SUITE DES NOTICES SUR LES FONCTIONS ELLIPTIQUES 21 JULY 1828

CARL JACOBI

Abstract. This is a translation of Carl Jacobi's *Suite des notices sur les fonctions elliptiques* published in Crelle's Journal in 1828. The full citation is given in reference [5]. The paper is a letter to August Crelle which continues Jacobi's advertisement of elliptic function results published earlier in the same volume of Crelle's Journal (see [4]). Some of the notation in Jacobi's paper has been modernized in this translation.

In this letter, Jacobi reformulates elliptic functions in terms of theta functions in order to obtain simplifications of elliptic function theory. He briefly discusses modular transformations in both elliptic function and theta function theory. He uses the Fourier expansions of the theta functions to derive a partial differential equation, specifically the heat equation, which the theta functions satisfy. He turns to an applications of an identity discovered by Poisson and concludes with a discussion of the application of elliptic functions to representations of integers as sums of squares.

I add the development of elliptic functions of the second and third kinds to the formulas given in my last letter [4]. Following Mr. Legendre, define:

$$\Delta(\phi) := \Delta(\phi, k) := \sqrt{1 - k^2 \sin^2 \phi},$$

$$\mathrm{E}(\phi) := \mathrm{E}(\phi|k) := \int_0^\phi \Delta(\phi)\, d\phi,$$

$$\mathrm{E} := \mathrm{E}(k) := \mathrm{E}(\tfrac{\pi}{2}|k),$$

$$\mathrm{F}(\phi) := \mathrm{F}(\phi|k) := \int_0^\phi \frac{d\phi}{\Delta(\phi)},$$

$$\mathrm{K} := \mathrm{K}(k) := \mathrm{F}(\tfrac{\pi}{2}|k).$$

Following my own conventions, if we set $\phi = \mathrm{am} \frac{2\mathrm{K}x}{\pi}$ and $q = \exp(-\pi\mathrm{K}'/\mathrm{K})$, then we obtain

$$\mathrm{KE}(\phi) - \mathrm{EF}(\phi) = 2\pi \frac{\sum_{n=1}^{\infty} nq^{n^2} \sin 2nx}{1 - 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nx} = 2\pi \sum_{n=1}^{\infty} \frac{q^n \sin 2nx}{1 - q^{2n}}. \tag{1}$$

Date: January 14, 2004.

1991 *Mathematics Subject Classification.* Primary: 33E05; Secondary: 05A19.

Key words and phrases. elliptic functions, elliptic integrals, Jacobi, theta functions, sums of squares.

Acknowledgments: Many thanks to Alan Lewis for pointing out several sign errors that I made in the course of transcribing and typesetting Jacobi's identities.

Mr. Legendre showed that the following expression, depending on elliptic functions of the second and third types, is symmetric in the angles A and ϕ:

$$\int_0^\phi \left[\frac{2k^2 \sin A \ \cos A \ \Delta(A) \ \sin^2\phi}{\Delta(\phi)(1-k^2\sin^2 A \ \sin^2\phi)}\right] d\phi - \frac{2F(\phi)\ (\mathrm{KE}(A)-\mathrm{EF}(A))}{\mathrm{K}}. \tag{2}$$

Setting $\phi = \mathrm{am}\frac{2Kx}{\pi}$ and $A = \mathrm{am}\frac{2K\alpha}{\pi}$, I obtain:

$$\log \frac{1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n(x-\alpha)}{1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n(x+\alpha)} - 4\sum_{n=1}^{\infty}\frac{q^n \sin 2n\alpha \ \sin 2nx}{1-q^{2n}}.$$

These equations are symmetric in x and α. The first covers all cases of elliptic integrals of the third kind in which α is ranges over complex values.

The elliptic functions can be replaced by the new transcendental function ϑ_4 defined by the following series expansion:[1]

$$\vartheta_4(x,q) := 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nx.$$

Letting $\omega = -\log q$, we obtain the following identities:

$$\vartheta_4(x+\pi,q) = \vartheta_4(x,q), \tag{3}$$
$$\vartheta_4(x+\imath\omega,q) = -q^{-1}e^{-2\imath x}\vartheta_4(x,q), \tag{4}$$
$$\vartheta_4(\tfrac{\imath\omega}{2},q) = 0, \tag{5}$$
$$\vartheta_4(x+\tfrac{\pi}{2},q) = \vartheta_4(x,-q). \tag{6}$$

Define:

$$\begin{aligned}\vartheta_1(x,q) :&= -\imath q^{\frac14}e^{\imath x}\vartheta_4(x+\tfrac{\imath\omega}{2})\\ &= \imath q^{\frac14}e^{-\imath x}\vartheta_4(x-\tfrac{\imath\omega}{2}).\end{aligned}$$

We obtain the series expansion:

$$\vartheta_1(x,q) := -2\sum_{n=1}^{\infty}(-1)^n q^{(2n-1)^2/4}\sin(2n-1)x.$$

In addition:

$$\vartheta_1(x+\pi,q) = -\vartheta_1(x,q), \tag{7}$$
$$\vartheta_1(x+\imath\omega,q) = -q^{-1}e^{-2\imath x}\vartheta_1(x,q), \tag{8}$$
$$\vartheta_1(\imath\omega,q) = 0, \tag{9}$$
$$\vartheta_1(x+\tfrac{\imath\omega}{2},q) = \imath q^{-\frac14}e^{-\imath x}\vartheta_4(x,q), \tag{10}$$
$$\vartheta_4(x+\tfrac{\imath\omega}{2},q) = \imath q^{-\frac14}e^{-\imath x}\vartheta_1(x,q). \tag{11}$$

[1] In this paper, Jacobi uses the notation Θ for ϑ_4 and H for ϑ_1. This differs by rescaling of the argument from the usages in the *Fundamenta Nova*. Since the notations Θ and H are conventionally reserved for the theta functions of the *Fundamenta Nova*, we use the more familiar notation here.

We can write the elliptic functions in terms of ϑ_1 and ϑ_4 (or in terms of either one of them):

$$\operatorname{sn}\frac{2Kx}{\pi} = \tfrac{1}{\sqrt{k}}\,\frac{\vartheta_1(x,q)}{\vartheta_4(x,q)}, \tag{12}$$

$$\operatorname{cn}\frac{2Kx}{\pi} = \sqrt{\tfrac{k'}{k}}\,\frac{\vartheta_1(x+\frac{\pi}{2},q)}{\vartheta_4(x,q)}, \tag{13}$$

$$\operatorname{dn}\frac{2Kx}{\pi} = k'\,\frac{\vartheta_4(x+\frac{\pi}{2},q)}{\vartheta_4(x,q)}. \tag{14}$$

The invariants k, k' and K are obtained from the following identities:

$$\sqrt{\frac{2\mathrm{K}}{\pi}} = \vartheta_4(\tfrac{\pi}{2}) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \tag{15}$$

$$\sqrt{\frac{2k'\mathrm{K}}{\pi}} = \vartheta_4(0) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}, \tag{16}$$

$$\sqrt{\frac{2k\mathrm{K}}{\pi}} = \vartheta_1(\tfrac{\pi}{2}) = q^{\frac{1}{4}}\vartheta_4(\tfrac{\pi}{2}+\imath\omega) = 2\sum_{n=1}^{\infty} q^{(2n-1)^2/4}. \tag{17}$$

The elliptic integral of the third kind (1) reduces to the simple expression:

$$\log\frac{\vartheta_4(x-\alpha,q)}{\vartheta_4(x+\alpha,q)}.$$

Consider the following striking identity, easily obtained using elementary trigonometry:

$$\vartheta_1(x,q)\vartheta_4(y,q) - \vartheta_1(y,q)\vartheta_4(x,q) = \vartheta_1\left(\tfrac{x-y}{2},\sqrt{q}\right)\vartheta_4\left(\tfrac{x+y}{2},\sqrt{q}\right) \tag{18}$$

i.e.:

$$\left[\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)^2/4}\sin(2n+1)x\right] \times \left[1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nx\right] \tag{19a}$$

$$-\left[\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)^2/4}\sin(2n+1)X\right] \times \left[1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nX\right] \tag{19b}$$

$$= 2\left[\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)^2/8}\,\frac{\sin(2n+1)(x-X)}{2}\right] \tag{19c}$$

$$\times\left[\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)^2/8}\,\frac{\sin(2n+1)(x+X)}{2}\right] \tag{19d}$$

From this identity, we obtain equations (12,13,14), Euler's elliptic function summation theorem, the derivative:

$$\frac{d}{dx}\operatorname{sn}\frac{2\mathrm{K}x}{\pi} = \frac{2\mathrm{K}x}{\pi}\operatorname{cn}\frac{2\mathrm{K}x}{\pi}\operatorname{dn}\frac{2\mathrm{K}x}{\pi},$$

and a number of other results.

The two theta functions can be formulated as infinite products:

$$\vartheta_4(x,q) = C\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2x + q^{4n-2}),$$

$$\vartheta_1(x,q) = 2\sqrt[4]{q}\,C\ \sin\ x\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2x + q^{4n}),$$

where C is a constant. Applying Côtes' theorem to these products gives the general theory for transformation and multiplication of theta functions, and, as a corollary, that of the elliptic functions. In effect, by Côtes' theorem, if n is any odd positive integer, then

$$\prod_{m=0}^{n-1}\vartheta_4(x+\tfrac{2m\pi}{n},q) = C'\vartheta_4(nx,q^n), \tag{20}$$

$$(-1)^{(n-1)/2}\prod_{m=0}^{n-1}\vartheta_1(x+\tfrac{2m\pi}{n},q) = C'\vartheta_1(nx,q^n), \tag{21}$$

where C' is another constant. If we denote by $\mathrm{K}^{(n)}$ and $k^{(n)}$ the quantities which depend on q^n in the same way as K and k depend on q, then using identity (12), we obtain:

$$\mathrm{sn}\frac{2\mathrm{K}^{(n)}x}{\pi} = \frac{1}{\sqrt{k^{(n)}}}\frac{\vartheta_1(x,q^n)}{\vartheta_4(x,q^n)}.$$

Now dividing equations (21,20), we obtain the identity:

$$\mathrm{sn}\left(\frac{2n\mathrm{K}^{(n)}x}{\pi},k^{(n)}\right) = (-1)^{\frac{n-1}{2}}\sqrt{\frac{k^n}{k^{(n)}}}\prod_{m=0}^{n-1}\mathrm{sn}\left(\frac{2\mathrm{K}(x+\frac{2m\pi}{n})}{\pi},k\right),$$

a general formula for transformation of elliptic functions, the same as one that I first established. Other real and imaginary transformations may be associated to the number n in a similar manner.

Since the elliptic functions are easily defined in terms of the theta functions, one might in turn attempt to express theta functions in terms of elliptic functions. This is accomplished by integrating equation (1):

$$\log\frac{\vartheta_4(x,q)}{\vartheta_4(0,q)} = \int_0^{\phi}\frac{\mathrm{KE}(\phi)-\mathrm{EF}(\phi)}{\mathrm{K}\Delta(\phi)}d\phi \tag{22}$$

where $\phi = \mathrm{am}\frac{2\mathrm{K}x}{\pi}$. From equation (2), we obtain:

$$\log\frac{\vartheta_4(0,q)}{\vartheta_4(2\alpha,q)} = \int_0^{A}\frac{2k^2\ \sin A\ \cos A\ \Delta(A)\ \sin^2\phi\ d\phi}{(1-k^2\ \sin^2 A\ \sin^2\phi)\ \Delta(\phi)} - \frac{2\mathrm{F}(A)\ (\mathrm{KE}(A)-\mathrm{EF}(A))}{\mathrm{K}} \tag{23}$$

where $A = \mathrm{am}\frac{2K\alpha}{\pi}$, as above.

Let us move on to other matters. Letting $\omega = -\log q = -\imath\pi\tau$, we obtain the Fourier series expansions:

$$\vartheta_1(x,q) = -2\sum_{n=1}^{\infty}(-1)^n e^{-(2n-1)^2\omega/4}\sin(2n-1)x,$$

$$\vartheta_4(x,q) = 1+2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\omega}\cos 2nx.$$

From these expansions, we obtain the partial differential equations

$$\frac{\partial^2}{\partial x^2}\vartheta_1(x,q) = 4\frac{\partial}{\partial \omega}\vartheta_1(x,q), \tag{24a}$$

$$\frac{\partial^2}{\partial x^2}\vartheta_4(x,q) = 4\frac{\partial}{\partial \omega}\vartheta_4(x,q). \tag{24b}$$

Among these two solutions ϑ_1 and ϑ_4 to the partial differential equation[2]

$$\frac{\partial^2 z}{\partial x^2} = 4\frac{\partial z}{\partial \omega},$$

we have the relations:

$$\vartheta_1(x,q) = \imath e^{-\imath x - \omega/4}\vartheta_4(x - \tfrac{\imath\omega}{2}, q)$$
$$\vartheta_4(x,q) = \imath e^{-\imath x - \omega/4}\vartheta_1(x - \tfrac{\imath\omega}{2}, q)$$

In general, if $z = z_1(x,\omega)$ is a solution to the differential equation $\frac{\partial^2 z}{\partial x^2} = 4\frac{\partial z}{\partial \omega}$, then it is easily verified that there is a second solution

$$z = z_2(x,\omega) = \exp(-\imath x - \omega/4) z_1(x - \imath\omega/2).$$

Now let

$$\begin{aligned} u :=& \sqrt{\frac{2k'\mathrm{K}}{\pi}} = \vartheta_4(0,q) \\ =& 1 + \sum_{n=1}^{\infty}(-1)^n q^{n^2}, \\ v :=& \sqrt{kk'\left(\frac{2\mathrm{K}}{\pi}\right)^3} = \vartheta_1(\tfrac{\pi}{2},q)\vartheta_4(\tfrac{\pi}{2},q)\vartheta_4(0,q) \\ =& -\sum_{n=1}^{\infty}(-1)^n(4n-2)q^{(2n-1)^2/4}. \end{aligned}$$

Applying the differential equation, we obtain the expansions:

$$\vartheta_1(x,q) = \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(2x)^{2n+1}}{(2n+1)!}\,\frac{d^n u}{d\omega^n}, \tag{25}$$

$$\vartheta_4(x,q) = \sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!}\,\frac{d^n u}{d\omega^n}, \tag{26}$$

If we divide these two equations, we obtain:

$$\mathrm{sn}\frac{2\mathrm{K}x}{\pi} = \frac{1}{2\sqrt{k}}\,\frac{\sum_{n=0}^{\infty}\frac{(2x)^{2n+1}}{(2n+1)!}\,\frac{d^n u}{d\omega^n}}{\sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!}\,\frac{d^n u}{d\omega^n}}, \tag{27}$$

[2] *i.e.* the heat equation

where values of $\frac{d^n u}{d\omega^n}$ and $\frac{d^n v}{d\omega^n}$ are obtained from the equation

$$\frac{d\omega}{dk} = \frac{-2}{k(1-k^2)\left(\frac{2\mathrm{K}}{\pi}\right)^2}, \tag{28}$$

easily deduced from known equations.

Mr. Poisson, in his researches into definite integrals, determined a number of properties of $\vartheta_4(x,q)$. The delicate methods of this noted geometer find delightful verification in the theory of elliptic functions. For example, in issue 19 of the *Journal of the Polytechnic School*, Mr. Poisson established the following identity:

$$x^{-1/2} = \frac{1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi x}}{1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi/x}} = \frac{\vartheta_4(x,-q)}{\vartheta_4(x^{-1},-q)}$$

If we let $x = \frac{\mathrm{K}'}{\mathrm{K}}$ and transform $k \mapsto k' = \sqrt{1-k^2}$, then we transform $x \mapsto \frac{\mathrm{K}}{\mathrm{K}'} = x^{-1}$. Starting with

$$\begin{aligned}\sqrt{\frac{2\mathrm{K}}{\pi}} &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}\\ &= 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi x}\end{aligned}$$

and transforming $k \mapsto k'$, we obtain

$$\sqrt{\frac{2\mathrm{K}'}{\pi}} = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi/x}.$$

Dividing these two results immediately gives Mr. Poisson's identity.

We return to the theory of transformation. If modulus k is transformed into modulus λ of order n, then we obtain an algebraic relation between k and λ whose degree in either variable is the sum of the divisors of n. Letting $q \mapsto q^{\frac{a'}{a}}$ where $aa' = n$ in the equation[3]:

$$\sqrt{k} = \frac{\vartheta_1(\frac{\pi}{2},q)}{\vartheta_4(\frac{\pi}{2},q)} = \frac{2\sum_{n=1}^{\infty} q^{(2n-1)^2/4}}{1 + 2\sum_{n=1}^{\infty} q^{n^2}}.$$

Let

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{\mathfrak{M}\,dx}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}}$$

be the differential equation satisfied by a rational expression of y in x, in which x appears with degree n: we can write $\mathfrak{M}$ as a rational function of k and λ by means of the general formula:

$$\mathfrak{M}^2 = \frac{n(k-k^3)d\lambda}{(\lambda-\lambda^3)dk}.$$

[3]The expression as given in [1] contains a typesetting error: the term involving q^9 in the denominator is omitted.

Eliminating λ using the modular equation, we obtain an equation of the same degree between k and $\mathfrak{M}$. These equations between k and $\mathfrak{M}$ display a property worth noting, namely, if n is a prime number, then we can express half the values of $\sqrt{\mathfrak{M}}$ by linear combinations of the other half. Letting $\mathfrak{M}$, $\mathfrak{M}'$, $\mathfrak{M}''$, $\ldots \mathfrak{M}^{(n)}$: denote the roots of the degree $(n+1)$ equation between $\mathfrak{M}$ and k, we obtain:

$$
\begin{aligned}
\sqrt{\mathfrak{M}} &= A\sqrt{(-1)^{\frac{n-1}{2}} n}, \\
\sqrt{\mathfrak{M}'} &= A + A' + A'' + A''' + \ldots + A^{\frac{n-1}{2}}, \\
\sqrt{\mathfrak{M}'} &= A + \alpha A' + \alpha^4 A'' + \alpha^9 A''' + \ldots + \alpha^{\left(\frac{n-1}{2}\right)^2} A^{\frac{n-1}{2}}, \\
\sqrt{\mathfrak{M}''} &= A + \beta A' + \beta^4 A'' + \beta^9 A''' + \ldots + \beta^{\left(\frac{n-1}{2}\right)^2} A^{\frac{n-1}{2}}, \\
&\ldots
\end{aligned}
$$

where α and β are the imaginary roots of $x^n = 1$. Then we can write the square roots of the $(n+1)$ roots by linear combinations of $\frac{n+1}{2}$ other quantities. This gives the theorem stated, one of the most important in algebraic theory of transformation and division of elliptic function. We obtain the same theorem by the relationship between equations giving $\lambda\mathfrak{M}$, $\lambda'\mathfrak{M}$, *etc.* in terms of k. A similar expansion for $n = 5$, $x = \lambda\mathfrak{M}$ is:

$$x^6 - 10kx^5 + 35k^2x^4 - 60k^3x^3 + 55k^4x^2 - [26k^5 + 256(k - k^3)]x + 5k^6 = 0.$$

Setting $x = y + k$, this reduces to:

$$y^6 - 4ky^5 - 256(k - k^3)(y + k) = 0.$$

Using elliptic functions, we can solve a problem posed by Euler in connection with Fermat's theorem which states that every non-negative integer is the sum of four squares. The solution is to show that the fourth power of a generating function of the form

$$\sum_{n=0}^{\infty} a_n q^{n^2}$$

contains every power of q. Specifically, I showed that

$$
\begin{aligned}
\left(\frac{2\mathrm{K}}{\pi}\right)^2 &= \left(1 + 2\sum_{n=1}^{\infty}\right)^4 \\
&= 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1 + (-q)^n} \\
&= 1 + 8\sum_{n=1}^{\infty} \frac{q^n}{(1 + (-q)^n)^2} \\
&= 1 + 8\sum_{n=1}^{\infty} \sigma_1(2n-1)\left(q^{2n-1} + 3\sum_{m=1}^{\infty} q^{2^m(2n-1)}\right)
\end{aligned}
$$

where $\sigma_1(n)$ is the sum of the divisors of n. Fermat's four square theorem follows as a corollary. In addition, we obtain theorems about the numbers of representations of a given number in terms of four squares from this and similar formulas[4]. (A similar result was stated in [3].) A careful

[4]For other examples, see Jacobi §40 of Jacobi's *Fundamenta Nova* or Smith [6] §127. Jacobi's formula for $\left(\frac{2\mathrm{K}}{\pi}\right)^2$ tells us that, if m is the sum of those divisors of a positive integer n which do not have 4 as a factor, then $8m$ is the

examination of the combinatorial algorithm which gives rise to these striking results leads to new methods in number theory.

Elliptic functions differ from ordinary transcendental functions in a fundamental way; specifically, they contain everything periodic in analysis.[5] While trigonometric functions have one real period and exponential (hyperbolic) functions have one imaginary period, elliptic functions have two fundamental periods:

$$\mathrm{sn}(u + 4\mathrm{K}, k) = \mathrm{sn}(u + 2\imath\mathrm{K}', k) = \mathrm{sn}(u, k).$$

Moreover, it is easy to show that an analytic function cannot have more than two fundamental periods, either one real and the other imaginary, or both imaginary. This latter case corresponds to an imaginary modulus k. The ratio $\frac{\mathrm{K}'}{\mathrm{K}}$ of the two periods determines the modulus: k may be found using equations (15) and (17).[6] It might be desirable to introduce this ratio as a modulus in place of the invariant[7] k. With respect to this quotient, I have found that:

Theorem. *k is fixed under a transformation of $\frac{\mathrm{K}'}{\mathrm{K}}$ to*

$$\frac{c\mathrm{K} + \imath d\mathrm{K}'}{\imath(a\mathrm{K} + \imath b\mathrm{K}')} = \frac{\mathrm{K}\mathrm{K}' - \imath(ac\mathrm{K}^2 + bd(\mathrm{K}')^2)}{a^2\mathrm{K}^2 + c^2(\mathrm{K}')^2}$$

where a, b, c, d are integers, a odd, c even, such that

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

a noteworthy theorem, one which may be considered one of the fundamental theorems of elliptic function theory – equally applicable to a much-studied class of multiple integrals of arbitrary order. I made an attempt at studying this thorny matter in a short note [2] in volume 2 of your journal.

You see, sir, that the theory of elliptic functions is a massive body of research whose tentacles reach most of algebra, the theory of integration and number theory. What a glorious accomplishment for the noted author of the *Traité des fonctions elliptiques*,[8] to have created this beautiful theory and to have kindled this flame for posterity.

Königsberg, 21 July, 1828.

References

[1] C. Jacobi. *Gesammelte Werke.* 8 volumes. Edited by C. Borchardt. Berlin, 1881–1891. Reprinted in New York (Chelsea, 1969). Available from the American Mathematical Society.

[2] C. Jacobi. De singulari quadam duplicis integralis transformatione. *J. für die reine und angewandte Mathematik* **2** (1827), 234-242. In collected works [1] **3**, 57–66.

[3] C. Jacobi. Note sure la décomposition d'un nombre donné en quatre carrés. *J. für die reine und angewandte Mathematik* **3** (1828), 191. In collected works [1] **1**, 247.

[4] C. Jacobi. Note sur les fonctions elliptiques, *J. für die reine und angewandte Mathematik* **3** (1828), 192–195. In collected works [1] **1**, 251–254.

[5] C. Jacobi. Suite des notices sur les fonctions elliptiques. *J. für die reine und angewandte Mathematik* **3** (1828), 303-310. In collected works [1] **1**, 255-263.

---

number of representations of n as the sum of four squares. Here a representation of n is a 4-tuple of integers the sum of whose squares sum to n. Changing the sign of an entry in a tuple or transposing two distinct entries of a tuple produces distinct representations.

[5] In the language of complex analysis, if a single-valued meromorphic function has more than two independent periods, then it is constant.

[6] Along with the definition $q := \exp(-\pi\mathrm{K}'/\mathrm{K})$

[7] The parameter $\tau := \imath\mathrm{K}'/\mathrm{K}$ is now commonly used.

[8] This of course refers to A. M. Legendre (1752–1833)

[6] H. Smith. *Report on the theory of numbers*. 1894. Reprinted by Chelsea (1965). (Also in collected works **1**.)

Eric Conrad, Department of Mathematics, The Ohio State University
E-mail address: econrad@math.ohio-state.edu