TABLE DES MATIÈRES

Chapter 1

The principle of uncertainty

This is to retrace a scientific journey, in this case the mine, but it's sort of secondary. What matters before everything is meetings, and above all scientific fields is concerned.

After entering the École Normale Supérieure, I decided not to pass the aggregation because I didn't want to start over cram. I had started, at the School, to do research in math but I really did find a topic that interested me only after getting out, with Quantum mechanics.

I always had with me the little book published by Heisenberg in 1930 (The Physical Principles of Quantum Theory) and I had been extremely inspired by the way he explained how he had discovered the mechanics of matrices which underlies the Quantum mechanics. I start there, because this discovery of Heisenberg played, throughout my journey, an absolutely essential role.

Before Heisenberg's discovery, there was a model for the atom, which was called "Bohr's atom" and which postulated electrons in stable circular orbit around the nucleus. And there were completely ad hoc rules, which had no conceptual justification but allowed to find, for example, how the spectrum of hydrogen was made. Heisenberg was taking care to calculate spectra of atoms, i.e. to determine mathematically the set of wavelengths present in the light emitted by the atom in question. By a contest of circumstances - chance plays an important role in science - he had been sent by his university to the island of Helgoland, in the North sea, to treat a serious hay fever: at the time, the only remedy was to take refuge in a place totally sheltered from pollens. It is a very small island, where he was staying with an old lady and had plenty of time to think and make calculations. He had developed his new mechanics but his theory seemed to him contradictory: energy conservation, which plays an essential role in classical formalism, posed a problem and had to remain true in its new formalism. So he made calculations with the system that he had created and he finally realized that the energy was well preserved! He describes this moment very vividly in his autobiography. It was then 3 or 4 in the morning, and he said that at that instant, he had a landscape in front of him that almost frightened him by its immensity. Instead of going to sleep, he climbed one of the peaks that borders the island and waited there for sunrise.

This discovery of Heisenberg was my starting point. When I graduated from the École normale, I was a student of Gustave Choquet and he had the idea to make me learn physics by sending me to Les Houches summer school, in 1970. There were Oscar lectures of Lanford, who explained what von Neumann had done after Heisenberg. What Heisenberg found was that when you do physics calculations for microscopic systems, like an atom interacting with light, a phenomenon quite extraordinary happens: you can no longer have the freedom you have usually to swap the order of terms in an equation. When we write $E = mc^2$, we might as well write $E = c^2m$, the result would be the same: this is the essential algebra rule that we say "of commutativity", which means that if we swap the two terms of a product, the result is unchanged. But Heisenberg found that, when working with a microscopic system and multiplying observable quantities, for example the position of a particle by its speed, or more precisely by its moment (its speed multiplied by its mass), we can no longer swap freely the terms of the product. The corollary is very well known: this is the Heisenberg's uncertainty principle, who says there is a limit to the precision with which we can simultaneously know two properties of the same particle associated with observables which do not commute. For example, if the more we know with great precision is its position, the less we know precisely is its speed, and conversely.

This is the physical part of what is going on, we will come back to it.

Consequence: there is a kind of permanent novelty, of freedom, in mechanics which means that when we repeat certain experiments at the microscopic level, we don't get the same result. For example, if we send an electron through a slit whose size is of the order of the electron wavelength, it arrives at a target placed beyond the slit, at a specific location.

But we cannot reproduce the experiment in such a way that the electron arrives again at this same precise place. All that we know, it's the probability that it will arrive at such and such a place. There is no way, it's the

Heisenberg's uncertainty principle that says, repeat the experiment so that the electron arrives exactly in the same place. So there is a kind of fantasy of the quantum that manifests itself at all times, every time we do such an experiment at the microscopic level.

Mathematically, it's a different story, because the discovery of Heisenberg taught physicists that they needed to be careful when handling these observable quantities in the framework of what has become quantum mechanics, that is to say the mechanics of microscopic systems. This may seem confusing but this is actually a phenomenon we are accustomized to: it manifests itself every day when we write. If we swap the letters used between them, like when we make anagrams, we change the meaning of the sentences: thus gravitational waves "includes the same letters but does not have the same meaning that" the distant stormy wind.

Both have the same value when working in Commutative algebra, where you can swap letters. For that, the sentences do not lose their meaning, we understood that we must do pay attention to the order of the letters in a sentence. And Heisenberg has shown that when we work at the microscopic level, we no longer have the right to simplify as we simplify in physics calculations ordinary. It's a major discovery because it has a considerable impact, not only in physics, but also in mathematics. As far as I'm concerned, I spent most of my existence of scientist to exploit it mathematically.

Max Born and Pascual Jordan understood that the calculations that Heisenberg made were what are called, in mathematics, calculations of matrices. No need to know what a matrix is. Which is essential is that matrices have this property, compared to ordinary numbers, not to switch between them. The product of two matrices in the order "ab" will generally have a different result of the product "ba". Born and Jordan understood that Heisenberg had rediscovered the matrices, but in a natural form, from observations.

Chapter 2

Specters

In common parlance, specters are ghosts or in any case report strange things. In physics, the word spectrum denotes a reality, just like in mathematics. One of the miracles which happened in the XXth century is that the spectra of physics have could be calculated as spectra in the mathematical sense in the most important physical examples.

The physical meaning of the spectra is understood in the way next: when, following Newton, we take the light that comes from the Sun and we pass it through a prism, it gives, once decomposed by its passage through the prism, a rainbow, that is, it breaks down into simpler elements which each correspond to one of the colors of the rainbow. But in refining this experience, we realized that we were watching at a place of the rainbow a black line, which is called the black line sodium. We considered this line as an isolated element, until what the German optician Fraunhofer had in the XIXth century the idea extraordinary to watch the rainbow obtained after the passage of the sunlight through a prism with a microscope. They are then saw that there was not a single black stripe, but did listed about five hundred. They are what physicists call an “absorption spectrum”, which looks a bit like a barcode. Years later, Robert Bunsen and Gustav Kirchhoff, among others, have noticed that by heating certain bodies, like sodium, you could get the same configuration, not with black stripes on a rainbow background, but with stripes bright on a black background. We then understood that these lines were a kind of signature of the chemical body in question, and succeeded in reproduce, with different chemical bodies, a number of lines that appeared in the spectrum of the Sun.

So these barcodes, these “absorption spectra”, appear like black lines when we look at the sunlight at through a prism, with the extraordinary precision that a microscope. But we observe lines that we do not manage to relate to a known item. This is where physicists and chemists stepped in to say they might be the ones of an unknown chemical body. And, as it comes from the Sun, we have it called “helium”.

Then occurs, at the beginning of the XXth century, the eruption of Vesuvius. With the same spectrometric processes, we analyzed the light from its lavas and helium was found there. That’s wonderful. That explains what what are spectra in the sense of physics, it gives their meaning and their importance: each of them is a signature. Each body different chemical has different signature, barcode different. When the chemical body is pure, its signature is not a superimposition of different signatures, it is also pure.

Obviously, we immediately tried to understand what was the nature of this signature. We looked for the available body on simpler, hydrogen, because it was more difficult for helium. We have took a while to figure out that if we looked at a general spectrum, not in wavelengths, but in frequencies, this one had a remarkable structure. It was actually a specter formed by the differences A - B between any elements A, B of a simpler set of frequencies. That is to say, you had to index frequencies that appear in a general spectrum by two indices like (a, b) or (c, d). Of course, if we take the difference (A - B) between A and B and that we add it to the difference (B - C), between B and C, this gives the difference between A and C.

This gave a general rule of composition for frequencies that appear in a spectrum called the “rule of Ritz-Rydberg composition”. Heisenberg’s genius is having builds its mechanics from this rule. He understood the thing following: if classical mechanics had been valid for a body microscopic, we wouldn’t have had this rule of composition but the rule of a group, that is to say that two frequencies u and v of spectrum add up to give a new frequency u + v of spectrum. We would have obtained, by a mathematical process that we called the Fourier transform, the algebra of observables.

Heisenberg had this wonderful idea of saying that these are the physics, the Ritz-Rydberg principle and chemistry which must prevail.

As this is what we find experimentally, we will base the observable algebra on this Ritz-Rydberg rule.

It was Born and Jordan who explained to Heisenberg that the mathematical structure he had found was well known to mathematicians, who call it matrices. A matrix is not nothing but an array, and instead of being indexed as a sequence by a single letter, it is indexed by two letters. When we multiply two matrices, we use the Ritz-Rydberg rule.

Shortly after this discovery of Heisenberg, Schrödinger made a another not extremely important. Thanks to him, we made the link between the spectra that appeared in physics and those that appeared in mathematics. Because, and it is remarkable, the word "spectrum" was already known in mathematics, by the Hilbert school for example, and known because of what are called operators and the spectrum operators. There is no question of explaining it precisely here, but it is something that has a perfectly defined mathematical meaning.

Schrödinger was the first to calculate the spectrum associated with hydrogen by mathematical calculation, while physicists apprehended him by measures. What is extraordinary is that Schrödinger's theory and Heisenberg's theory are the same, which gave rise to a formalism carried out by one of the greatest mathematicians of the time, who was not only mathematician: John von Neumann.

Von Neumann understood that there is a mathematical formalism existing, developed by the school of Hilbert, and which uses as a framework common mathematics what is called Hilbert space. This space, consider it as a kind of abstract, unique joker, which will play an essential role in everything that follows. There is only one Hilbert's only space and it's going to be the seat of mechanics quantum. This is the most suitable framework that we know up to present.

We know the Euclidean space, the plane, we also know the space of dimension 3. To go to Hilbert space, you have to do some not difficult, and even if we do not understand all the details, it is necessary know that it exists. The first step is to move from a space real to a complex space, which is not too difficult yet understand. We are very used to real numbers, but they are not not very flexible and easy to manipulate to do physics. We had need to add to the real numbers another number, baptized "Pure imaginary number", which checks that its square is equal to 1. he is very valuable for physics and in particular for electromagnetism.

The next step is much more difficult to accept: the Hilbert space has an infinity of dimensions. It is thanks to this that an incredible number of wonders will appear.

The first of these wonders is that there is a coincidence between Heisenberg's point of view (which is extremely practical, extremely concrete, because the observables he discovered become operators, that is, something that acts in this Hilbert space) and that of Schrödinger (who discovered how we could calculate the spectrum of a chemical element, which also manifested by an operator in Hilbert space). That seems very mysterious, but if we understood something about the nature, on reality, on quantum mechanics, it's good that the corresponding mathematical scene is that of Hilbert space and that the actors are the operators in this space.

Chapter 3

Operator algebras

This very beginning of history took place from 1925 to the 1930s.

Von Neumann then gave his formalism to quantum mechanics. But he didn't stop there. He asked himself the question, absolutely fundamental, concerning subsystems of a quantum system. He understood that ordinary quantum mechanics is formalized through Hilbert space. With a collaborator, Murray, he tried to understand what it meant to have a subsystem, that is, not knowing all the information about a quantum system. This is what we called algebras of operators, and with them that my mathematical existence has begun.

For recall, it was after having gone to the summer school of Les Houches, in 1970, that I was spotted by an American organization as a "promising young mathematician". Therefore, they invited me the following year to Seattle for a conference. I was so young married man and we took this opportunity to visit the United States.

We didn't really like the plane, so we decided to join Seattle by train, crossing Canada, four or five days across large, somewhat monotonous plains. I looked for first stop at Princeton to buy a math book to read during the trip and ended up spotting one from a Japanese author who seemed interesting to me. I only took this one and reading it absolutely fascinated me. Arrived in Seattle, I went to the Battelle Institute to learn about the program. What did I read ? The author of the book was there and gave a series of lectures!

Because of this, then, I applied what Brutus said in Julius Caesar of Shakespeare:

There is a tide in the affairs of men,
Which, taken at the flood, leads on to fortune;
Omitted, all the voyage of their life
Is bound in shallows and in miseries.

I decided that I would not go to any other conference than this Japanese's one and that I will work on the subject he exposed. Returned in France, I went in September to the only seminar that existed on operator algebras: the one of Jacques Dixmier. This one has explained that, that year, his seminar would be on another subject, which a priori had nothing to do with that of the Japanese. He asked who among the audience wanted to make a presentation. I carried myself voluntary, and he gave me to read an article on tensor products infinite. When I got home, by train, I realized that there was an extraordinary link between the article Dixmier had given to me and the works of the Japanese, and it was this confluence that was the point of departure of my thesis.

I wrote a small letter to Dixmier, half a page. He had me replied that what I had written was incomprehensible and that it was necessary that I give details. I gave them to him, then went to see him, and he said to me, "Go for it!" "That's how I happened engaged. Ultimately, the starting point of my career, it is this link with the work of the Japanese. Which, in fact, was two. The one who found the theory in question, which is called the theory of modular algebras, was called Tomita. But deaf since the age of two years old, he had trouble communicating, and it's Takesaki, another Japanese mathematician, who shaped and communicated his theory. It was the latter who spoke in Seattle.

I immediately linked Tomita's theory to work on type III factors made by Araki and Woods. What I found, a few months after defining general invariants using the theory of Tomita is that there is a phenomenon quite miraculous of independence which makes it possible to calculate these invariants.

The evolution over time does not depend on the choice of a state of algebra, provided that we work modulo interior automorphisms: there is automatically an evolution over time which is not completely canonical, but canonical modulo these interior automorphisms! A von Neumann algebra is precisely an algebra like the one Heisenberg had discovered, that is to say non-commutative, in the sense that we no longer have the right to swap between them the terms of a product. To summarize, when you don't know all the information about the quantum system, this partial knowledge is at the origin of an evolution which miraculously emerges from the very fact that our knowledge is imperfect. It allowed me not only to write my thesis, but to completely decanulate all these algebras which seemed extremely mysterious, and to understand their structure. Something was still missing: how did this miraculous appearance of time could be related to physics.

It was totally mysterious in my work, which was purely mathematical. I was missing this element and that would come much later.

Chapter 4

Mille-feuille cake

So I found these results, then, after my thesis, I found others results, very important, on the same algebras. Then I was invited to the Institute of Advanced Scientific Studies (IHES) in Bures-sur- Yvette. And there, I was shocked: I had worked on a subject when even quite specialized and I did not know at all the extent of the rest of math. When I got to IHES, people were talking about things that I didn't understand. I was immersed in a totally different environment from the specialized environment I was used to. My situation was a little embarrassing because I wanted to absolutely participate in this development of mathematics, which seemed so important - and it was.

Grothendieck has already left, but someone at IHES played a crucial role for me: Dennis Sullivan. He had this particularity quite extraordinary to ask newcomers about their research in math or physics with extremely naive questions. One had the impression that he hardly understood. But after a while, his interlocutor realized that it was him-self who didn't understand what he was talking about. His power was absolutely incredible, and it was he who taught me the differential geometry. I understood at that time that I had a considerable asset: there was a way to fabricate the algebras that I had classified, those of von Neumann, from objects of well-known differential geometry called foliations.

What I had done so far could be illustrated with objects that people doing differential geometry could perfectly understand.

What is a foliation pastry? A mountain can have a stratified appearance, that is, strata of smaller dimensions make it up. Another typical example of a mille-feuille foliation pastry, which results from a stack of sheets. The structure of a mille-feuille cake is very simple. It is made up of two parts: leaves themselves, and the set of all those leaves. A set of sheets, in a notebook for example, is very simple since it is simply indexed by the page number. But in math, a foliation can have a structure much more complicated, like a spool of thread in which the thread, instead to be rolled up so that after a finite number of turns it comes back on itself, is rolled up in an irrational way. That is say it never comes back on itself it will keep on wrapping indefinitely. What is extraordinary is that, whatever the flipping, the resulting algebra is always non-commutative.

There are other examples. In a conference I attended in the 1980s, Roger Penrose explained that he had discovered very explicit quasi-periodic tiling. Because if it is relatively simple to pave a space with hexagonal tiles, for example, since you can give a tiler the recipe to do it, the quasi-periodic tiling is more complicated, and in particular has the next peculiarity: they can have a pentagonal symmetry that no classic paving can have. What explained Penrose, it was that these tiling has a quantum side: when we takes two, we can superimpose parts as large as we wants, although they are not identical. This kind of almost coincidence, but never complete, has a quantum aspect that it had well felt intuitively. I realized at that time that the Penrose paving space had the same characteristics typical of the leaf space of a foliation, and that, thanks to the associated von Neumann algebra, it really corresponded to the Quantum mechanics.

Chapter 5

Non-commutative geometry

So that's kind of the starting point for non-commutative geometry. It is an almost direct consequence of Heisenberg's discovery.

Descartes explained that one can makes the geometry of plane entirely algebraic. For example, if we want to demonstrate that the three medians of a triangle intersect, we can use the axioms of geometry. But there is another way to demonstrate this theorem: algebraic calculations. It is then a question of calculating the barycenter of three points, using the coordinates of each of them in the plane, and the theorem is immediately demonstrated.

What is the advantage of transforming a geometric problem into an algebraic problem? For example, demonstrate geometrically in dimension 5 the analog of the fact that the medians intersect will be difficult, while the calculation is immediate. We calculate the barycenter, and the demonstration is made.

It was Descartes's idea, these coordinates, and that was the basis of algebraic geometry for years. These coordinates called "Cartesian" commute. But the coordinates in what we calls the phase space, which corresponds to the microscopic system, they no longer switch: it is the Heisenberg's discovery. This is what led me to develop the geometry for spaces whose coordinates no longer commute and which are therefore calls non-commutative geometry.

One would think, and it would be normal, that to generalize the geometry to a case where the coordinates no longer commute would feasible. It's actually quite tricky and finds its justification essentially by quantum mechanics. But if there had only been that, it would not have been enough for me. What motivated me is what I found in my thesis, i.e. the fact that such spaces have something extraordinary: they generate their own time. They are not static like ordinary spaces, but dynamic: they evolve over time.

After this initial discovery, I told myself that this extraordinary property of generating its own time made this geometry was necessarily extremely different from classic geometry, and all the more interesting. After the foliation finding, I had enough examples and when we try to develop a new theory, besides a good reason, you must have a large amount of examples. Indeed, if we have too much little, we risk developing a completely formal theory which will make no sense. The meaning is given by the variety of examples.

From the start, I understood that the most famous foliation gave the most exotic factors. I also understood that the associated von Neumann algebra only perceived one side relatively crude of the non-commutative space in question. The fact that these leaf spaces came from geometry gave them many other structures from differential geometry and that it was necessary to understand the non-commutative case.

It was the starting point for a whole development during the 1980s, in which one of the most important contributions was the cyclic cohomology, which I found, and which now plays an essential role in many other areas.

This made it possible to understand and develop the analog of the differential geometry in the non-commutative framework, to find the analog of de Rham's complex, cohomology etc. There have been all kinds of surprises, for example the Godbillon-Vey invariant appeared miraculously in this completely different setting. However, I still had this frustration of not knowing how to link this emergence of time with physics.

Chapter 6

Emergence of time and thermodynamics

In the meantime, I have always continued, as a hobby, as a somewhat parallel task, getting interested in physics, about which I read a lot. But not just any physics: quantum physics of course but beyond that, what's called the field theory. Around 1994, I was invited for several months at the Newton Institute in England at a session whose subject was gravitation. I went there because I wanted to complete my knowledge. On the spot, I got a little bored because there had almost no collective activity. One day I saw an ad for a conference whose title seemed to me extremely pretentious: We know what quantum space-time is. As we do not know, in fact, still not what it is, I got stuck with the speaker a bit. The conference was by Carlo Rovelli. We then discussed at length and I glimpse that he had an extraordinarily philosophical point of view.

I thought it was great, because in our community, people are overwhelmed by technique, by their specialization, and that there is ultimately very few philosophical discussions, unlike the time of Einstein and Heisenberg. I dared to explain to him what I found in my thesis: this extraordinary emergence of the time. He then left me without saying anything to return a few minutes later with two articles he had written the year previous. For purely philosophical reasons based on his thinking about what would happen if we tried to quantify gravitation, he placed himself at a level called "semi-classical", that is to say say not yet quantum. His idea was that when we write the Wheeler-De Witt equations, we find that when we try to quantify gravitation, time disappears. And it disappears because what's called the Hamiltonian, which normally generates evolution over time, is one of the constraints. We do not know more what we talk about when we talk about time.

Carlo had tackled this problem and, by mere philosophical reflection, he had an idea: the only way that time can emerge, it's from thermodynamics, because we let's bathe in a kind of thermodynamic bath, a bath of the 3 degree Kelvin radiation from the Big Bang. The idea is therefore not completely abstract, it relates to something very concrete. And it is this heat bath which would have caused the passage of the time.

His idea was very attractive because the passage of time such as we know wears us out. When time goes by, what we collide, it is wear and tear. And this wear comes from the temperature, because we're in a thermodynamic bath.

To implement his idea, he wrote an equation, and I recognized it right away when he showed it to me, the semi-classical limit of the equation used to have this magic flow intervening in the quantum. It was then that the junction was made. I had tried to understand how this emerging time could be related to physics. I tried to do it using quantum fields theory, but I didn't get there because the real where it happens, it's not in field theory, but in gravitation, when we try to quantify it.

We wrote an article in common, but it has neither Carlo Rovelli's philosophical qualities, nor my mathematical qualities. It was more to take a date to show that we had recognized these equations, but we haven't gone far enough in interpretation of the result.

This interpretation, I will try to explain it, because it has played an essential role in the development of non-commutative geometry. The paradigm I had arrived at in the years 1980 for non-commutative geometry can be explained very simply from quantum precisely and what is producting there.

Chapter 7

Variability

The idea, almost easier to explain, and more fundamental than the coincidence found with Carlo Rovelli, is as follows: the quantum has this extraordinary fantasy, this extraordinary imaginative power, which means that every time we repeat an experience microscopic, we get a response that we can neither predict nor reproduce. We are touching on a central problem which I will call “variability problem”.

Normally if you ask someone about what the fundamental variability, everyone, not just the physicists, responds that the only variability is the passage of time.

Any variability can be reduced to the fact that time is passing. If we look carefully, we realize that almost all of physics is written in terms of what is called a differential equation, i.e. that we write that the derivative of a physical quantity with respect to the time is given by a certain relation with other quantities.

All of physics is written on this paradigm, and all of understanding that we have of variability is thought in these terms.

Let’s do a little math excursion to try to understand how mathematicians have sought to formulate this that it is only a variable and how this formulation was dethroned by the quantum. When you ask a mathematician what is a real variable, he will say this: it is a set and an application of this set in real numbers. It may seem a little obscure, but that’s the standard answer. We can then point out to the mathematician that there are variables that take only discrete values, for example the age of a person, that will be expressed only as an integer, and others who take continuous variables. The two cases are quite different.

There can be no coexistence in mathematics between a discrete variable and a continuous variable. Indeed, a discrete variable takes only a countable number of values (we can enumerate them one by one) while a continuous variable takes a not countable number of values so the set where it takes its source cannot be the same as that associated with a discrete variable. It is a fact. The first wonder is that formalism of quantum mechanics that von Neumann has developed solves this paradox of the non-coexistence of the discrete and the continuous. It is resolved, as I said above, because Schrödinger found that the spectra were spectra of operators in the space of Hilbert. In this same Hilbert space, on the same stage in somehow, some operators will have a discrete spectrum, like the integers, and others a continuous spectrum, that is to say that they can take all the real values between zero and infinity. The only nuance is that the two operators cannot switch.

Thus, the formalism of the operators in Hilbert space solves the paradox.

This formalism provides the framework for non-commutative geometry.

And it is thanks to it that we will be able to try to understand the emergence of time.

Chapter 8

Length unit

How does this formalism generalize the geometry in such a way that it absorbs everything that quantum has brought us? This is where the link with physics absolutely appears fundamental.

At the time of the French Revolution, there were in France as many definitions of unit of length as cities, or almost. There must have been about a thousand. At the entrance of a village, we found something about a meter which defined the unit of length in use at this location. The fabric of a merchant be a multiple of this length to be able to be sold there. It was very annoying.

We then sought to unify the unit of length, for France in particular. Scientists were first asked to give one valid definition which is not dependent on the place. They reflected, and took the biggest object at their disposal: the Earth. They have considered the circumference and then defined the unit of length as being a portion of this circumference: the forty millionth part. Since it is impossible to directly measure the entire circumference of the Earth they used an angle by pointing some stars. They knew very precisely the angle having summits the center of the Earth, Dunkirk and Barcelona, it was their so easy to calculate the total length from the measurement of the distance between Barcelona and Dunkirk, which had to be measured directly. They sent a team of scientists, Delambre and Méchain, to make these measurements. The expedition was adventurous, the France and Spain being in conflict. While they were at the top of a hill, equipped with a telescope to make measurements by triangulation, they had a lot of trouble explaining to the soldiers enemies that they weren't spying on. But they had success. The result was a platinum bar supposed to be exactly the length of the forty millionth part of the Earth circumference. It was deposited near Paris, in Sèvres.

This unit of length, as such, was not very practical.

Difficult to measure a bed by comparing it to this bar. A lot of replicas were therefore made. And then it happened, in the 1920s, a completely fabulous phenomenon, a phenomenon that is the exact parallel of the transition from ordinary geometry to non-commutative geometry.

A physicist made very precise measurements by comparing the platinum bar with the wavelength of a spectral line of the krypton, and he realized that the unit of length... was changing length! It is very annoying to have a unit that is not stable! It was therefore decided, after a fairly long time, to use what had allowed to see the change as a new unit: the line krypton orange. But it was not practical. It would have been better to take a unit that is in the order of microwaves, which have been incidentally discovered by accident (people who worked on the radar noticed that their chocolate bar had melted). There is fortunately a chemical body, cesium, which has what is called a hyperfine transition: in the outer layer of a cesium atom, there are two states so close that their energies are very close too. This means that the correspondent transition, the one I was talking about Heisenberg, between the two energy levels indexed by two indices, is such that its frequency is very small and, therefore, that the associated wavelength is big. For this transition, we get a wavelength of about 3 centimeters, that is to say it takes about 33 times more to get 1 meter. We will therefore have a measuring instrument which will be able to make precise measurements on a fabulous routine basis.

There is now a commercially available device, based on this wavelength, which measures a length with twelve decimal places given.

On the other hand, if we want to unify the metric system throughout the Galaxy, this will be a problem because cesium is not necessarily present on other star systems. A chemical body with a sufficiently high atomic number is indeed produced only in supernovae, and even super-supernovae. I think that one day or another we will be able to base the unit of length, not on cesium, but on hydrogen or helium. Why ? Because they are present practically everywhere in the Universe.

Chapter 9

Infinitesimals

What is happening at the mathematical level? Exactly the same thing. The geometry was based by Riemann on a measurement of lengths which exactly corresponds to the way of measuring of Delambre and Méchain. It consists, when we take two points in a geometric space, to consider the shortest path between these two points. In doing so, we do not need to measure this length, that of the element of infinitesimal length, which we call ds, whose Riemann gives the formula only for the square, what we call ds^2 .

What we call geometry in the Riemannian form is so a geometry based on the element of infinitesimal length, which is expressed in the form of what is called g, μ, ν. Mathematical content doesn't matter, what should be remembered is that it is something extremely concrete and which corresponds exactly to the way of measuring of Delambre and Méchain.

In physics, we had to replace the paradigm of unit of length given by the standard meter by the spectral paradigm, which precisely corresponds to a spectrum. The way it happened in non-commutative geometry is exactly parallel: the infinitesimals have their place among operators in the space of Hilbert. Some operators are infinitesimal. They have a discrete spectrum, but which decreases towards zero, and corresponds exactly to the definition Newton gave of infinitesimals.

What is new is that infinitesimals can no longer switch with continuous variables. The crucial point is that there are an infinitesimal which is characteristic of a geometry. This infinitesimal was introduced by physicists when they founded field theory and quantum theory, this is what they call the "propagator" for fermions. Physicists therefore have developed in their theory an entity which is an operator in the space of Hilbert and who has all the properties to embody the element of infinitesimal length. We can clearly see the gain both in physics and in mathematics. In physics, this allows for a system of length measurement, based on the spectrum of hydrogen, which is really universal. We can exchange with a visitor from another stellar system without having to bring him to Sèvres for showing him the stallion.

In mathematics, it's exactly the same thing. When we takes the propagator of fermions as the length element of a geometry, as it doesn't switch with coordinates, since coordinates are of continuous value, it has the property of not being able to be located and to be present everywhere. It is no longer located somewhere. If it had switched with the coordinates, the fact that it is infinitesimal, it would have been somewhere. But the fact that it doesn't switch allows him to be everywhere.

This gives a new geometry of spectral nature, that is to say it manifests as a spectrum. It's very new since usually when we talk about a geometric space, we think of it as a whole with a distance, a structure, which are given to it locally. It's not like that a space of non-commutative geometry will manifest. He will manifest by its spectrum.

Chapter 10

The music of forms

A new phenomenon appears: this manifestation by a spectrum can be understood musically. If I take any shape, a drum, a sphere or any other, we know since the XIX$^{\text{th}}$ century thanks to Helmholtz that a range is associated to it. And since Mark Kac and his famous talk *"Can we hear the shape of a drum?"*, it is formalized in mathematics. What does this mean ? You might think that when you hit a drum, the sound product will always be the same. It is a serious mistake. In the XVIII$^{\text{th}}$ century, the vibrations of the drum were observed by putting sand underneath.

When the drum vibrates, the sand is concentrated where the vibration is the smallest. We thus observe that the vibration of the drum is of this or that shape depending on where it was struck.

Two parameters actually qualify this vibration exactly: how many oscillations if we start from the center towards the circumference? and how much when we go around the drum? If we know these two parameters, for example three oscillations from the center towards the circumference and four when we go around, the vibration is then perfectly defined. It will produce a particular frequency. The simplest vibration produces the lowest frequency. And, at each time the value of the parameters is increased, the vibration becomes more acute. We can calculate the frequency of these vibrations mathematically. We get a range. We finally realize that each form has a range. They are not always different. Some forms, different, however have the same range. So we lose a bit of information. But, basically, we essentially knows an object from its range.

A space is manifested by a range, and this is the starting point of noncommutative geometry. It includes space from its range. There is an additional invariant, which must be known to really understand the whole space: what are the agreements possible? A point in space gives more information than the scale, it gives an agreement on the notes of this one. If we know all agreements, we recognize the space.

When I found this result, I gave a lecture at the Collège de France on the link that there was between forms and music.

I called it “music of forms”. In preparing this conference, I asked myself a question: there are many different forms (sphere, disc, square, rectangle, etc.), would there be one who allows us to make music as we know it? I tried different forms and I realized that it was a disaster. For example, if you want to play Au clair de la Lune, none of the forms I have mentioned gives convincing results.

Why ? The ear is sensitive to multiplication by two. If you double the frequency of a note, the ear will hear something nice, a resonance between the two frequencies. And that corresponds to something very concrete: it's the transition to the octave.

The ear is also very sensitive to tripling.

As it is sensitive to doubling, we can also multiply by three halves, which is like playing a C and a G.

Now let's think a bit about it. To fall back on my feet, I would like to multiply by two enough times to be the same thing as to multiply by three a certain number of times. It is impossible, because when we take a power of two we always have an even number, and when we take a power of three, we always have an odd number. Which is true and amazing, and this is the basis of music as we know it is that multiply nineteen times twice, it's almost the same as multiplying twelve times by three. There is a better way to say it: the twelfth root of two is almost equal to the nineteenth root of three.

How does it manifest? This is what I call the guitar spectrum. On the neck of a guitar there are the frets, these lines perpendicular to the neck that produce a sound specific. They are not regularly spaced. It's not at all an arithmetic progression. These are the powers of this number: $2^{1/12}$. By taking this number, we get exactly the frets positions on the neck of a guitar.

The spectrum of the space we are looking for is therefore given to us by the guitar spectrum. Could it be a sphere of dimension 2? No, because mathematical theorems say how the range of a shape develops when we take more and more notes, and the dimension of the space we are looking at is intimately linked to the way notes develop. On the spectrum of the guitar, a little mathematical calculation indicates that the dimension of the corresponding space must be zero. More specifically, it must be smaller than any positive number. Therefore we cannot find it among the spaces we know. And it is a non-commutative space; this is called the non-commutative sphere, which was found by physicists. So we see where non-commutative geometry gives freedom to find spaces far more extraordinary than the one we have with ordinary spaces.

Danye Chéreau, my wife, Jacques Dixmier and I have written a book, *The Specter of Atacama*, which evokes the spectrum received by the large Atacama Desert Observatory (ALMA: Atacama Large Millimeter Array) and who seeks to understand what this spectrum represents. And we still haven't understood. It is connected to one of the most difficult conjectures in mathematics. It's fascinating to see that a space as we know it produces a spectrum. But there are examples where space is perceived by its spectrum, and we don't know what this space is; it remains mysterious.

All this geometric side has been considerably developed and allowed us, with my collaborators Ali Chamseddine and Walter van Suijlekom, to understand why, in physical reality, there is not only gravitation, but also the standard model; why are there other forces of physics that appear naturally. In the context of non-commutative geometry, we may, by purely geometric reasoning, fall by miracle that it's easier to describe space-time, the geometric space in which we are, by non-commutative variables. This simpler way of describing it requires other forces beyond gravitation than pure gravity in this new space. These other forces correspond exactly to what we are measuring, i.e. the forces of the standard model.

It is a very elaborate theory and extremely satisfactory in aesthetic and conceptual level. A part is missing on which we are working: it exists at the level of what is called the first quantified, and does not yet reach the level of what is called the quantum gravitation, that is to say in which we would really quantify fields.

But back to the essence of quantum. It's something absolutely fundamental, which is not yet fully understood.

Chapter 11

The ticking of the Divine clock

We human beings reduce any variability to the passage of time. And we are still looking to write a story. It is one of the ways we understand things, and a story is of course written over time.

But when we try to understand quantum, we are faced with paradoxes. The most typical is what we calls in French entanglement. Einstein never accepted the quantum, although it was he who practically initiated it with the photoelectric effect. In one of his poems, Alfred Brendel tells Einstein arriving in heaven, realizing that God is playing dice and then asking for the address of hell. Because Einstein never accepted the random side of quantum and he gradually built a number of counterexamples, paradoxes. The first one gave rise to a wonderful story. Einstein imagines a Swiss cuckoo hanging from a spring. The cuckoo must emit a photon to a precise moment. Thus, it will indicate the exact time when it issued it. In weighing the cuckoo before and after the emission of the photon, we would know exactly its mass, and therefore that of the emitted photon, or rather its energy, since energy is equal to mass. So, as we will know the energy of the photon emitted with limitless precision, we will contradict Heisenberg's uncertainty principle, which says that we cannot know time and energy simultaneously. That gave place to an extraordinary episode, with a triumphant Einstein who explains its paradox. And Bohr who follows him with an absolutely confused mine because Einstein's argument seemed unstoppable to him. But the story did not end there.

What was Einstein's idea? Since we were going to measure cuckoo's mass (before and after), the gravitational constant would be involved in calculating mass from weight. So it was impossible for the Planck constant to stand out by itself as the principle of uncertainty requires it, it seemed impossible to eliminate the constant of gravitation! Bohr did not sleep overnight, and returned in the morning with a wonderful response. He showed that one obtained exactly the Heisenberg's uncertainty principle in using Einstein's theory of gravitation! According to this theory, and it has been measured since, time does not pass in the same way when we change altitude and this change involves again the gravitational constant. If we do the math, we realize that the two gravitational constants are eliminated. Only Planck's constant remains, and we get the principle of uncertainty as stated by Heisenberg.

This episode was a defeat for Einstein, but it didn't confessed defeated. Five or six years later, he wrote an article with Podolsky and Rosen, an article that was almost never cited at first, now its quotes are growing exponentially. It was the first time that quantum entanglement appeared. Einstein proposed to create two particles at a given location, these two particles having exactly, by conservation of the moment, opposite moments. These particles propagate. We measure the position of one and the moment of the other. As they are causally separated, these two measures are independent. So we get a contradiction with the principle of uncertainty since, on the one hand, we measures the position, on the other side, the speed and by symmetry we deduct position and speed for both: it's won.

This paradox is much deeper and much more difficult to eliminate than the one Bohr had solved. Basically, what people say, is that when you take a measurement on one side, there is an action, which Einstein called *spooky action at a distance*, which affects the other side almost instantly, and therefore goes much faster than the speed of light. Alain Aspect has had experiences that showed that the action spreads at least ten thousand times faster than the speed of light, which is incredible.

We think we are solving the paradox by saying that, if there is indeed an action, we cannot transmit information with it, but we're still hungry. What I claim is that the reason of this paradox is that we try to write a story in relation to the time. And when we try to write a story over time, we necessarily get a contradiction. Why ? Because the fact that one is outside the cone of light shows that there is no causal relationship between the two measures, this means that, depending on the benchmark that we are going to take, one event occurs before the other or the other before the one. There can therefore be no causal relationship between both. It is strictly impossible. What I claim is that the real meaning of quantum entanglement is that the quantum hazard on one side and the quantum hazard on the other are not completely independent.

The basic idea that has not yet been realized is to try to understand how the quantum hazard generates the passage of time. In *The Quantum Theater*, our first book with Danye Chéreau and Jacques Dixmier, we had found a sentence which should be remembered simply because it well expresses the problem: "The hazard of quantum is the ticking of the Divine clock". That means that true variability does not come from the passage of time, but from this fantasy, this constant imagination, the quantum. It's here that things vary, and time is just an emerging phenomenon.

We should have a much more precise philosophical reflection, much deeper, which would say that the hazard of quantum is not completely random, not completely independent, when one takes distant points, but that there are going to be correlations between the hazard of quantum at one point and the hazard at another point when there is quantum entanglement. We should be able to define correctly the quantum hazard, and we have the mathematical tool which allows us to make time emerging. This tool is what I found in my thesis from what the two Japanese men had done.

So we have the tools. But it lacks a philosophical reflection, exercise little appreciated by physicists in our time. Back to Einstein, Dirac, Heisenberg, Schrödinger, philosophical reflection was an essential ingredient of the discipline. For instance, Heisenberg and Dirac had the extraordinary chance to do a boat trip from California to Japan and they were able to chat indefinitely. Our time is very crowded by all kinds of external disturbances. We no longer know boredom which was fundamental in creative power. We now know the extraordinary success of Einstein's theory or quantum theory. And there is still stagnation. We live in a period where we are constantly disturbed, by fifteen daily mails or such report to make. We no longer have time to be bored, and we no longer have the will to do it. *The Specter of Atacama* is a praise of boredom, in a way. The hero of the book is faced with this spectrum that he doesn't understand and, instead of staying at the observatory, he fled to the far south to live on an island almost deserted for a while. And he manages to find this fundamental state of the soul, which is that of boredom.

It is a difficult state to appreciate these days. It must be admitted that the CNRS is one of the rare institutions that allows find. So Vincent Lafforgue, who just had a very big price, does exactly that. He is able to isolate himself to think about a problem for years, practically in underwater state. This ability gives it the necessary depth to make great discoveries. It is a miracle that the CNRS allows. It is a very different system from that of ERC (European Research Council) or NSF in the United States. Young people are there constantly writing articles, and must constantly show that they are productive. It is a perversion which has for consequence of creating scientific feudalities and which does not authorize the diversity. We must preserve this incredible chance which allows some to isolate themselves and find this state of fundamental research so important, so creative and so impossible to appreciate, to judge in the short term. What's terrible about these selection methods on project is that researchers are asked to say, in advance, what they will find. It's ridiculous, in physics like in math. If we knew what we were going to find, discipline would lose interest. Which is really interesting, which is really exciting, it's just to look at a problem and then, at the bend of a path, to find something that we were absolutely not waiting.

I recently had to give a talk at the Collège de France on the mathematical language. I was wondering what I was going to talk about. And then finally I chose to talk about Morley's theorem. Morley has found this result by accident. He was looking for more complicated things, and he fell on it! The wording is very simple.

We take any triangle. We cut each of its angles in three equal parts. And then we intersect the lines two by two corresponding. Morley's theorem says that the triangle obtained in such a way is equilateral.

It's a shame to corset research in a more and more administrative straight jacket, because, ultimately, it encourages researchers to confine themselves to small problems in which they can make small strides, and in no way favors large discoveries.