a certain precision.

So, what Galois theory says, it says something extraordinary : it says that when you take an equation like that, which is irreducible, that is to say that we cannot factor it into a product of 2 factors with rational coefficients for example. So when an equation is irreducible, what Galois theory says is that there is a group which operates on the roots, here on the 5 roots, and which means that we cannot, if you want, isolate a root. In other words, there is an ambiguity between roots ; this group turns the roots. And any relationship that is checked between the roots, any rational relation which is checked between the roots, for example, with the computer, you can see that this relationship, is almost verified, the fact that $E = 4C^2 + 2D^2$, you can verify it. In fact, what Galois theory says is that there is a group that swaps these roots, that is, they can move from one to the other. And so that if a relationship like that takes place, it will take place for them permuted roots. And what Galois theory says is that by this group, you can transform any root into any other. So what Galois says after, actually what I'm telling you in particular here is that it's impossible to have this relationship. Why is it impossible to have that relation ? Because if you have this relationship, you can see that the 5 roots are real. This is not at all difficult to demonstrate. Therefore you have 5 roots which are real. But suppose you have a relationship like the one I wrote : $E = 4C^2 + 2D^2$. Well at that time, as E can become any of the other roots, C and D will be other roots too. And you can see that all the roots should be positive, since they are sums of squares. And so it's not possible. It is not possible. So it's extraordinary !

It tells you that without calculating and without getting your hands dirty, or anything either, you know that this relationship is not possible. In other words, with the computer, it will tell you "But it's true, it's true !". It will say with decimals and all that. No ! Galois says "it's not possible, this relationship is not true !". And it is not true by pure thought, it is extraordinary ! This is something extraordinary ! Because he understood that behind an equation, there is not only the numerical value of the roots. No. There are the relationships between the roots that can exist, and what does Galois theory is to detect exactly all the relationships between roots. And they are detected by a group. So don't believe people who will tell you that it was Galois who invented group theory. No, the people like Lagrange, etc., knew what groups were like before him. But Galois is the first modern mathematician. That is to say, he is the first mathematician who had this dazzling, if you will, who did that certain things like that are true without having to calculate or what whatever, okay. We have an abstract theory, it's the ambiguity theory and solving an equation is gradually reducing the ambiguity that there is, so that finally, on the equation, we can affirm such a root, and such root, etc. Okay. So that's the ambiguity theory. And here, in this case, we can calculate what the Galois group is. So the group of Galois, you see the 5 roots, they are indicated here. The group of Galois, he will swap them. But it swaps them if you want in a transitive way, that is if we iterate these permutations, if for example, I take the root which is above in the middle, it will go on the first ; and afterwards, if I look where the first goes, it goes on the last ; after, if I look at the last, she goes on the penultimate ; if I look at the penultimate, it goes on the second. So you see that you've gone all around the turn.

So good, and that is always true, that is to say whatever the equation you take, Galois tells you that if it was reducible, there is a group that swaps the roots. So there are a lot of mathematicians who believe they know Galois theory, because they say that Galois succeeded to demonstrate that an equation is solvable by radicals if and only if his group of Galois is solvable. But in fact Galois, at 17, had better than that, he had... a theorem...

I'm going to scare you but don't worry, we'll move on to another subject right away. So what Galois demonstrates is that if we take a equation he calls primitive. It's a certain technical definition, for that it be solvable by radicals, it is necessary and sufficient that we can index the roots by a Finite field. It was Galois who invented the Finite fields. That's enough funny because the French are modest, because the Anglo-Saxons call those Finite fields the Galois fields. If we translate this literally into French, it gives Galois body. But in France, we don't use this terminology : we are talking about a Finite field. And then the Galois theorem, which he had when he was 17 years old, that for a primitive equation to be solvable by radicals, it is necessary and sufficient that one can index its roots by a finite field, so that the Galois group, then hold on, is either contained in the semi-direct product of the affine group of the finite field by the Frobenius, by the powers of the Frobenius. Okay, okay, good. *(laughs)*

And so when I prepared my presentation for the Academy, I have noticed that in fact, Galois knew countless things and that he knew for example, now what's called the theory of Sylow, which is a theory that was developed perhaps 50 years after the death of Galois. So that's to tell you a little bit about how well he managed to see so far. And at the end of my talk, I will show you a text by Grothendieck and this is a text that is fundamental because it applies marvelously to the case of Galois, and this is a text on creativity, on the discovery and the fact that true creativity, it asks precisely if you want to return to that spirit of the child which is both free but also who does not accept if you want the weight of knowledge that we put on him. So we'll come back to that, okay.

So now I come to another subject, because I don't want to neglect physics. And another subject that is so close to my heart - which is, if you like, another great discoverer, in the XX$^{\text{th}}$ century, if you like, is the discovery of quantum mechanics. Now let's move on to Heisenberg and quantum mechanics. So we take a break if you want. What I tried to do was to choose subjects that each show you a new concept that has been discovered, either by doing mathematical research or by doing research on nature, on physics. But each of these notions is a notion which has a meaning, which has an absolutely fundamental meaning.

So the story of Heisenberg, it's actually connected to a place, and that place is an island which in German is called Helgoland ; in French, we translate Heligoland. It is an island in the Nordic countries. And it's an island that has a particularity, I no longer know if this particularity is still true today ; in any case, it had a particularity in the 1925s, which was that it had no pollen. There were no trees, there were no sources of pollen. So what is the connection with Heisenberg ? The link is that Heisenberg was a physics student, finally a student, he already had a lot of experience... He was in Göttingen, I think. And at one point, it was in the month May he got a terrible allergy, with fever if you want. So his head was swollen, well all that, and so at the time, the only one remedy, we did not give antihistamines, the only remedy was to send it to Heligoland. So he was sent to this island. They told him to stop teaching, etc. and they send him to this island. And it happened on this island. He was housed by an old lady in a house, perhaps one of the barracks up there. And then at the time, he was looking for... *(little interrogative circumspect noise)*. At the time, he was looking for...

He was trying to... At the time, quantum mechanics was at a prehistorical state, that is to say that we had decided on what we call certain principles, which were used to calculate energies and all that, but I mean that's was absolutely not a real theory. And Heisenberg was thinking about a problem. Basically, his problem would take too long to explain, if you want, the idea, basically, at the time, we conceived the atom as a small solar system. But it didn't work. Because what's going on in a system like the solar system, is that, for example if the electron revolved around the nucleus, it emits energy, and therefore in fact, its orbit should shrivel up on the nucleus. And that is not what is actually happening. So there were things like that that didn't stick at all. And so, Heisenberg reflected on this. It is based on experimental results, this called the Ritz-Rydberg principle. And hey, he had this calculation he wanted to do and when he was on this island, he started doing this calculation. There were things he didn't understand, all that. And then one morning at 4 a.m. in the morning, everything worked ! He had this extraordinary revelation ! And instead to go to bed, he went to climb on one of the rocky peaks *(laughs)* which are on the edge of the island. He settled upstairs and waited for the sunrise. And in his memoirs he describes in an extraordinary way if you will, this enlightenment which he had and he really says, and it is true, that he had all of a blow before the eyes an immense landscape which was revealed to his eyes, but it was an intellectual landscape, of course ; this landscape was the essence of discovery he made, if you will, it's something incredible ! He discovered that when we do calculations, this is Heisenberg, let's come back to it. What he discovered is that you see, when you do physics, well for example, you write $e = mc^2$ or stuff like it. You could write $e = c^2 \times m$, it's kif-kif, these are numbers. Okay, well, I mean, it doesn't change anything. What Heisenberg found, it's something incredible. Heisenberg found that if you try to manipulate position and time, we talk about speed, but we have to talk about the moment : the moment is the product of speed by mass, okay ? So if you try to manipulate both position and time, at the very least microscopic scale of a tiny thing, an atom or something like that, well you can always do whatever you want, you won't never to fault what is called the Heisenberg uncertainty principle, okay, which is $\Delta x \Delta p$... where Δx is the uncertainty about the position, Δp is the uncertainty about the time. Well that's always bigger or equal to $\hbar/2$, what is $\hbar$, this is the constant that Planck had introduced at the beginning of the century, to explain certain physical phenomena.

So there, I have to tell you a little story, about the uncertainty principle because hey, *(whispering)* I think there is a book by the way, which is not bad, by the way... But in fact, concerning the principle of uncertainty, if you really want to feel how this principle has disturbed folks, there's a story I need to tell you. What is sure is that Einstein didn't believe it. However, Einstein is behind the quantum theory, I mean, it was Einstein who had the idea that the photon had energy levels that were quantum. So Einstein didn't believe it. So Einstein had imagined a device.

At the time therefore, Heisenberg found his principle of uncertainty towards the late 1920s. At that time, there were what were called Solvay congresses ; these were meetings of physicists, in small numbers, and of course, they were talking to each other.

So there was a Solvay congress, I believe it was in 1830, or something like that. And so Einstein had imagined the following thing ; he had imagined to fault the principle of uncertainty, but not on the position and time, but on $\Delta t \Delta E$; that is to say that... *(sigh, sigh)*... Time is the dual variable of energy, just as position is the dual variable of the moment. And the principle of uncertainty gives you something similar to $\Delta t \Delta E$. Something like $\hbar$ or $\hbar/2$, it depends units. So Einstein didn't believe that. And Einstein had imagined... Of course, he always did the same thing, that is to say that when he did not believe something, he imagined a thought experiment. An experiment of thought, what does that mean ? That means I will make you a very coarse drawing : but in fact, we can very well imagine that this experience be made more and more precise. Okay ? So the drawing very rude, it was as follows : *(drawing the device on the board)* there, we will put a small spring, and then here, we will put a box. And then we will put like a sort of cuckoo. And then with this, there will be time here, okay ?

That was his system, and then there is a kind of thing. And then there are... That's the system.

So what was Einstein's idea ? Einstein's idea is that Δt, well, we will control it since we have time here, okay. So this is the t therefore. And ΔE now ? So what does that mean, what do I have says here *(showing a place in the drawing)* ? It means there will be a moment given where the cuckoo is going to "Touc !". It will emit a photon. And we will know what time he emitted it since there is this thing that marks the hour, okay. So Δt, *(noise to express that we don't know what...)*. So now ΔE. Well the photon, that, Einstein, he knows it, the photon, it weighs $h\nu$, where ν is the frequency of the photon. So this is e if you want, it's energy. So when the photon comes out, this thing, it becomes a little lighter... *(seeing that he may seem to have lost some understanding of his audience)* Do you know the story of the truck that was carrying holes, right ? You don't know it ? It was in the mountains, okay, then at some point, the driver, he felt heavier, he backed up, he fell into the hole, okay. *(laughs)*.

Okay, I'll start again, okay ? So here, once the photon has been emitted, okay, this thing gets a little lighter, so it's okay, if you want, the needle, it will go up a little bit, and looking at how much it went up, we're going to know ΔE so in fact, Einstein said "bah we're going to know ΔE, we are going to know Δt, with as great a precision as we want. So we will not have the principle of uncertainty". So he said that. And it made terribly scared to Bohr who was chatting with them, because Bohr of course believed in the principle of uncertainty, it frightened him terribly because... What was the reason he was afraid ? The reason for which he was afraid of is that when you do the math with this system, proposed by Einstein, what will intervene is the gravitational constant because you see, the clock, when it goes up a bit, it's in the gravitational field, so when you go to get how much energy has decreased, you will use the gravitational constant, so obviously, the gravitational constant, it absolutely does not fit in the $\hbar$ by Planck etc. Planck's theory, it is completely separate from the gravitation. So Bohr said to himself, it's done !

So there is an extraordinary photo, on which we see Einstein going out very proudly of the Solvay congress hall and we see Bohr following him a little like a little dog, and who is, well... And then what happened is that it is not the end of the story. The end of the story is absolutely wonderful, because what happened was that Bohr came back to his hotel. Obviously, he didn't sleep, he didn't sleep all night long because, and he found the answer... And the answer is fantastic. The answer is absolutely fantastic, because, if you will, well, it seemed impossible, impossible ! Why ? Because, as I said there will be the gravitational constant when you go to do the math and that is impossible for it to work ! It is impossible to find the $\hbar$. Where does it come from ? What Bohr found overnight, he found the same that has laid Einstein, in fact, to general relativity. *(Alain Connes writes with the chalk formulas on the board).* At the time ! That, you know, now, this year in november there are going to be a bunch of celebrations of the discovery of general relativity by Einstein. It's been exactly 100 years. That's why there are going to be all these celebrations. So that's exactly 100 years. And so it was ten years, or even more, fifteen years before the story in question. What does that have to do with the thing ?

What this has to do with the thing is this : what says general relativity ? It says the passage of time if you write the metric you have what's called Minkowski metric, in fact, which is due to Poincaré, so space-time if you want. When that is the space-time of special relativity, and if you look at the metric of the space-time of general relativity, as a first approximation, which happens, is that the metric does not change for the usual coordinates : we are in a Euclidean space. However, it changes for the passage of time, and the way it changes is that the coefficient dt^2 is multiplied by 1+ twice the Newtonian potential $V(x, y, z)$.

Don't worry, it's not... Good. What does it mean ? It wants to say that time passes differently depending on altitude, ok ? But the clock has changed altitude a little bit *(laughter)*. So its time has passed differently. You do the calculation and you find the principle of uncertainty of Heisenberg. It's incredible ! It means that Bohr, if Einstein hadn't not discovered general relativity *(bursts of laughter)* fifteen years before, he would have been right, okay ?... Nobody would have believed that the uncertainty principle was valid. But, because of general relativity, that he even had invented, he was beaten, he was put in default. So the next day morning, Bohr came back, triumphant, I mean, it's extraordinary! It is really extraordinary, but if you want, all that is to try to make you feel the fact that none of these notions was accepted at first. Not at all! Absolutely not. There is always an absolutely terrible resistance, to things that are new like that... And so, what's unbelievable in quantum, which is mind-boggling in quantum, if you want, and that I think it has not really passed into knowledge. Yes, then, good. I'll talk about it after that, I'll talk about it afterwards. I will come back to it. What's amazing in quantum, if you like, is the fact that, and that, it comes from the Heisenberg uncertainty principle, is that, unlike the classical physics, when you do an experiment in quantum, you cannot reproduce the experience. It is something fundamental. When I was trying to tell you, if you want, that I was going to explain to you concepts or concepts... These are concepts that make such a break with the classic view, if you will, that it's a huge difference. What do I mean by that? What I mean by that is that if you do a quantum experiment, for example you send a photon, and this photon, it will go through a very small slit which is little close to the size of its wavelength. And afterwards, you will receive it on a target. Well, the fact that you receive the photon at a given location x, this experience is not reproducible. That is to say, you can redo experience with as much precision, given the same initial conditions, etc., the end result will not be the same. This is incredible! And it won't be the same because of Heisenberg's uncertainty principle.

So you can say to me, "Okay, well, okay, well, I don't care record, there is a bit of a hazard, what! A microscopic hazard, I don't care!". But no! Now what happens is that the fact that there is this basic uncertainty, if you will, well, it was used to produce random numbers. That is to say that there are Swiss who made a device that works, now it's with an LED lamp, you know the little LED lights there, like that. So these little lamps send out photons on a target, that's it. We look at where the photon arrives, it hits one of the target's tiles. And from there, we make a number and as it is a quantum phenomenon, that is to say that it is a phenomenon which is not reproducible, it produces random numbers, which are so random that even if an attacker wanted to reproduce the same thing, it means if he knew all the data on the system, he would not be able to reproduce the same number. Whereas with a computer, if you fabricate random numbers, if the attacker knows your system of manufacturing, it will happen to reproduce them, the random numbers, okay? So it's phenomenal, it's phenomenal! So from there, if you want, this extraordinary truth, in fact, brings out an idea, which we have started to to exploit and this idea is this : you see, we are used in physics to attribute any variability to the passage of time, that is to say that well... I once remember my teacher, I had a teacher, I don't know anymore if it was in Math. Sup.. He told me to go on the board, so I go on.

He questions me. And then he does this to me *(gesture of a curve drawn in the air)* - Ouhouh ? ! *(laughs)*. It looks like this... He says to me "Mr. Connes, what is the variable ?". So, I was doing kinematics. I'm thinking... And then after a while, I answer him : "it's time !". It was the right answer ! You see, normally there are a lot of things that are variable. And all physics is written by noting d/dt of something equal something else... All physics is written as a function of time. And in fact, if we think enough, at the conceptual level, we realize in fact, that quantum mechanics immediately causes paradoxes, very very violent paradoxes, very very strong, if you will, and which come precisely because we attribute variability to the passage of time.

And there is a fundamental idea that is hard to get by, but that we tried to popularize etc., and this idea is actually that the real variability is the quantum variability and that time in fact emerges from that variability. It means that time is only a secondary phenomenon, it is only an emerging phenomenon, which results from quantum variability, but which is not not at all fundamental okay.

So to try to get this idea across, in fact, I'm not going to give all the details, we wrote a book, therefore, with Danye Chéreau and Jacques Dixmier, we wrote a book together, called Le Théâtre quantique. And in this book, you'll see an introduction to that idea, which is, we hope, understandable although a bit cryptic obviously, that is to say that we don't give all the details etc. But the idea comes from another quite extraordinary mathematician Von Neumann.

That is to say after the discovery of Heisenberg, if you will, after the great discovery of Heisenberg, of course, mathematicians formalized what Heisenberg had found. It took a while. What Heisenberg had found, so I remind you, was that you can't swap them letters, variables like $e = mc^2$, you cannot write $e = c^2 m$. We can't do that, okay ? Then there are people who will say to you "Wow la la la la ! What's going to be complicated all this !". But in fact, no, back to Heisenberg.

You see these two sentences, so this is an anagram that has been found by Jacques Perry-Salkow, who is quite extraordinary and who was the birth of the book I showed you. But what does an anagram ? It means that if I had the right to swap letters, I would will hold the same result : not terrible ! *(laughs)* has *a*2*bcd*... So you see, in the commutative, it gives you the same result. But of course, we we are all used to paying attention to the order of the letters... language ! Language is made for that. And the discovery of Heisenberg can incredibly simple saying to herself : she can say to herself by saying that Heisenberg, he found that we had to pay attention to the order of the letters, when we do calculations with microscopic variables, it's wonderful ! It's something absolutely wonderful, okay ! Okay so Von Neumann worked out on that, he found that he needed a mathematical formalism called the formalism of Hilbert spaces is a fairly complicated thing.

So, you know, in my introduction, I said that I was going to talk about Von Neumann algebras, okay. So right there I'm talking about it, okay. I don't give you too much detail, of course, not too much detail about the types and all that. But now I'm going to tell you about another mathematician, and that was the starting point, really, of my work, my thesis, etc., and which is the tool which made it possible to have, the essential tool which makes it possible to give a sense of this idea that the passage of time emerges from the hazard of quantum. So the reason I'm showing you his photo is that unfortunately he died on October 9 at the age of 91 ; his photo was taken when he came to Bures-sur-Yvette exactly 30 years ago. he spent a year in Bures-sur-Yvette 30 years ago, and why he's a person absolutely extraordinary ? He's an extraordinary character because that for example he was in the military at the time of the war between Japan and the United States, but he was deaf since the age of 2. So there was a time when all of his fellow believers were running for shelter, because there was a bombardment. Tomita did not move, and when his military friends came back to see him, they said "but are you crazy ?...". They were shaking him, and he said to them "What bombing ?". It was at that point, he was known like that. And then there was an episode where the officer who commanded them said that he wouldn't be on the expedition that they were going to do because as he was deaf, it was rather a problem. So he stayed and everyone else was dead. And apparently, but I'm not so sure, apparently, it was the next on the list of suicide bombers when the war ended.

Then he had a teacher, it must be said that shortly after the war, when he was at the university, instead of lecturing, instead of going to listen to the classes, the students were going to plant potatoes so there were starvation. They were going to plant potatoes near the university. So he had a teacher. He had a teacher to do his thesis, his teacher was called Ono, and his teacher, the first time where Tomita went to see his teacher, because he wanted to do a thesis ; his teacher takes a big big book, I don't have brought any with me. The teacher gave him a book, oh ! more than that, twice that easy, okay, he gave it to Tomita and he said "Read this book and come back and see me when you'll have understood everything". So it's okay like that. So during 2 years, they do not see each other. And then after 2 years, by chance, Tomita rmet his teacher in the corridors of the university. His teacher remembered when even : "So this book, is it going on ?". And Tomita replies "I lost it after a week..." *(laughs)* He was an absolutely great guy. He told stories that were absolutely great. He made an absolute awesome. Only, since he was deaf, if you will, it was very very very difficult to communicate with him. It was really very very difficult, most of the time, he would turn off his device. *(laughs)*. Me, that was the starting point of my work if you want. The start of my work was the fact that so Tomita and then after Takesaki, who had resumed work of Tomita, had found that, on a Von Neumann algebra, like Von Neumann had defined them, there was an evolution but which depended of a state. And then, what I demonstrated in my thesis, is that in fact, it did not depend on a state and it was enough to have non-commutativity, that is, it was enough to have an algebra, to do calculations in the which you pay attention to the order of the terms, so that there is an evolution in time, so that there is a passing time. So then, there were a lot of consequences of that, of course. And in fact, the bulk of my work was if you want to develop geometry for spaces which, unlike Descartes'spaces, because Descartes if you want, had managed to understand that there was a duality between geometry and algebra because Descartes understood that one could encode a geometric space by coordinates and then do algebraic calculations instead of do geometric calculations. The simplest example possible : if you want to demonstrate that the 3 medians of a triangle intersect. Well there are several ways of doing it, but the simplest way is to do the calculation of the barycenter. You take the coordinates then you calculate the third of the sum of the coordinates. What is the advantage of the algebraic demonstration on the geometric proof ? You can of course do a geometric demonstration of the fact that the 3 medians of a triangle meet. But suppose I ask you to demonstrate it in dimension n? *(laughs).* While the algebraic demonstration, it is obvious, you do $1/n$ times the sum of the coordinates and then that's it, it gives you the intersection point and then it's over. So you see the power of this back and forth, between geometry on one side, and algebra on the other. So what Heisenberg discovered was that there were incredibly natural spaces in which precisely, the coordinates do not commute. And these spaces correspond to observables on a microscopic system. And so for me, the main part of my work was to develop the geometry for sorts of such spaces. So time is still pretty short, instead of talking to you about my work, I'm going to tell you about another mathematician absolutely extrordinary, whose name is Alexandre Grothendieck, and who died a few years ago, and the reason I'm going to tell you about him, it's not because that I want to describe topos theory to you, because that is a marvellous theory but it wouldn't pass, I don't want to talk about it. After maybe... But it's mainly to explain to you, to show you what Grothendieck says about creativity and this absolutely necessary need to find, when you are faced with a very very difficult problem, your soul of fant and this kind of, precisely, openness, sensitivity, etc. who is too often completely erased, completely erased by the weight of knowledge. So this is what Grothendieck says, I will read it with you, and then we'll stop there. So this is Grothendieck when he was young. He had an extremely tumultuous life too. So this is what he writes. He writes : *"In our knowledge of the things of the universe, be they mathematical or others, the renovating power in us is none other than innocence. It is the original innocence, which we all received in sharing at our birth, and which rests in each of us, object often of our contempt, of our most secret fears. It alone (therefore this innocence) unites hum militancy (of course, research is a school of humility, everyday school humility) the humility and boldness that make us eencounter the heart of things that allow us to let things get into us, and to soak up it."* (This is the first thing he says. Then he says :) *"This power there, (That's very, very important now.) This power power is by no means the privilege of extraordinary gifts."* You see, when sometimes we attend exhibitions, on mathematics, you have the impression that wow! These are aliens these people, no, you shouldn't have that fear at all, absolutely not. He is coming on the contrary too often that overly intelligent people have an immediate reaction and that this immediate reaction in fact is false. That is to say they tell you "it's not going to work for such and such a reason...". In fact, if they had thought more, they would have noticed that it works, okay. So this what

Grothendieck says is that : *"This power there is by no means the privilege of extraordinary gifts, of a cerebral power, let's say out of the ordinary, to assimilate and to handle with dexterity and ease, an impressive mass of facts, ideas and known techniques. These gifts are certainly precious and sources of envy surely for the one who, like me, was not filled like this at birth beyond measure..."*. There, it is really ironic, ironic, I don't like to say the biggest because that the biggest, what does that mean..., we cannot compare different things but it had a phenomenal influence on the mathematics of the XX$^{\text{th}}$ century. A phenomenal influence. So hear him say that... it's reassuring, let's say ! *"These gifts are certainly precious and sources of envy surely for the one who like me was not fulfilled thus at his birth beyond measure. It's not these gifts, however, or ambition even the most ardent (ambition is not enough) ambition, served by a flawless will, which make these invisible and imperious circles cross that enclose our universe. Only innocence crosses them, without knowing it, nor without worrying, at the moment when we find ourselves alone listening to things, intensely absorbed in a breeze."* So what he explains is that there is nothing more fruitful than to grasp a question and think about it, but in this way, in a way completely independent of the weight of science, etc. Okay. Good sure, well, to get to the problem, you have to know a number of things but then you have to think about it like that. And so he goes on to say : *"Discovery is the privilege of the child. I want a small child to speak, the child who is not yet afraid of making a mistake, of looking silly, of don't be serious..."*

For example, a while ago, there will be questions, so okay, beware of that, you shouldn't be afraid. There is even a Chinese proverb who says, "If I ask a question, I look silly for 5 seconds ; if I don't put it down, I look silly for the rest of my life.". So this is what he says, therefore, not to be serious, not to be like everyone else. And it's However, there is a typically French attitude that is quite in an assembly : we are afraid to ask a question, except when we know the answer *(laughs)*. *"Neither is he afraid that the things he looks at will have the bad taste to be different from what he expects from them, from what they should be, or rather from what it is understood that they are, that is to say, what the majority of people will tell him they would be ; he ignores dumb consensus and flawless, which are part of the air we breathe, that of all people sensible and well known as such. God knows if there were any, sensible people and well known as such, since the dawn of ages ; our minds are saturated of a heterogeneous knowledge, tangle of fears and laziness, cravings* *and bans, general information and push button explanations...".*

A typical example is what is called the butterfly effect, the number of people who brooded over it without knowing that it was an idiocy, it's something considerable. But I mean, it went on, it went on for a long time. So I continue... *"enclosed space where information, cravings and fears come to pile up, without never let the offshore wind rush in, except for the know-how of routine. It would seem that the main role of this knowledge is to evacuate a person. living concept, a knowledge of the things of this world.".* That is what counts, it is this living perception. For example, for loving math, you have to do it, of course. And whatever the problem you are watching, but what's important is that you do it, it's not that you take like... If somebody tells you a theorem, for example, if you want, you shouldn't have too much demonstration. There must be search for it yourself, even if you can't find it. You will win. Why ? Because if you search for it, by yourself, when it is given to you will say, even if you can't find it, well you will say "but of course, that was it, and that was it !". If you don't look for it and we give it to you, it comes in through one ear and it comes out through the other, and then you forgot it after half an hour. So it's very very important to do it, okay. So therefore... Its effect is above all that of immense inertia. He talks about weight of this common knowledge, often overwhelming. The little child discovers the world as he breathes. The flow and the reflux of his breathing make him welcome the world into his delicate being and make him project into the world that welcomes it. The adult also discovers, in these rare moments he forgot his fears and his knowledge, when he looks at things or himself with wide open eyes, eager to know, with eyes new, child's eyes. I hope you feel the most important in what I said. It is that it doesn't apply at all to math ; which means you wanted to do the humanities, you wanted to do the linguistics, whether you want to do anything, maybe even art, if you will, it's crucial that you get the message. And that you have understood that, in particular in mathematics, they have a far greater scope than calculating with numbers, calculating with numbers, etc. It's not at all that, it's a kind of version of the philosophy which is much harder because indeed, to arrive to a new concept like Grothendieck's concept of topos, it took years and years of reflection... But it gives you thought tools absolutely fundamental. And I don't have time to talk about it, but the concept of topos, it's a concept that shows you that the notion of truth, when we say for example in a current way of something that it is true or that it's wrong, well, when you look in a topos, it's a universe which is different from the universe, well one thing may be partially true and partially false, it may be true from a certain point of view, it may be false for another point of view, etc. So it gives a tool of thinking which is incredibly well suited in fact to life, to politics, to 36 things, but which has not yet passed into the common domain. It's a concept which is still a concept in the mathematical field, and which is not has not yet passed into the common domain. And we would gain a lot if you want, if all of these wonderful things that have been discovered, now become part of the common domain. So my scribe was going in that direction okay, to try to make you see, to a little surreal way if you will, that there are these wonderful things but that good of course, you have to make an effort to learn them and an effort to know them. Here.

Questions to the speaker

- Thank you so much. Perhaps, therefore, we do what we said. If you have questions, clarification on what was said, therefore, questions that you shouldn't be afraid to ask, I have some, but I'm sure you have some too...

- You talked about the butterfly effect... and that it didn't exist.

- I didn't say it didn't exist. But I said it was a vast illusion. Because what I mean is like saying that there is a butterfly that will fly, then the plane that follows another plane does not go to take off ; there is a colossal damping effect. Of course we can make a mathematical system which depends on few variables and which is such that when we move a variable a little bit, it will change the results. But from there to make believe that a small butterfly which flies at a place, it will create, I don't know, me, a hurricane in another place, it's ridiculous... Well we can remember where it comes from, it comes from the fact that there are differentials equations in mathematics, which are such that if we change a whole little bit the initial conditions, it changes the result considerably that becomes exponentially larger. That is true. But it's true in a particular model. This is true in a model, in which there is no depreciation, as it occurs in nature. In nature, fortunately, depreciation occurs, because otherwise, in the nature, we would look at the butterflies everywhere, and then we would be scared *(laughs)*. Fortunately it is like that. But it's common sense, it's common sense. But we have seen perhaps I do not know how many politicians or people who repeated the butterfly effect without understanding anything, since if they had understood anything, they would have realized that it was, huh, good... This is a typical example of people repeating things without understanding them, simply because they say to themselves : "Ah yeah, he is a powerful person who said it, so it must be true, what !"

- Thank you.

- It was a question about the fact that you often said that the physicists expressed everything as a function of time, and which we often considered that it was the variable...

- fundamental.

- and you said that in fact it turns out that the real variable is the quantum variable, and I didn't understand how time flows from this variable.

- This is quite a story. Basically, this is the story of my trajectory. That is to say what actually happens, but it is explained a bit in our book, but it is above all well explained in a talk I gave at IHES in May, and which I think must be on the IHES website, go and listen to this presentation, I could say two words about it. But basically, it is that Von Neumann created the Von Neumann algebras as being systems where we have a partial knowledge of reality. And with work by Tomita, then my work during the thesis, we understood that if we had a system that has a partial knowledge of reality, at that time, there is a time that emerges. That is to say, there is an evolution over time. Since everything is quantum, and the knowledge we have of reality is actually partial, that's what, with the work I did with Carlo Rovelli, that's what should explain the passage of time, that's what we called thermodynamic time. This idea of thermodynamic time, it is well explained in our three-authors book.

- Suddenly, my question is somewhat similar to Constantin's question concerning time. So suddenly, the fundamental constant, there is no longer fundamental constant, since ultimately everything is based on quantum variability ?

- There is one thing I did not mention but I had transparencies so I can actually show them. It's important, it's so important, it's this idea of variables. Because ultimately we come back to the idea of variables. What is a variable, you see ? What are we teached in class that a real variable is... A real variable, it is an application which goes from a set X into reals. It is like that we are told what a real variable is. Now if we look at this definition of a real variable, we see in fact, with a little reasoning, we realize that we cannot have the coexistence of what we call continuous variables, variables that take for example an interval of values, and discrete variables, which take discrete values. *(He draw an interval and points on the board)*. And the reason why we cannot have this coexistence, is that if we take a continuous variable, the set X must at least have the cardinality of the continuous but if it has the linearity of the continuous, we cannot have a discrete variable, because it there will be points that will be reached too many times. We can't have that. The extraordinary value of quantum formalism, as von Neumann has developed it, is that in quantum formalism, everything is solved : to say that in quantum formalism, in fact, a variable is the spectrum of a self-attached operator in Hilbert space, it's a bit complicated but here, for operators in Hilbert space, we can have operators which have a discrete spectrum and operators which have a continuous spectrum and which coexist. And then, what is extraordinary, is that in fact, it joins exactly Newton's thought. That is to say, Newton, in his writings, when he was trying to define what an infinitesimal is for example, he write exactly the right sentence, which corresponds to the quantum. That is to say, he said a variable is infinitesimal. First he said what was a variable. Now quantum formalism gives exactly the right answer by comparison to Newton. This is the first thing. And so now what happens is that once we have this formalism, of what it is that a variable, we realize that of course, the discrete variables can coexist with continuous variables only by non-commutativity, and we see that it is this non-commutativity that creates the passage of time, okay ? So in fact, the $\hbar$ still exists in fact, Planck's constant is still there, but this is what is extremely striking, it's that time should not be considered as a gift fundamentally born, but as an emerging datum, and that if we had a absolute knowledge of everything, time would not pass. It's amazing to think that, okay, that is, the reason why we feel that time goes by, etc., it's because we have a partial knowledge of the universe, okay. That's what is great if you want with this game of physics. In the book we wrote with Danye Chéreau and Jacques Dixmier, I must tell you that Danye Chéreau is my wife *(laughs)*, what we do is we found a very striking sentence we used to express the idea I just told you. We said "The hazard of quantum is the ticking of the divine clock". You know, Einstein said "God doesn't play dice". So that's the answer. The hero of the book's response to this Einstein's joke is that "the quantum hazard is the ticking of the Divine clock". That is to say, it is because there are constantly these little tricks completely random, pop! pop! pop! that happen, that time past. Nature has a phenomenal imagination. And that's what gives the grow up to measure random numbers, it's amazing, it means "It takes the pulse of nature", pop! Come on, it's a random number, pop! another... You can always try to reproduce them. Well well that's incredible, it's the sentence that says it's, "the quantum hazard is the ticking of the divine clock".

- To try to put that a little bit clear, suddenly, if, let's admit, well, we can always hypothesize, if for example, precisely, this nature was quantum stable, if it did not move, time would not pass?...

- Well, no, precisely, it keeps moving!

- But let's admit that we imagine that it does not move. It means that time would not pass?...

- Ah yes! No, no, no, that's not it; if we knew it completely, if we had all the knowledge, there time would not pass. Time flies because we have partial knowledge, it's thermodynamics, Thermodynamics tells us, by engineering of Boltzmann, it tells us that entropy, for example, is partial knowledge about things, okay. So the passage of time is related to that. But nature keeps moving, okay?!... *(laughs)*

- I have a question because you said that for your thesis, you were inspired by Tomita who showed non-commutativity...

- No, no, that's not it. Well yes yes, it's a detail... ! *(frank laughter)*.

- Could you just explain what are called types ?

- Ah yes, the guys ! ! Of course, of course, absolutely, well the three types... So the three types. Where are they, the 3 guys ? The first type, that's him... *(He shows a photo, laughter.)* The 3 types, therefore : type I is a quantum system such as in fact, Hilbert space of the quantum system breaks into a tensor product of two spaces, that is to say that it is really the simplest case that we can imagine, and that was what people had imagined it would always be the case. They always imagined that when you take a subsystem from a quantum system, we could break the Hilbert space into a tensor product of two sub-systems, so that the first system corresponds to the first space of Hilbert, to the operators in the first Hilbert space, and the other to operators in the second Hilbert space. So what von Neumann and Murray discovered is that there were actually two other types. It is to say that there was a way to have quantum subsystems that weren't at all a splitting of Hilbert space into a tensorial product. So the first type, there were the real dimensions. And then the Type III that remained was the others. And before Tomita, we had no tool to attack type III, okay. So what Tomita found is that in type III, there was this group $\sigma_{t\phi}$ and then what I found in my thesis, me, afterwards, was that the group that Tomita had found, in fact, it was unique modulo the interiors, that is to say it defined a real evolution, independent of everything else. So that gave a lot of invariants, etc., it made it possible to unblock everything. Okay ! So but it's incredible because von Neumann had defined these quantum subsystems so completely, how to put it, it's in von Neumann's writings, completely abstract. And we never would have thought at the time of von Neumann that it would have been linked to time, to the passage of time, I mean, it's absolutely amazing. It means depth of quantum. Heisenberg discovered that it came from non-commutativity, von Neumann reformulated it as operators in Hilbert space, the problem of subsystems arose, and from there comes the passage of time, it is fabulous !

- Could you explain to us why the entropy principle results from the partial knowledge that we have of the world ?

- This is Boltzmann and poor Boltzmann was so misunderstood in his day he ended up committing suicide. He had an idea absolutely... He have the following formula engraved on his grave $S = klogn$. That is engraved on the tomb of Boltzmann. He committed suicide near Trieste. This formula, what does it say ? It's one of the simplest formulas but one of the most difficult to understand. What is the integer n ? It's the number of microscopic realizations of a macroscopic state. I have to tell you a little bit about history : history is during the period where people discovered the steam engine, and then there were locomotives and all that, and so what people had discovered was that they had a way to turn heat into energy, movement, everything what they want. And that's how the railway started, etc. And they asked themselves the question of what we called the yield of machines and all that. And so of course, if you will, there were amounts of heat dq so that were between two systems etc. But we noticed fast enough that if we took two different paths to go from one point to another, from one state to another, the integral of $\int dq$ if you want, it wasn't preserved, that is to say that we cannot define the amount of heat of an object. On the other hand, we realized that if we divided dq by what we call the absolute temperature, well, that quantity if you want, it was well defined, that is to say that whatever path we took, between a state and another, the whole thing gave the same result. And it's which had made it possible to define entropy. But this entropy, it was defined for macroscopic systems which were given by the temperature, the pressure, the volume, well I don't know what, if you want a number of macroscopic quantities, there had no interpretation, none, and it was called entropy. It was called entropy, S. But this entropy had no philosophical meaning. since precisely, that's what we're talking about, okay ? And the incredible Boltzmann genius, it was this formula $S = klogn$. $dq + ds = logn$, that is to say what Boltzmann understood is that each time we take a macroscopic state so a given volume etc., we can have the same macroscopic state, from totally different microscopic states. That is to say that the simplest example is to take red balls and white balls, and stack them in a tank. And you have for instance 50 red balls and 50 white balls. You can see that you can stack them in 36 different ways, okay. But the macroscopic state matching will tell you that there are half of the red balls and the half white balls and that's it. The rest, you don't care. Well, what Boltzmann understood and which is incredible, is that entropy, which was defined completely ad hoc by the people who made steam engine systems and all that, well it was just the logarithm of the number of microscopic realizations of a given macroscopic state. Of course, there had to be a constant ahead. That's what we call Boltzmann constant, it's normal that it bears its name. So this Boltzmann constant, it's not the same as Planck's constant, and it is, well, obviously it must have the dimension of an entropy etc. Okay. But this is the most incomprehensible formula, and the most brilliant whoever that is, this formula of agreement. And it is very difficult to understand. What is very difficult to understand is that the laws of physics, not particle physics, but the laws of ordinary physics are invariant when we change t to $-t$. And if you want, which is very difficult to understand is that one of the fundamental principles of thermodynamics is that entropy increases. So we say : "But the time, which way does it go ?". That haunted people for years and years. And Boltzmann, he had understood countless things simply because of this idea. This is a wonderful example, with a very simple formula, but precisely if you want, that is also a very important thing that I would not have wanted to forget to tell you, which is that there are a number of notions in mathematics or concepts in physics like that, which have an extraordinary power, and this quality is to put thought in motion. This formula this is a typical example, you look at this formula, you try to understand it, there you go, your thinking is moving now. She has an extraordinary potential to set thought into motion. Because you can say to yourself "Why is it increasing ?". Basically, the explanation of Boltzmann's reason why it is increasing is that, in general, we will go to states that have more and more microscopic realizations, that is, which are more and more likely. And then to put it on a solid foundation is another story...

- So you've told us a lot about physics, and we know that in this moment, theoretical physics becomes a bit of a mathematicians haunt, for example with string theory, and suddenly I was wondering if you, in a sense, could consider yourself more like a physicist doing math ?

- It's a good question, I had friends who, knowing my opinions on string theory, said I was a kind of machine in which we put money, then, but if we put 1 euro, I will speak for 10 minutes against string theory. So I'm going to spare you. No, but I will quote a sentence from Hadamard. I have to find it already *(laughs)*. Wait, I have to find it... So, I think I'll get there. Here, it is a sentence on the link between mathematics and physics ; what says Hadamard, to characterize the depth of mathematical concepts that come straight from physics, he says (I say it in English so, but it's very easy to translate into French) :
  "... not this short lived novelty, which can too often only influence the mathematician left to his own devices, but this infinitely fecund novelty, which springs from the nature of things"[1]

  So here's the answer. The answer is that there is no on one hand mathematics, and on the other hand physics. It's the same thing : we try all to understand, okay. And precisely, there is this extraordinary depth in certain mathematical concepts which come directly from physics. Like Heisenberg. It is inexhaustible because it came from what we see, it came from experience, it came from physics, it is nature which speaks to us, which tells us something okay. This is priceless ! But this is not the case with string theory, because string theory is a deviance which came from abstract mathematics, etc., and it has no contact with experience.

- There is another mathematician, whose name is Carlo Rovelli *(precision d'Alain Connes : "He's a physicist, he's a physicist" (laughs))* and he says that for him, the beauty of physics is a simple idea, which opens us up to a completely new world and at the same time this world is real, this world is correct. And I was wondering for you, what do you think about mathematical beauty...

- That's a good question *(sigh)*. First of all, there are a lot of people who, and I think it's true, who will tell you that the notion of beauty is a

1. *"non une brÃ¨ve nouveautÃ© qui souvent influence le mathÃ©maticien rivÃ© Ã ses propres prÃ©occupations, mais une nouveautÃ© infiniment fÃ©conde qui jaillit de la nature des choses."in Jacques Hadamard, PrÃ©face Ã lâintroduction au calcul tensoriel et au calcul diffÃ©rentiel de G. Juvet, Albert Blanchard, Paris, 1922..*

very relative notion, that is to say that everyone has her own different notion etc. of course. But hey, I admit that for me, the mathematical beauty, it is when, after terrible, terribly complicated calculations, we arrive to the same thing, we arrive at the result, but by an idea of incredible simplicity, a bit like the egg of Columbus okay. For me, that's it, the mathematical beauty, for me, beauty is the simplicity of an idea, but actually, of an idea that will... well for example, I don't know... when we were talking about Galois, I will give you an example of this beauty. that said Galois ? Galois says when we take an equation, we take a polynomial equation. So the first thing we're going to do is to find a function of the roots, which when we swap the roots, will take... Well, for instance, we take an equation of degree 5. It is necessary that when we permute the roots arbitrarily, this function takes 120 different values (5 ! different values). So how does he do it, Galois, to find such a function ? It's very simple : he says "if I call the roots A, B, C, D, E, okay, I take A plus 1,000,000 times B + 1 trillion of times C etc. Obviously, when I swap them, it will only take different values. It will take 120 different values". It's the first thing. Second thing, what does Galois say ? He says "well, now let's take for equation the equation which has these 120 roots as roots. We take this equation and we break it down into irreducible factors. We can express the roots of the original equation based on these irreducible factors and we will obtain, by taking these irreducible factors, permutations of roots of the equation. Theorem, that's the mathematical beauty. Theorem : The group of permutations obtained does not depend on any of the choices we have done. That is, if, instead of taking A more than 1 000 000 times B etc., I had taken 1 000 001 or whatever, I would have obtained the same group. That, it's mathematical beauty, it's something incredibly beautiful. Why ? Because that means we gave a recipe, which looked completely arbitrary, and we arrived at an invariant, we arrived at a group, which is a characteristic of the equation, which will give all the results that we want, and that is of a biblical simplicity, at the end, that is to say that the how it is defined is biblically simple. For me, that's it, mathematical beauty. But this is an example, I mean, define it abstractly, if we gave an abstract definition, it is obvious that we could find a counterexample... In fact, if you're looking for general things about beauty in math, on things like that, read Grothendieck's *Crops and Sowing.*

Because that, it is... Grothendieck was not only a mathematician, in fact, he was a literary, and he was someone who was capable in his writings, to go very very far in the analysis of what mathematics is, of what that beauty in mathematics is, etc. So he wrote 1500 pages, these 1500 pages, you can find them on the internet, okay. And don't listen to people who will tell you he's crazy because it's not true, it's not true : he was someone who was wonderfully intelligent, and who wrote wonderfully as a literary. He has an extraordinary vocabulary, etc. I made a presentation, at the seminar of Antoine Compagnon, on Grothendieck and Proust, comparing them precisely, and I think it's available this talk, maybe on the Collège de France website or on my website. Therefore, because I mean, because it's very striking, it's very striking to see that these are two individuals who have achieved something that few people both succeed in, which is not only a work, for Grothendieck, but also if you want what Grothendieck says, what he explains is that in fact, if we want to be realized of course, well, it's good to make a real analysis but in fact the main difficulty we have, is to understand yourself, and to understand yourself, it seems silly *(laughs)*, right ? And to understand yourself, you have to basically self-analyze, that's what Grothendieck did and in a way, this is what Proust also did in his book. They are also people who from one point on, stopped living and spent the rest of their life to re-analyze and understand their past life, okay, etc. And in both cases, the result is wonderful. Therefore the best answer, I think, is to go to *Crops and Sowing*, and reading it, not leafing through it, you really have to read it, you have to read it carefully, you see, like the passages that I read to you quite time.

- Please specify : *Crops and Sowing*, this is the book that Grothendieck wrote and which we have had access to for a short time, finally...

- No, not a long time ago, we had access to it for a very long time, but well, he wrote other books. He's a character, all these characters have extraordinary lives. Grothendieck had an extraordinary life because that : in 1970, he left the Institute of High Scientific Studies (IHES) and he became again, because that was his fundamental temperament I think, a bit of an outcast if you want. And from 1991 he took refuge in a village in the Pyrenees. And no one had any news of him, but he continued to work, he continued to write. And then, not only he wrote *Crops and Sowing*, but he also wrote another wonderful text, which is called *The key to dreams*. And it's a mystical text, but I'll say it's the same, I mean, it's the same... But it's extremely interesting, but I think for example, for people who are literary women, these texts have an infinite value. There are theses to do on this, there are 36 things to do, of course...

- So I saw somewhere on the internet I think, I'm not sure of my sources, that in fact you thought that mathematics existed were without men in fact, that even it was not an invention made by men, and I have a little trouble understanding that actually, because often, we see math as something very abstract that would not exist without men inventing them, so can you explain that ?

- Okay, I can give you the answer. The answer is very simple. You take chemistry. This is a subject that I, myself, discuss when I was in Math Sup and Math Spe, so you have all this stuff. So we have the compound bodies then we have the simple bodies. Simple bodies, there is the periodic table of the elements. The periodic table of the elements, unbelievable but true, there is the Pauli exclusion principle, and a very small equation, which gives it to you. That's enough for me. Why ? Because let's imagine that there is another planetary system etc. If they are intelligent beings, they will understand chemistry, that there are simple bodies, they will be the same, they will not have... I mean they will not have chemical bodies, they will not have simple bodies different from ours, so they're going to understand simple bodies. And then, if they're really smart, they'll try to find, good, they will have the periodic table of the elements. They will try to find what is the abstract origin of the periodic table of the elements. Well if they are really smart, they will find the same thing, they will find that there is the Pauli exclusion principle. And then there is this little equation... What does it mean ? It means that, behind the apparent, how say, arbitrary of the world around us, there are some incredibly simple rules, chemistry, iteration since trees are all governed by that iteration, and that in fact, there is a way of understanding the world, which instead of being chaos, if you will, is something much more structured, and that is structured by mathematics. And there is no reason why, of course, people give the same names to the mathematical concepts they will have used, but it is quite clear that they will use the... , if they are different beings and they will have 1, 2, 3, 4, 5. They will not say it from the same way, but they will use mathematical language, this language will be in correspondence with ours, as the Chinese language is in correspondence with ours. So, it is in this sense, it is in this absolutely fundamental sense, that I say that mathematics preexists, why ? Because it would be incredibly pretentious to say that we invented whole numbers. So at that time, why would chemistry have already used these things to exist ? It seems completely stupid. So actually, what I'm saying, is that, when we found, when Watson and Crick found the DNA structure in double helix, they didn't invent it, nobody will believe that they invented it of course, they discovered that. It was a reality, and this reality, it pre-existed to them. Well, for math, it's the same, it's exactly the same, that is to say that we discover, a bit like an explorer will discover something. This explorer, he has free will, he can go to such and such a place. This is what makes people believe people actually it's like art. But no ! It's not art, it's exploration. And it's an exploration all the more... how to say ? real, that this reality, it resists. If it was art, we could say anything. This is not the case. This is not the case. There is a terrible resistance... And so an example, another typical example, is that if for example, I write an equation etc., and then good, for example, Galois said that the calculations we had to do to follow his method were impossible to do and in his day it was impossible to do. And for showing that it was impossible to do, when I gave my presentation to the Académie, I explained to people the Galois method and I asked them "Can you give me an idea... ?", Well, because each root is expressed as a polynomial according to the roots of the auxiliary equation. I asked them, I gave them an equation like the one that I gave you a while ago, I told them "can you give me the order of magnitude of the coefficient of order 0 of the polynomial which expresses the first root ?"... 1 million out of 1 million, or something like that. No, the answer is a 500-digit number over a 500-digit number ! The computer does it now, and the computer checks that Galois was right ! So to say that Galois invented it, I mean, it's a bit big, what ! No no no ! No no no ! We discover, we discover, but exactly as the electron microscope was necessary to find out the double helix structure of DNA, the mathematician invents tools conceptual to successfully perceive this reality. But he invents conceptual tools of course. But it's a reality that is there, it resists, it is completely tangible and it governs nature. It's more fundamental, for me, this reality is more fundamental than the nature that is around. It pre-exists that, okay. If you want, it would be the same... I think that it would not be fair to say that nature is only written in the language of mathematics ; math is more than that, it's more than that. Nature is consubstantial with mathematics. We don't realize it because you're not smart enough to give an account of the explanation which is behind all these phenomena. If we realized more, we would know much better. And it's all more true with quantum. I mean, the quantum there is obvious. The quantum, it's a reality that we only perceive through mathematics, we absolutely does not perceive it otherwise. That is to say that the people who make experiments with quantum, in quantum optics, they understand that it's that Hilbert space, they touch it like that, okay, it's incredible, really incredible !

- Thank you.

- Are you ready to answer a few more questions ?

- Yes, it's okay, it's okay.

- For us to have more facilities for example with the abstract concepts, in mathematics or in physics, or even with reasoning, what would you recommend in education and teaching, to present, in primary school or even in secondary ?

- I will answer, first not in primary school, nor in the secondary, I answer for you, because it is the most useful. So for you, what I advocate is the following. To answer a question, even a complicated math question, you leave everything in the air, you're going to do a walking tour, okay. And the question, you keep it in your head, okay, you keep it in your head, and you reflect. Obviously, it can be different depending on the individual, but for me, it's the big secret. That is to say a calculation, as complicated as it is, you can say to yourself "Oh ! I will never get there if I try like that !" No ! You go for a walk, and you think about the structure of the thing, and after when you come back, well, you will see that it improves things drastically. So now, in primary school, I don't know, me, everything I can tell you, it's my own experience, because I don't know any else. But my own experience is that when I was a kid, when I was 5 years, my father forced us to do calculations. We were in the garden with him, he was with us, and he made us do operations, and at the time, we did the four operations, that is to say we did the division, at 5 years old, we was multiplying, we hadn't waited for the sixth to learn the division, and all that, we did that. And after another experience which happened to me... And I loved it, it was probably also my relationship with my father, it scared me at the same time, but I was happy to make him happy. Anyway, I don't know, so I don't know, I don't know how to explain that : I will say that I think there was an extraordinary virtue in doing operations like that, that is, memorizing the multiplication table and then we didn't forget the multiplication table, if we made multiplications and additions all day long, we did not forget it, we knew it afterwards. And it became absolute automatism. So there was that, and I liked it a lot. Another story than I have, it is that once, that, I find it absolutely extraordinary, once, I met a friend I hadn't seen for a while, we were playing soccer together, in time, and then maybe 8 or 9 years later, I take the TGV to go to, I think it was in Rennes or something like that, and then I go to my place of TGV and then, I looked at my number, and I see someone next to it kept his number and he was my boyfriend. We started to chat etc. And then then, the usual discussion, you have children, he begins to explain to me he has a son, and that his son is weird. It must be said that my boyfriend is literary. I said "why ?". Well you know, well first, he had been sick when he was young and then once, when he was 5, we were together, we were on the beach and then he looked painful ; I was worried, I mean for an hour he was there instead of going for a swim he was a little white and then after an hour, he comes to see me, so he's my boyfriend who tells, he comes to see me, and he says to me : "Daddy there is no greater number !". I said to him "Look, your son, he's great !" *(laughs)*. He tells me. "Ah ? Yes of course !". I asked him if his son had found a demonstration and he had found a demonstration, which is not the usual demonstration, it was not adding 1, it was multiplying by 2 or something like that, whatever, he had found a demonstration. It's amazing, but afterwards, he said to me, "you know, he had problems at school" *(frank laughter)*. So he told me about his problems at school. So that will answer your question for primary school. He was in elementary school, so we had him posed the following problem : it was "a florist has 120 flowers, she makes 4 bouquets of 17 flowers, how many flowers are left ?", okay. So he had had "zero, has no sense of operations". He was not stupid, he has found 120 since she didn't give them *(laughter from all)*. When he told me that, I said "well listen, your son, he's a mathematician". And so, we organized therefore with his father a meeting at Tea Caddy in Paris, it is a charming place. And her son at the time was 12 years old. So, things have evolved luckily, it took a while, and now he's a great mathematician who is a teacher at Orsay. So incredible, incredible, incredible ! So I believe that what matters is what Grothendieck says is to go back to this childhood state and to ask the right questions, and then not to hesitate to be against the current etc. Above all I mean that the moment we become mathematician, this is the moment when we are able to tell the teacher that he is wrong and why, that means being able to resist its authority, to say that we have reflected, and that we do not agree, and then to be sure of ourselves, because that one has thought for oneself, it is hyper-important.

- No actually, I have a question, even if I think you have a little answered this question by your comments, but I would really like to know for you what is the purpose of the work of a scientist finally if you think that rather, it is to increase, to advance fundamental knowledge, or to make it accessible to its public for possible application.

- There are these two aspects, which we should not mix at all, there are these two aspects. I think the real motivation is to move forward basic knowledge. In other words, the real motivation which must be precisely independent of any authority, of any desire to knowledge, etc., the real motivation is to try to understand, understand where we are, where we landed, okay, that's it, it's very simple to understand, that's where we are.

- No, but on this question of the popularization of knowledge and in particular mathematical knowledge. To hear you, there is a risk : the effect butterfly is one. I was going to say what I had learned, for example, from entropy, what philosophers, what some philosophers can do of entropy, there is sometimes indeed a great danger of a kind of... and at the same time, you seem to be saying that there is a need. So if we basically asked "how to avoid the danger and respond to the need, would you be care... ?

- Yes, yes, that is a very good question. There was Sokal, there was Deleuze. There was Lacan, I don't know if you know, but Lacan said in a seminar that the number $\sqrt{-1}$ is the symbol of the male sex, okay. This is what we call the imaginary pure number ! ! *(laughs)*. You had to do it anyway, huh ? ! And in addition, he once did a seminar, where he had a theorem, and his theorem was that "Don Juan is compact". Someone had told him the definition of a compact space in mathematics. So that, of course, it's stupid, okay, it's absolutely stupid. And what is it ? These are poorly understood math concepts that are used as a psychological authority over others, that is, they are used because people will not understand and the butterfly effect is a blatant example, as a psychological authority because people when they don't understand, they are in a position of inferiority, their understanding is blocked, and if you want, they are impressed etc. So there is this terrible way to use math which is precisely to use large words, like a kind of psychological power over crowds. So that, it is to be banished at all costs. However, I'm sorry if you want, that concepts as beautiful as Grothendieck's concept of topos, not be better known by people who would need it because like I tell you we are all now victims of scientism which consists to believe that a thing is true or false when in reality there are situations that are much more subtle than that, much more subtle than that and who ask for a thought tool that the concept of topos gives and it's a notion which is delicate, which is difficult, which requires, to know it, to understand it, mathematical knowledge. So what I will say if you want is that there is a beautiful boulevard that is open. This boulevard is to learn enough math to use it afterwards in the right way in other areas, but first you have to start to learn enough math, that's the price to pay, it's absolutely necessary okay.

- My question was precisely in relation to the question of truth : by hearing you, one has the impression that really mathematics, it allowed to reach this truth with quantum physics, and I wanted to know, I believe it was Einstein who said that "the world is a bit like a closed box", you can just see what's going on, but you can never be sure that what we find is true. And what do you think of that, of the idea that maybe all we are explaining are theories that are ultimately false, for example, Einstein, who put everything back in cause in physics... ?

- There is always indeed the possibility of a theory above, which will simplify what's on the front floor etc. But we can still see that progress, from this point of view. What I was trying to get across, it's the extraordinary subtlety, the richness of nature, where you are. The fact that each time we will have surprises and we will have extraordinary surprises. Since at the end of the XIX$^{\text{th}}$ century, there were physicists who said that we had understood everything. And precisely, we were before the quantic era, before all that, before general relativity. It's true, perfectly true, what you say. But, I will insist more on the wonderful imagination of the nature, what, I mean, we are surely very, very far away, there may be civilizations, there are surely civilizations, in other inhabited planets, in which people have gone much further than us. That's really possible and they would take us for primitives. It is possible, it is absolutely possible.

- You said that mathematics were not in the domain of known, and globally, the sciences. But don't you think that this is because mathematicians, physicists and others do not participate not enough to debate, for example, when defining scolar programs, we hear philosophers, historians...

- That's right, there is truth in there, there is truth. But on another side, that's not really the problem. I will not say that the problem comes from fact that it is not popularized enough. I think the problem comes more from the slow absorption by the whole of society of elaborate notions. For example, I take a typical example, which for me is important. You see, when the printing press was discovered, the notion of number has been transmissible. That is to say, there have been books, etc., etc. Now, we are at the point where it is no longer the number that is transmissible, but it is the notion of function, graph, etc. And there is a vocabulary that has passed in the general public, for example, when we say that we are going to reverse the unemployment curve *(laughs)*. So that involves the cancellation of the second derivative. If we were told “we’re going to cancel the second derivative...”, there would be something to laugh about *(laughs)*. Well, well, you kind of see... So it’s sure that, if you will, there are now some mathematical notions, including the notion of growth function, decrease, primary derivative, cancellation of first derivative, etc. that have gone out to the general public, okay, but there are much more subtle notions, like the notion of topos, like the notions that come from quantum etc. who have more trouble to pass in the general public. How to get them across to the general public ? No doubt through education, but at that point, we should be very braver, in the school system, it’s obvious, it’s absolutely obvious. We should not be in the current renunciation which is dismal. I knew that in my time, well I mean, we didn’t stop, I did not stop and my friends did not stop, making problems of geometry. We came home, and we made problems, and they were not easy problems at all. And now it’s over ! Now we learn recipes, we learn to apply recipes ; of course it’s much easier for a teacher. One day when you have children, you will notice one thing that is absolutely fundamental is that when you have a small child, you have a choice. The little child, he tries to do something, you have two possibilities : the first possibility, is to do this thing for him, you believe you are helping him, in fact, you don’t help him at all, you harm him by doing that ; the second thing is to be patient, and to wait for it to happen by itself. And there you do something really useful, okay. So good, in what we do in the current school system, which is to learn ready-made recipes for solving ready-made problems is doing things instead of the child. That’s exactly what we do, it’s exactly that thing. So that the real discovery of work, the real discovery of school, it must be between 11 and 12 years old. And it must be done by drying in front of problems which we are not given the solution for, but which we ask you to find, okay, where you are asked to dry. And from the moment where at that age, we understood what real work is, it’s okay, it’s okay, it’s OK. And that is not the case in the current school system. Of course, far from there, terribly. Of course. There is Laurent Lafforgue who tried, and of all his strength, to go in that direction, well, he spent a lot of energy ; to say that he got there, that would not be true, I will say, in any case, there are people who made a colossal effort to go the right way, now afterwards, there is a terrible resistance from the system.