The Cartesian Oval and the Elliptic Functions p and σ

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By Clara Latimer Bacon.

The object of this paper is to show that the leading properties of the Cartesian oval, as developed in such texts as Williamson's "Differential Calculus," Salmon's "Higher Plane Curves" or Loria's "Spezielle Algebraische und Transcendente Ebene Kurven," may be readily deduced from the Weierstrass elliptic functions p and σ, and that they give a geometric interpretation to the standard formulæ of these functions.

Greenhill has used these functions in connection with the Cartesian oval (*Proc. London Math. Soc.*, t. XVII, 1886, and also his treatise on "Elliptic Functions," Arts. 236, 248 and 249) and has deduced the relations between the focal radii, though we have given a more symmetric form to these relations by the introduction of the triple focus.

Professor Frank Morley published a note on the subject some years ago in the Haverford Studies. (See also Harkness and Morley, "Treatise on the Theory of Functions," p. 336.)

When the u-plane is mapped on the x-plane by means of the equation

$$du = \frac{dx}{\sqrt{Q}},$$

where Q is a quartic in x, to lines in the u-plane parallel to the real or imaginary axis, correspond in the x-plane bicircular quartics whose real foci are the zeros of Q, provided these zeros are concylic or anticyclic. (Greenhill, *Camb. Phil. Proc.*, t. IV; Franklin, American Journal of Mathematics, Vol. XI, 3 and Vol. XII, 4.)

The problem of reducing this form to the Weierstrass form which we use, where one focus is at infinity, is analogous to the reduction of the bicircular quartic to the Cartesian. (Salmon, "Higher Plane Curves," Art. 280.)

1. *Preliminary Examples in Mapping.*

We let $x = pu$. Then

$$\frac{dx}{du} = p'u = -\sqrt{4p^3u - g_2pu - g_3} = -\sqrt{pu - e_1 \cdot pu - e_2 \cdot pu - e_3}.$$

(Harkness and Morley, "Introduction to Analytic Functions," p. 259, (16).)

Hence

$$du = -\frac{dx}{\sqrt{x-e_1 \cdot x-e_2 \cdot x-e_3}}.$$

We may suppose g_2 and g_3 to be real and

$$\Delta = g_2^3 - 2g_3^2 > 0.$$

Then e_1, e_2 and e_3 are real.

Let $e_1 > e_2 > e_3$. Since, when $x < e_3$, $\frac{du}{dx}$ is imaginary and, when $x > e_1$, $\frac{du}{dx}$ is real, we have a real period $2\omega_1$ and a purely imaginary period $2\omega_3$. Hence the parallelogram of periods is a rectangle. We write for symmetry

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

Let u describe some path in the u-plane and trace the corresponding path of x in the x-plane.

For example, let u move from 0 to ω_1 along the real axis. Then pu or x is real and decreases from infinity to e_1. If u bends to the left to avoid the critical point ω_1 in a quarter circle about it, x turns to the left and describes a semicircle about e_1, as the isogonality breaks down at this point and the angle which u makes about ω_λ is one-half that which x makes about e_λ. And so, as u describes the perimeter of the parallelogram 0, ω_1,—ω_2, ω_3, x traverses the real axis from plus infinity to minus infinity. (Burkhardt, "Functionen Theorie," Vol. II, § 86, IX.)

As u goes from 0 to $2\omega_1$, passing to the left in a semicircle about ω_1, x moves from plus infinity to e_1, then in a circle to the left about e_1 and back to plus infinity.

Any point within the parallelogram 0, ω_1, $-\omega_2$, ω_3 is mapped into a point on the negative half of the x-plane. The map of the whole period parallelogram doubly covers the x-plane, since pu is an elliptic function of the second order.

2. *The Map of a Vertical Line and the Equation of the Curve.*

Suppose u to describe a line parallel to the axis of imaginaries and at a distance α from it. Let

$$u = \alpha + i\beta$$

and

$$\bar{u} = \alpha - i\beta.$$

Then the equation of the line is

$$u + \bar{u} = 2\alpha, \tag{1}$$

u and $\bar{u}$ being conjugate and α real.

Since

$$pu = \frac{1}{u^2} + \frac{g_2 u^2}{2^2 \cdot 5} + \frac{g_3 u^6}{2^2 \cdot 7} + \cdots$$

(Tannery and Molk, "Théorie des Fonctions Elliptiques," IX, 3) and g_2 and g_3 are real by hypothesis,

and

$$\left.\begin{aligned} x &= p(\alpha + i\beta) = pu \\ \bar{x} &= p(\alpha - i\beta) = p\bar{u} \end{aligned}\right\} \quad (2)$$

are conjugate complex numbers. (Burkhardt, Vol. II, § 85; Vol. I, § 71.) To find the curve in x which corresponds to the vertical through α traced by u, eliminate β between equations (1) and (2) by means of an addition theorem which gives the relation between $p(u+v)$, pu and pv.

To obtain this theorem we have

$$p'^2 u = 4p^3 u - g_2 pu - g_3 .$$

Consider

$$p'u - cpu - c',$$

where c and c' are constants. It is an elliptic function with the same periods as pu. It has an infinity of the third order at $u=0$. Therefore it has three zeros u, v and w, such that

$$u + v + w = 0.$$

For each of these points

$$(cpu + c')^2 = (p'u)^2 = 4p^3 u - g_2 pu - g_3 ,$$

or

$$4p^3 u - c^2 p^2 u - (2cc' + g_2) pu - (c'^2 + g_3) = 0.$$

Then

$$S_1 \equiv pu + pv + pw = \frac{1}{4} c^2,$$

$$S_2 \equiv pu\, pv + pu\, pw + pv\, pw = -\frac{1}{4}(2cc' + g_2),$$

$$S_3 \equiv pu\, pv\, pw = \frac{1}{4}(c'^2 + g_3).$$

Therefore

$$S_1\left(S_3 - \frac{1}{4} g_3\right) = \frac{1}{16} c^2 c'^2 = \frac{1}{4}\left(S_2 + \frac{1}{4} g_2\right)^2,$$

from which we get an addition theorem:

$$\left(S_2 + \frac{1}{4} g_2\right)^2 = S_1 (4S_3 - g_3).$$

In this theorem substitute

$$u=\alpha+i\beta,$$
$$v=\alpha-i\beta,$$

whence

$$w=-2\alpha.$$

For the sake of brevity, write A for $p2\alpha$ or $p(-2\alpha)$. Then

$$S_1=x+\bar{x}+A,$$
$$S_2=x\bar{x}+A(x+\bar{x}),$$
$$S_3=Ax\bar{x}.$$

Therefore

$$\left(x\bar{x}+A[x+\bar{x}]+\frac{1}{4}g_2\right)^2=(x+\bar{x}+A)(4Ax\bar{x}-g_2). \tag{3}$$

The coefficient of x^2 is

$$(\bar{x}+A)^2-4A\bar{x}\equiv(\bar{x}-A)^2.$$

Speaking projectively, this gives the tangents at the circular point I or J and the equation is that of a bicircular quartic, with cusps at I and J; that is, a Cartesian oval.

The cusp tangents at I and J,

$$x-A=0$$

and

$$\bar{x}-A=0,$$

meet at the triple focus A corresponding to the point $u=2\alpha$.

To obtain the single foci, we have the two equations:

$$x\text{-discriminant}=0,$$

and

$$\bar{x}\text{-discriminant}=0.$$

Rearranging the equation of the curve according to the powers of $\bar{x}$, we have:

$$\bar{x}^2(x-A)^2+\bar{x}\left(\frac{1}{2}g_2x+\frac{1}{2}g_2A-2Ax^2-2A^2x+g_3\right)$$
$$+A^2x^2+\frac{1}{2}g_2Ax+\frac{1}{16}g_2^2+g_3x+g_3A=0.$$

Therefore the $\bar{x}$-discriminant is

$$^*(4A^3-g_2A-g_3)(4x^3-g_2x-g_3)=0.$$

Hence the single foci are e_1, e_2 and e_3.

The curve depends on the single constant $p2\alpha$ or A. It will therefore include the representation of four lines in the rectangle; namely, those which

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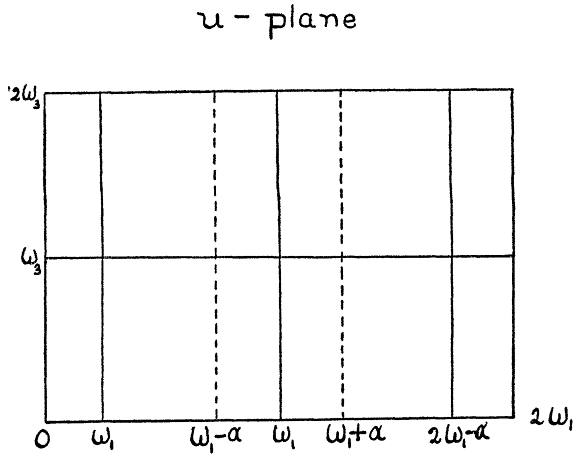
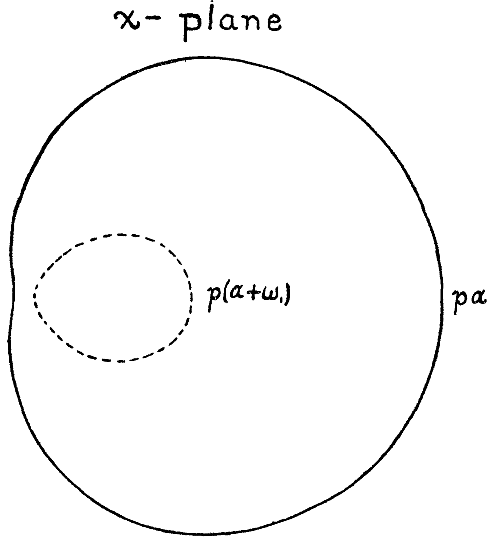
* If $\alpha=\frac{1}{2}\omega_1$, and so $p2\alpha$ or A causes the first factor of this equation to vanish, the Cartesian oval becomes a circle counted twice (see p. 266) and the foci accordingly are arbitrary.

meet the real axis at the four points α, $\omega_1-\alpha$, $\omega_1+\alpha$, $2\omega_1-\alpha$. If u is any one of these points, $p2u=p2\alpha$.





Let $i\beta=v$. Since $p(\alpha+v)=p(2\omega_1-[\alpha+v])=p(2\omega_1-\alpha-v)$, the outer oval corresponds to the first and fourth lines, the p-function moving in clockwise direction from the real axis as v moves up from α and in anti-clockwise direction as it moves up from $2\omega_1-\alpha$.

Similarly the inner oval, taken twice in opposite directions, corresponds to the second and third lines.

In the same way, to lines parallel to the real axis correspond Cartesian ovals with the same foci, as may be seen by interchanging α and v in this discussion.

Since at every point $u=(\alpha+v)$ in the u-plane there is a line parallel to the axis of reals and one parallel to the axis of imaginaries, so through every point in the x-plane pass two orthogonal Cartesian ovals corresponding to these two lines.

3. *The Inversion of the Curve into Itself.*

Among the formulæ for the p-function is

$$(p[u+\omega_1]-e_1)(pu-e_1)=e_2-e_1\cdot e_3-e_1.$$

(Harkness and Morley, "Introduction to Analytic Functions," p. 261, (20).) Therefore, if by quasi-inversion we mean ordinary inversion followed by reflection in the real axis, $\alpha+v+\omega_1$ is the quasi-inverse of $\alpha+v$, and the inverse of $\alpha-v$ since the curve is symmetric with respect to the real axis. Hence the curve is its own inverse with respect to a circle with center e_1 and radius $\sqrt{e_2-e_1}\,\sqrt{e_3-e_1}$ (a circle with radius real).

As α approaches $\frac{1}{2}\omega_1$, the two verticals through α and $\omega_1-\alpha$ approach coincidence. Therefore, if $\alpha=\frac{1}{2}\omega_1$, the two ovals coincide and each is the above-mentioned circle.

An algebraic proof of this statement may be added. Substituting in equation (3), p. 264,

$$2\alpha=\omega_1$$

and therefore

$$p2\alpha=e_1,$$

and remembering that

$$-\frac{1}{4}g_2=\Sigma e_1e_2$$

and

$$g_3=4e_1e_2e_3,$$

we get

$$[x\bar{x}+(x+\bar{x})e_1-\Sigma e_1e_2]^2=(x+\bar{x}+e_1)(4e_1x\bar{x}-4e_1e_2e_3),$$

or

$$(x-e_1)(\bar{x}-e_1)-(e_2-e_1)(e_3-e_1)=0,$$

which is a circle with center e_1 and radius $\sqrt{e_2-e_1}\cdot\sqrt{e_3-e_1}$.

Likewise

$$[p(u+\omega_2)-e_2][pu-e_2]=(e_1-e_2)(e_3-e_2),$$

or, more generally,

$$[p(u+\omega_\lambda)-e_\lambda][pu-e_\lambda]=(e_\mu-e_\lambda)(e_\nu-e_\lambda),$$

where

$$\lambda,\ \mu,\ \nu=1,\ 2,\ 3.$$

Therefore the curve has three inversions into itself, one about each focus as center. (But the circle with e_2 as center is nullipartite, since its radius, $\sqrt{e_1-e_2}\cdot\sqrt{e_3-e_2}$, is imaginary.)

These three circles are orthogonal, since

$$(e_\mu-e_\lambda)(e_\nu-e_\lambda)+(e_\lambda-e_\nu)(e_\mu-e_\nu)=(e_\nu-e_\lambda)^2.$$

But the bicircular quartic in general inverts into itself from any one of its four foci, and the four circles of inversion are all orthogonal. The fourth focus of the Cartesian oval is at infinity and the real axis is the circle of infinite radius orthogonal to the other three circles of inversion. Hence inversion with respect to the fourth focus amounts to reflexion in the real axis. (S. Roberts, *Proc. London Math. Soc.*, t. III, p. 108.)

At first thought, it might appear possible to obtain any number of points on the curve from a given point by means of repeated quasi-inversions. But in fact we get a set of only four points. For, if

$$[p(u+\omega_\lambda)-e_\lambda][pu-e_\lambda]=(e_\mu-e_\lambda)(e_\nu-e_\lambda)$$

and

$$[p(u+\omega_\mu)-e_\mu][pu-e_\mu]=(e_\nu-e_\mu)(e_\lambda-e_\mu)$$

and

$$[p(u+\omega_\nu)-e_\nu][pu-e_\nu]=(e_\mu-e_\nu)(e_\lambda-e_\nu),$$

we get, by eliminating pu from the second and third equations,

$$[p(u+\omega_\mu)-e_\lambda][p(u+\omega_\nu)-e_\lambda]=(e_\mu-e_\lambda)(e_\nu-e_\lambda).$$

Hence, if we say that pu and $p(u+\omega_\lambda)$ are quasi-inverse as to e_λ, then $p(u+\omega_\mu)$ and $p(u+\omega_\nu)$ are also quasi-inverse as to e_λ.

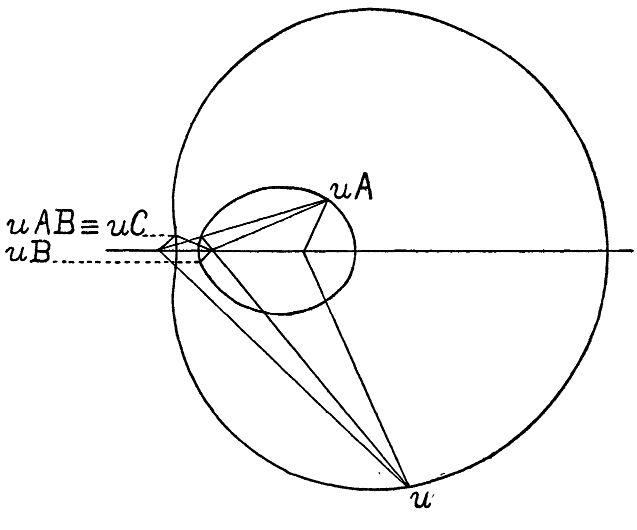
If we call inversion as to e_λ I_λ $(\lambda=1, 2, 3)$ and reflexion in the real axis I_4 and let

$$\begin{aligned} A&=I_1I_4=I_4I_1=I_2I_3=I_3I_2, \\ B&=I_2I_4=I_4I_2=I_1I_3=I_3I_1, \\ C&=I_3I_4=I_4I_3=I_1I_2=I_2I_1, \end{aligned}$$

where A, B and C are evidently the three quasi-inversions, then 1, A, B, $C\equiv AB$ form a "four group,"

$$1=A^2=B^2=(AB)^2.$$

(Klein, "Ikosaeder," pp. 5, 13; Cayley, "Mathematical Papers," Vol. II, p. 125.)



If we reflect these four points in the real axis, we get four other points, which, together with these, are the eight points of intersection of this curve with the confocal orthogonal Cartesian oval which passes through the point $\alpha+v$; that is, the curve which is the map of the line in the u-plane through $\alpha+v$ parallel to the real axis. (For convenience we denote a point on the curve by its parameter.)

4. *Focal Radii.*

If ρ_1 and ρ_2 are the focal radii of the curve, $k\rho_1+\rho_2=$constant is a well-known property of the curve, which may be derived in the following manner.

A focal vector of any point on the curve is

$$r=p(\alpha+v)-e_1,$$

and of the conjugate point is

$$\bar{r}=p(\alpha-v)-e_1.$$

Then the focal radius is

$$\rho_1=\sqrt{r\bar{r}}=\sqrt{p(\alpha+v)-e_1}\ \sqrt{p(\alpha-v)-e_1}.$$

But

$$\sqrt{pu-e_1}=\frac{\sigma_1 u}{\sigma u}.$$

(Harkness and Morley, "Treatise on the Theory of Functions," p. 307.) Hence

$$\rho_1=\frac{\sigma_1(\alpha+v)\cdot\sigma_1(\alpha-v)}{\sigma(\alpha+v)\cdot\sigma(\alpha-v)}.$$

Therefore ρ_1 is an elliptic function with the same periods $2\omega_1$, $2\omega_2$ and $2\omega_3$ of the second order, with infinities at $\pm\alpha$ and residues $\mp\frac{\sigma_1 2\alpha}{\sigma 2\alpha}$. Likewise

$$\rho_2=\frac{\sigma_2(\alpha+v)\cdot\sigma_2(\alpha-v)}{\sigma(\alpha+v)\cdot\sigma(\alpha-v)}$$

is an elliptic function with infinities at $\pm\alpha$ and residues $\mp\frac{\sigma_2 2\alpha}{\sigma 2\alpha}$. Let the distance to the triple focus $p2\alpha$ from a single focus e_1 be λ_1. Therefore

$$p2\alpha-e_1=\lambda_1,$$

$$\sqrt{p2\alpha-e_1}=\sqrt{\lambda_1}=\frac{\sigma_1 2\alpha}{\sigma 2\alpha}.$$

Hence the residues of ρ_1 at $\pm\alpha$ are $\mp\sqrt{\lambda_1}$, of ρ_2 at $\pm\alpha$ are $\mp\sqrt{\lambda_2}$. Now if two homoperiodic elliptic functions, f_1u and f_2u, of the second order have the same infinitudes, c and c', but different residues, r_1 and r_2, at an infinity c, then

$$\begin{vmatrix} f_1u & r_1 \\ f_2u & r_2 \end{vmatrix}=\text{constant}.$$

(Harkness and Morley, "Introduction to Analytic Functions," p. 259, II.) Therefore

$$\begin{vmatrix} \rho_1 & -\sqrt{\lambda_1} \\ \rho_2 & -\sqrt{\lambda_2} \end{vmatrix}=k;$$

that is, there is a linear relation between the focal radii. We proceed to find k.

$$k=\sqrt{\lambda_1}\sqrt{p(\alpha+v)-e_2}\sqrt{p(\alpha-v)-e_2}-\sqrt{\lambda_2}\sqrt{p(\alpha+v)-e_1}\sqrt{p(\alpha-v)-e_1}.$$

Or expanding,

$$k=\sqrt{\lambda_1}\sqrt{p2\alpha-e_2+p'2\alpha(v-\alpha)\ldots.}\sqrt{\frac{1}{(\alpha-v)^2}\ldots.-e_2}$$

$$-\sqrt{\lambda_2}\sqrt{p2\alpha-e_1+p'2\alpha(v-\alpha)\ldots.}\sqrt{\frac{1}{(\alpha-v)^2}\ldots.-e_1}$$

$$=\frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{(\alpha-v)}\left[1+\frac{1}{2}p'2\alpha(v-\alpha)+P(v-\alpha)\ldots.\right]$$

$$-\frac{\sqrt{\lambda_2}\sqrt{\lambda_1}}{(\alpha-v)}\left[1+\frac{1}{2}p'2\alpha(v-\alpha)+P(v-\alpha)\ldots.\right].$$

Therefore as v approaches α

$$k=\frac{1}{2}\sqrt{\lambda_1}\sqrt{\lambda_2}\left[\frac{p'2\alpha}{\lambda_1}-\frac{p'2\alpha}{\lambda_2}\right],$$

or since $p'2\alpha=-2\sqrt{\lambda_1\lambda_2\lambda_3}$,

$$k=\lambda_1\sqrt{\lambda_3}-\lambda_2\sqrt{\lambda_3}.$$

Hence

$$\sqrt{\lambda_1}\rho_2-\sqrt{\lambda_2}\rho_1=\sqrt{\lambda_3}(\lambda_1-\lambda_2),$$

or, more generally,

$$\sqrt{\lambda_\nu}\rho_\mu-\sqrt{\lambda_\mu}\rho_\nu=\sqrt{\lambda_\lambda}(\lambda_\nu-\lambda_\mu).$$

Two of three such equations give

$$\sqrt{\lambda_1}\sqrt{\lambda_2}\rho_3-\lambda_1\lambda_2=\sqrt{\lambda_2}\sqrt{\lambda_3}\rho_1-\lambda_2\lambda_3=\sqrt{\lambda_3}\sqrt{\lambda_1}\rho_2-\lambda_3\lambda_1.$$

(Compare with Greenhill's result, "Elliptic Functions," Arts. 248 and 249.) From these equations, too, we obtain Panton's form (Math. Quest. 2622, *Ed. Times*, t. XI, p. 56).

$$\sqrt{\lambda_1}(\lambda_2-\lambda_3)\rho_1+\sqrt{\lambda_2}(\lambda_3-\lambda_1)\rho_2+\sqrt{\lambda_3}(\lambda_1-\lambda_2)\rho_3=0.$$

The bifocal equation of the Cartesian oval, as ordinarily given, is

$$l\rho_1\pm m\rho_2=n$$

(Williamson, "Differential Calculus," p. 375), in which the upper sign belongs to the equation of the inner oval and the lower to that of the outer oval.

Since for the outer oval, which maps $\alpha+v$, v varying from 0 to $2\omega_3$, each of the quantities ρ_1, ρ_2, $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ is positive, our form

$$\sqrt{\lambda_2}\rho_1-\sqrt{\lambda_1}\rho_2=\sqrt{\lambda_3}(\lambda_2-\lambda_1)$$

is consistent with that given above. That it holds also for the inner oval may be explained in either of two ways:

(a) We may consider the inner oval as the map of $\alpha'+v$, where $\alpha'=\alpha+\omega_1$. In that case ρ_1, ρ_2 and $\sqrt{\lambda_2}=\dfrac{\sigma_2 2\alpha}{\sigma 2\alpha}$ remain positive, but $\sqrt{\lambda_1}=\dfrac{\sigma_1 2\alpha}{\sigma 2\alpha}$ has passed a zero (ω_1) of $\sigma_1 2\alpha$ and is therefore negative. Hence the equation agrees in sign with the one usually given.

(b) We may consider α as fixed and take as equations of the inner oval

$$x=p(\alpha+\omega_1+v)$$

and

$$\bar{x}=p(\alpha-\omega_1-v),$$

in which case $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ are unchanged and must be taken positively. To study the signs of ρ_1 and ρ_2, let us look at the limiting case when v approaches zero. Then

$$\rho_1=\frac{\sigma_1(\alpha+\omega_1)\cdot\sigma_1(\alpha-\omega_1)}{\sigma(\alpha+\omega_1)\cdot\sigma(\alpha-\omega_1)}$$

is positive, since $\sigma_1(\alpha+\omega_1)$ is negative, $\sigma_1(\alpha-\omega_1)$ positive, $\sigma(\alpha+\omega_1)$ positive and $\sigma(\alpha-\omega_1)$ negative. (Tannery and Molk, "Théorie des Fonctions Elliptiques," t. 1, p. 201.) But

$$\rho_2=\frac{\sigma_2(\alpha+\omega_1)\sigma_2(\alpha-\omega_1)}{\sigma(\alpha+\omega_1)\sigma(\alpha-\omega_1)}$$

is negative, as $\sigma_2(\alpha+\omega_1)$ and $\sigma_2(\alpha-\omega_1)$ are both positive. Therefore our form of the equation holds equally well for the inner oval.

Similarly it may be shown that each of the expressions

$$\sqrt{\lambda_\lambda}\sqrt{\lambda_\mu}\,\rho_\nu-\lambda_\lambda\lambda_\mu=\frac{1}{2}\,[\rho^2-(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\,],$$

where ρ is the distance of any point on the curve from the triple focus. For

$$\rho^2=[p(\alpha+v)-p2\alpha]\,[p(\alpha-v)-p2\alpha],$$

which is an elliptic function of the second order with infinities at $\pm\alpha$ and residues $\pm p'2\alpha$. Hence

$$\begin{vmatrix}\rho_1 & -\sqrt{\lambda_1}\\ \rho^2 & p'2\alpha\end{vmatrix}=k,$$

or

$$\rho_1 p'2\alpha+\rho^2\sqrt{\lambda_1}=k,$$

or

$$k=p'2\alpha\sqrt{p(\alpha+v)-e_1}\,\sqrt{p(\alpha-v)-e_1}+\sqrt{\lambda_1}[p(\alpha+v)-p2\alpha\cdot p(\alpha-v)-p2\alpha].$$

Taking, as before, the limiting value as v approaches α, we find

$$k=-\frac{\sqrt{\lambda_1}}{2}\,\frac{p'^2 2\alpha}{\lambda_1}+\sqrt{\lambda_1}\,\frac{p''2\alpha}{2}.$$

But since $p'^2 2\alpha = 4(p2\alpha - e_1)(p2\alpha - e_2)(p2\alpha - e_3)$,

$$2p'2\alpha \cdot p''2\alpha = 4p'2\alpha[(p2\alpha - e_1)(p2\alpha - e_2) + (p2\alpha - e_2)(p2\alpha - e_3) + (p2\alpha - e_3)(p2\alpha - e_1)].$$

Therefore

$$p''2\alpha = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$$

and

$$k = -2\sqrt{\lambda_1}\,\lambda_2\lambda_3 + \sqrt{\lambda_1}(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1).$$

Hence

$$\rho_1\sqrt{\lambda_2\lambda_3} - \lambda_2\lambda_3 = \frac{1}{2}[\rho^2 - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)].$$

Similarly we may obtain a relation between the distances of any point on the curve from the double tangent and from a single focus. The equation of the double tangent is

$$x + \bar{x} + A = 0,$$

as may be seen from equation (3), p. 264 (where $A = p2\alpha = p(-2\alpha)$). If we denote by T the distance to a point of the curve from the double tangent, then

$$2T = p(\alpha + v) + p(\alpha - v) + A.$$

But if $u + v + w = 0$,

$$pu + pv + pw = \frac{1}{4}\left(\frac{p'u - p'v}{pu - pv}\right)^2 = (\zeta u + \zeta v + \zeta w)^2.$$

(Harkness and Morley, "Treatise on the Theory of Functions," pp. 300, 304). Therefore

$$\sqrt{2T} = \zeta(\alpha + v) + \zeta(\alpha - v) - \zeta 2\alpha = -\frac{1}{2}\left[\frac{p'(\alpha + v) - p'(\alpha - v)}{p(\alpha + v) - p(\alpha - v)}\right],$$

which is an elliptic function of the second order with infinities at $\pm\alpha$ and residues at ∓ 1. Hence

$$\begin{vmatrix} \sqrt{2T} & 1 \\ \rho_1 & \sqrt{\lambda_1} \end{vmatrix} = k,$$

or

$$\sqrt{\lambda_1}[\zeta(\alpha + v) + \zeta(\alpha - v) - \zeta 2\alpha] - \sqrt{p(\alpha + v) - e_1}\,\sqrt{p(\alpha - v) - e_1} = k.$$

Taking, again, the limit as v approaches α, we find

$$k = \frac{1}{2}\frac{p'2\alpha}{\sqrt{\lambda_1}} = -\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{\lambda_1}} = -\sqrt{\lambda_2\lambda_3}.$$

Hence

$$\sqrt{\lambda_1}\sqrt{2T} = \rho_1 - \sqrt{\lambda_2\lambda_3},$$

or

$$\sqrt{\lambda_\lambda}\sqrt{2T} = \rho_\lambda - \sqrt{\lambda_\mu\lambda_\nu}.$$

From these we could have obtained the formulæ derived previously on p. 269,

$$\sqrt{\lambda_1}\sqrt{\lambda_2}\,\rho_3 - \lambda_1\lambda_2 = \sqrt{\lambda_2}\sqrt{\lambda_3}\,\rho_1 - \lambda_2\lambda_3 = \sqrt{\lambda_3}\sqrt{\lambda_1}\,\rho_2 - \lambda_3\lambda_1.$$

5. *Another Definition of the Curve.*

The equation of the tangent to the curve at any point $p(\alpha+v)$ is

$$\begin{vmatrix} x & \bar{x} & 1 \\ p\,(\alpha+v) & p\,(\alpha-v) & 1 \\ p'(\alpha+v) & -p'(\alpha-v) & 0 \end{vmatrix}=0.$$

Hence the equation of the normal at the same point is

$$\begin{vmatrix} x & \bar{x} & 1 \\ p\,(\alpha+v) & p\,(\alpha-v) & 1 \\ p'(\alpha+v) & p'(\alpha-v) & 0 \end{vmatrix}=0.$$

To find the point N where the normal cuts the real axis, let

$$x=\bar{x}=N.$$

Hence

$$N=p(\alpha+v)-p'(\alpha+v)\left[\frac{p(\alpha+v)-p(\alpha-v)}{p'(\alpha+v)-p'(\alpha-v)}\right].$$

But, since

$$\begin{vmatrix} 1 & p(\alpha+v) & p'(\alpha+v) \\ 1 & p(\alpha-v) & p'(\alpha-v) \\ 1 & p2\alpha & -p'2\alpha \end{vmatrix}=0$$

(Burkhardt, "Functionen Theorie," II, p. 62), or

$$p(\alpha+v)=p2\alpha+[p'2\alpha+p'(\alpha+v)]\left[\frac{p(\alpha+v)-p(\alpha-v)}{p'(\alpha+v)-p'(\alpha-v)}\right],$$

$$N=p2\alpha+p'2\alpha\left[\frac{p(\alpha+v)-p(\alpha-v)}{p'(\alpha+v)-p'(\alpha-v)}\right].$$

But from p. 271,

$$\sqrt{2T}=-\frac{1}{2}\left[\frac{p'(\alpha+v)-p'(\alpha-v)}{p(\alpha+v)-p(\alpha-v)}\right].$$

Hence

$$N=p2\alpha-\frac{p'2\alpha}{2\sqrt{2T}}=p2\alpha+\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2T}},$$

or

$$N-e_1=\lambda_1+\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2T}}=\frac{\sqrt{\lambda_1}\,(\sqrt{2\lambda_1 T}+\sqrt{\lambda_2\lambda_3})}{\sqrt{2T}}.$$

Therefore

$$N-e_1=\frac{\sqrt{\lambda_1}}{\sqrt{2T}}\rho_1$$

(see p. 271) and

$$N-p2\alpha=\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2T}}.$$

Therefore

$$\frac{N-e_1}{N-p2\alpha}=\frac{\sqrt{\lambda_2\lambda_3}}{\rho_1}.$$

If we draw from $p2\alpha$ a line perpendicular to the tangent at x, and therefore parallel to the normal, cutting ρ_1 at τ_1, then

$$\frac{x\tau_1}{\rho_1}=\frac{N-p2\alpha}{N-e_1}=\frac{\sqrt{\lambda_2\lambda_3}}{\rho_1}.$$

Therefore

$$x\tau_1=\sqrt{\lambda_2\lambda_3}.$$

Similarly

$$x\tau_2=\sqrt{\lambda_3\lambda_1},$$

and

$$x\tau_3=\sqrt{\lambda_1\lambda_2}.$$

Hence we deduce the theorem:

The perpendicular from the triple focus on the tangent at a point x of a Cartesian oval meets a focal radius ρ_λ at a point τ_λ such that $|x-\tau_\lambda|=$ a constant.

Conversely, if a line, revolving about a fixed point $p2\alpha$, cut off on two other straight lines, revolving about two points e_1 and e_2 collinear with $p2\alpha$, constant distances $x\tau_1$ and $x\tau_2$, then x lies on a Cartesian oval. For by the theorem of Menelaus

$$x\tau_2\cdot\lambda_2\cdot\rho_1-x\tau_1=\rho_2-x\tau_2\cdot x\tau_1\cdot\lambda_1,$$

or, if we take the constants $x\tau_2$ and $x\tau_1$ to be $\sqrt{\lambda_3\lambda_1}$ and $\sqrt{\lambda_2\lambda_3}$, respectively,

$$\sqrt{\lambda_1\lambda_3}\cdot\lambda_2\cdot\rho_1-\sqrt{\lambda_2\lambda_3}=\rho_2-\sqrt{\lambda_1\lambda_3}\cdot\sqrt{\lambda_2\lambda_3}\cdot\lambda_1.$$

Hence we have

$$\sqrt{\lambda_2}\,\rho_1-\sqrt{\lambda_1}\,\rho_2=\sqrt{\lambda_3}\,(\lambda_2-\lambda_1).$$

Therefore x lies on the Cartesian oval. Therefore we may define a Cartesian oval as the locus of the vertex of a triangle with two of its sides of fixed lengths, and passing always through two fixed points, while the base revolves about a fixed point collinear with the other two.

If we had measured the constant distances from e_1 and e_2 along the focal radii, instead of from the point x, we should have had Newton's definition of the curve; namely, the locus of a point whose distances from the circumferences of two circles (with centers at e_1 and e_2) are in a constant ratio.

6. *Intersections of Circles with the Curve.*

If

$$x=p(\alpha+v)$$

and

$$\bar{x}=p(\alpha-v)$$

are the parametric equations of the Cartesian oval, and the equation of any other curve is

$$R(x, \bar{x}) = 0,$$

the intersections of the two curves are given by the zeros v_i of

$$f(v) = R[p(\alpha+v),\ p(\alpha-v)],$$

whose infinities are $\pm\alpha$. Hence

$$\Sigma v_i \equiv 0, \tag{1}$$

from Abel's theorem for integrals of the first kind. (Picard, t. II, p. 396.)

Let $z = \alpha + v$. Then

$$\phi(z) - k = R[pz, p(2\alpha - z)] = A\Pi \frac{\sigma(z - z_\nu)}{\sigma(z - c_\nu)},$$

where z_ν is a zero of $\phi(z) - k$ and c_ν is an infinity, or pole, of $\phi(z) - k$. Accordingly

$$\mathscr{D} \log R = \Sigma\zeta(z - z_\nu) - \Sigma\zeta(z - c_\nu).$$

But

$$\mathscr{D} \log R = \frac{\phi'(z)}{\phi(z) - k}.$$

Hence, when $z = 0$,

$$\frac{\phi'(0)}{\phi(0) - k} = -\Sigma\zeta z_\nu + \Sigma\zeta c_\nu, \tag{2}$$

which is a form of Abel's theorem for elliptic integrals of the second class (Harkness and Morley, "Treatise on the Theory of Functions," p. 458) since

$$\zeta u = -\int pu\,du = \int \frac{pu\,d(pu)}{\sqrt{4p^3u - g_2 pu - g_3}}.$$

Let the curve be a circle with center c. Its equation is

$$x\bar{x} - \bar{c}x - c\bar{x} + \lambda = 0,$$

and its intersections with our curve are given by the zeros of

$$p(\alpha+v)\,p(\alpha-v) - \bar{c}p(\alpha+v) - cp(\alpha-v) + \lambda = 0,$$

an elliptic function of the fourth order with double infinities at $\pm\alpha$. Therefore

$$v_1 + v_2 + v_3 + v_4 = 0,$$

or the circle meets the curve in four points, the sum of whose parameters is zero.

We proceed to apply equation (2) to find a second relation,

$$\phi(z) - k = pz \cdot p(2\alpha - z) - \bar{c}pz - cp(2\alpha - z) + \lambda - k = A\Pi \frac{\sigma(z - z_\nu)}{\sigma^2 z \sigma^2(2\alpha - z)}.$$

Therefore

$$\frac{\phi' z}{\phi(z)-k}=\Sigma\zeta(z-z_\nu)-2\zeta z-2\zeta(z-2\alpha),$$

$$pz=\frac{1}{z^2}+3S_4 z^2+\ldots\ldots,$$

$$p(z-2\alpha)=p2\alpha-zp'2\alpha+\frac{z^2}{2}p''2\alpha-\ldots\ldots,$$

$$\phi(z)-k=\frac{1}{z^2}p2\alpha-\frac{1}{z}p'2\alpha+\frac{1}{2}p''2\alpha-\ldots\ldots-\bar{c}\left(\frac{1}{z^2}+\ldots\ldots\right)-cp2\alpha\ldots\ldots+\lambda-k,$$

$$\frac{\phi' z}{\phi(z)-k}=\frac{-\frac{2}{z^3}(p2\alpha-\bar{c})+\frac{1}{z^2}p'2\alpha+P_0(z)\ldots\ldots}{\frac{1}{z^2}(p2\alpha-\bar{c})-\frac{1}{z}p'2\alpha+\frac{1}{2}p''2\alpha-cp2\alpha+\lambda-k\ldots\ldots}$$

$$=\left[-\ ^{-}+\frac{p'2\alpha}{p2\alpha-\bar{c}}+\ldots\ldots\right]\left[1+\frac{zp'2\alpha}{p2\alpha-\bar{c}}+\ldots\ldots\right].$$

Therefore

$$\Sigma\zeta(z-z_\nu)-\frac{2}{z}\ldots\ldots-2\zeta(z-2\alpha)=-\frac{2}{z}-\frac{p'2\alpha}{p2\alpha-\bar{c}}-P(z),$$

whence, if $z=0$,

$$\overset{4}{\Sigma}\zeta z_\nu-2\zeta 2\alpha=\frac{p'2\alpha}{p2\alpha-\bar{c}},$$

or, substituting for z its value $\alpha+v$,

$$\overset{4}{\Sigma}\zeta(\alpha+v)=2\zeta 2\alpha+\frac{p'2\alpha}{p2\alpha-\bar{c}}.$$

Similarly

$$\overset{4}{\Sigma}\zeta(\alpha-v)=2\zeta 2\alpha+\frac{p'2\alpha}{p2\alpha-c}.$$

These equations give the center of the circle. If c is at infinity, the circle is a line and we have, for the intersections of the line with a Cartesian oval,

$$\overset{4}{\Sigma}\zeta(\alpha+v)=2\zeta 2\alpha$$

and

$$\overset{4}{\Sigma}\zeta(\alpha-v)=2\zeta 2\alpha.$$

Therefore

$$\Sigma[\zeta(\alpha+v)+\zeta(\alpha-v)-\zeta 2\alpha]=0.$$

But from p. 271 we have

$$\zeta(\alpha+v)+\zeta(\alpha-v)-2\alpha=\sqrt{2T},$$

where T is the distance from any point of the curve to the double tangent.

Therefore, if a straight line be drawn, cutting a Cartesian oval in four points, the sum of the square roots of twice the distances of these points from the double tangent is zero.

But we found on p. 271 that

$$\sqrt{2T}=\frac{\rho_1-\sqrt{\lambda_2\lambda_3}}{\lambda_1},$$

whence, if we denote the distance of any of the points $\alpha+v_\nu$ from the focus e_1 by $\rho_{1\nu}$, we have

$$\Sigma\frac{\rho_{1\nu}-\sqrt{\lambda_2\lambda_3}}{\sqrt{\lambda_1}}=0,$$

or

$$\Sigma\rho_{1\nu}=4\sqrt{\lambda_2\lambda_3}.$$

Therefore the sum of the distances of the points in which a straight line cuts a Cartesian oval from any focus is constant, a well-known property.

It follows as a corollary to this that the points of contact of the double tangent lie on a circle with center e_1 and $\sqrt{\lambda_2\lambda_3}$.

7. *Bitangent Circles.*

In the case of bitangent circles the equation

$$\Sigma v_i\equiv 0$$

reduces to

$$2v_1+2v_2=0.$$

Four cases are to be distinguished, when $v_2=-v_1$, and when $v_2=-v_1+\omega_\lambda$ and the coincident point $v_2=-v-\omega_\lambda$ $(\lambda=1,2,3)$. Hence there are four systems of bitangent circles.

In the first case,

$$\Sigma\zeta(\alpha+v)=2\zeta 2\alpha+\frac{p'2\alpha}{p2\alpha-\bar{c}}$$

becomes

$$\zeta(\alpha+v)+\zeta(\alpha-v)-\zeta 2\alpha=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-\bar{c}},$$

and its conjugate

$$\zeta(\alpha-v)+\zeta(\alpha+v)-\zeta 2\alpha=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-c}.$$

Therefore $c=\bar{c}$, or the center lies on the real axis, as is evident geometrically from the symmetry of the curve.

*Among the circles of this system are the bitangent line, two circles with fourfold contact with the inner oval, one at $\alpha+\omega_1$ and one at $\alpha+\omega_2$, and two with the outer oval, one at α and the other at $\alpha+\omega_3$, and three point circles with imaginary contact, the three single foci. The fourth point circle is at infinity.

In the other case we have

$$2\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)+\zeta(\alpha-v-\omega_\lambda)-2\zeta 2\alpha=\frac{p'2\alpha}{p2\alpha-\bar{c}},$$

or

$$2\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)+\zeta(\alpha-v+\omega_\lambda-2\omega_\lambda)-2\zeta 2\alpha=\frac{p'2\alpha}{p2\alpha-\bar{c}},$$

or

$$\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)-\zeta\omega_\lambda-\zeta 2\alpha=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-\bar{c}}.$$

Likewise

$$\zeta(\alpha-v)+\zeta(\alpha+v-\omega_\lambda)+\zeta\omega_\lambda-\zeta 2\alpha=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-c}.$$

The following formula is one given by Professor Morley (*Proc. London Math. Soc.*, Ser. 2, Vol. IV, p. 390):

$$\zeta(a)+\zeta(b)+\zeta(c)-\zeta(a+b+c)=\frac{\sigma(b+c)\sigma(a+c)\sigma(a+b)}{\sigma a\cdot\sigma b\cdot\sigma c\cdot\sigma(a+b+c)}.$$

To prove it, suppose a to be variable. Then

$$\zeta(a)+\zeta(b)+\zeta(c)-\zeta(a+b+c)$$

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*The relation of these bitangent circles is well brought out in a study of sphero-conics.

Suppose a sphere to be intersected by a quadric cone with the vertex at the center. Through this intersection pass a pencil of quadrics, among which are four degenerate quadrics or cones, three besides the cone already mentioned. These three cones have their vertices at infinity, or are cylinders whose axes are mutually perpendicular. From one of the foci of the sphero-conics project the figure on the plane of the great circle of which that focus is the pole.

The two ovals of the sphero-conic project into the two ovals of the Cartesian; the great circle, on which the four foci lie, into the real axis with the three foci lying on it and the fourth at infinity. Through each element of the four cones passes a tangent plane which intersects the sphere in a circle bitangent to one of the sphero-conics (or both) with real or imaginary contacts. These project into bitangent circles of the Cartesian and we get the four systems corresponding to the four cones. The sphere and each of the other quadrics have four common tangent planes (imaginary in every case but one), and since these planes cut out the point circles on the sphere, they touch it at the four real foci of the two sphero-conics which we have already said project into the four foci of the Cartesian oval (one being at infinity). The circle cut out by the other tangent plane through the focus, from which we are making the projection, to the quadric which has real common tangent planes with the sphere, projects into a straight line, namely, the bitangent line.

The tangent planes of the cone with vertex at the center, and also of the cylinder whose elements are parallel to the axis of this cone, determine circles whose projections touch different branches of the Cartesian oval. In the former case, the circles enclose the inner oval; in the latter, they lie between the two ovals. (See Darboux, "Sur une Classe Remarquable de Courbes et de Surfaces.")

is an elliptic function with two infinities, $a=0$ and $a=-b-c$, and two zeros, $a=-b$ and $a=-c$. Therefore it is equal to

$$k\frac{\sigma(a+b)\sigma(a+c)}{\sigma a\cdot\sigma(a+b+c)}.$$

Multiplying both sides of the equation by a, and taking the limit when $a=0$, we get

$$k=\frac{\sigma(b+c)}{\sigma b\cdot\sigma c}.$$

Therefore

$$\zeta(a)+\zeta(b)+\zeta(c)-\zeta(a+b+c)=\frac{\sigma(a+b)\sigma(b+c)\sigma(c+a)}{\sigma a\cdot\sigma b\cdot\sigma c\cdot\sigma(a+b+c)}.$$

Hence

$$\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)-\zeta(\omega_\lambda)-\zeta 2\alpha$$
$$=-\frac{\sigma(2\alpha+\omega_\lambda)\sigma(\alpha+v-\omega_\lambda)\sigma(\alpha-v)}{\sigma(\alpha+v)\sigma(\alpha-v+\omega_\lambda)\sigma\omega_\lambda\cdot\sigma 2\alpha)}=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-\bar{c}},$$

and in the same way

$$\zeta(\alpha-v)+\zeta(\alpha+v-\omega_\lambda)+\zeta\omega_\lambda-\zeta 2\alpha$$
$$=\frac{\sigma(2\alpha-\omega_\lambda)\sigma(\alpha-v+\omega_\lambda)\sigma(\alpha+v)}{\sigma(\alpha-v)\cdot\sigma(\alpha+v-\omega_\lambda)\sigma\omega_\lambda\cdot\sigma 2\alpha}=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-c}.$$

Multiplying the two equations, we get

$$-\frac{\sigma(2\alpha+\omega_\lambda)\sigma(2\alpha-\omega_\lambda)}{\sigma^2\omega_\lambda\cdot\sigma^2 2\alpha}=\frac{\frac{1}{4}p'^2 2\alpha}{p2\alpha-\bar{c}\cdot p2\alpha-c}.$$

But the left-hand member of this equation is equal to $p2\alpha-e_\lambda=\lambda_\lambda$. (Harkness and Morley, "Introduction to Analytic Functions," p. 104.) Therefore

$$\lambda_\lambda=\frac{\frac{1}{4}p'^2 2\alpha}{p2\alpha-\bar{c}\cdot p2\alpha-c}=\frac{\lambda_\lambda\lambda_\mu\lambda_\nu}{p2\alpha-\bar{c}\cdot p2\alpha-c};$$

hence

$$(\bar{c}-p2\alpha)(c-p2\alpha)=\lambda_\mu\lambda_\nu.$$

Therefore the locus of the center of each of these three systems of bitangent circles is a circle whose center is $p2\alpha$, the triple focus, and whose radius is $\sqrt{\lambda_\mu\lambda_\nu}$. For example, the locus of the center of the bitangent circle touching the curve at $\alpha+v$ and $\alpha-v+\omega_1$ is a circle with a triple focus as center and $\sqrt{\lambda_2\lambda_3}$ as radius.

That this set of circles is orthogonal to a circle, with radius equal to

$$\sqrt{e_2-e_1}\cdot\sqrt{e_3-e_1}$$

and center e_1, follows from the fact developed on p. 265, that $p(\alpha+v)$ and $p(\alpha-v+\omega_1)$ are inverse points as to this circle, and the theorem of elementary geometry, that the product of a secant to a circle from a point without and its external segment is equal to the square of the tangent from the same point.

Hence we get the theorem of Casey in the *Transactions of the Royal Irish Academy*, 1669 (Williamson, "Differential Calculus," p. 383): "If a circle cut a given circle orthogonally, while its center moves along another given circle, its envelope is a Cartesian oval."

Among the circles of this bitangent system, orthogonal to the circle with e_1 as center, are the circle touching the oval at α and $\alpha+\omega_1$ and the one touching it at $\alpha+\omega_2$ and $\alpha+\omega_3$. The center of the first circle is $\frac{1}{2}(\alpha+\alpha+\omega_1)$ and of the second is $\frac{1}{2}(\alpha+\omega_2+\alpha+\omega_3)$. Therefore the center of the circle of centers is

$$\frac{1}{4}(\alpha+\alpha+\omega_1+\alpha+\omega_2+\alpha+\omega_3).$$

But we have just proved that this point is the triple focus. Hence the triple focus is the centroid of the four points in which the real axis cuts the Cartesian oval, which gives a simple method for the construction of the triple focus.

As a check for this statement, we proved that, if a straight line cuts the curve in points whose parameters are v_r,

$$\Sigma\zeta(\alpha+v_r)=2\zeta 2\alpha.$$

If the line is the real axis we have

$$\zeta\alpha+\zeta(\alpha+\omega_1)+\zeta(\alpha+\omega_2)+\zeta(\alpha+\omega_3)=2\zeta 2\alpha.$$

Hence, by differentiation,

$$p\alpha+p(\alpha+\omega_1)+p(\alpha+\omega_2)+p(\alpha+\omega_3)=4p2\alpha,$$

or $p2\alpha$ is the centroid of these four points.

Among the circles of the system, with centers on the circle with radius $\sqrt{\lambda_1\lambda_2}$ orthogonal to the circle with center e_3 and radius $\sqrt{e_2-e_3}\cdot\sqrt{e_1-e_3}$, is the circle touching the curve at α and $\alpha+\omega_3$ and one touching at $\alpha+\omega_1$ and $\alpha+\omega_2$, also two with fourfold contact to the outer oval at the points $\alpha\pm\frac{1}{2}\omega_3$, and two to the inner oval at $\alpha+\omega_1\pm\frac{1}{2}\omega_3$. These are the points of contact of the tangents from e_3 to the curve; for since $p(\alpha-v+\omega_3)$ is the inverse of $p(\alpha+v)$ as to the circle with center e_3, mentioned above, if $v=\frac{1}{2}\omega_3$ we have

$$p\left(\alpha-\frac{1}{2}\omega_3+\omega_3\right)=p\left(\alpha+\frac{1}{2}\omega_3\right)$$

the inverse of $p\left(\alpha+\frac{1}{2}\omega_3\right)$; that is, the inverse of itself. Hence the points of contact of the bitangent circle come together where the orthogonal circle with center e_3 cuts the curve, and the focal vector e_3 is tangent at this point. A similar proof holds for the points $\alpha+\omega_1\pm\frac{1}{2}\omega_3$. The circle with e_3 as center cuts both ovals orthogonally. It, counted twice, is the Cartesian oval corresponding to $\alpha=\frac{1}{2}\omega_1$ in our system. Hence it cuts every oval of the system orthogonally.

The last system of bitangent circles is the one with centers on the circle with center $p2\alpha$ and radius $\sqrt{\lambda_1\lambda_3}$. Here we have one circle touching the outer oval at α and the inner at $\alpha+\omega_2$, and another touching the outer at $\alpha+\omega_3$ and the inner at $\alpha+\omega_1$.

8. *Relation of Tangents to Bitangent Circles.*

If $x\bar{x}-\bar{c}x-c\bar{x}+\lambda=0$ is the equation of a circle, and x_1 and $\bar{x}_1$ are the coordinates of a point, and the tangents are drawn from the point to the circle, the square of the distance from the point to the circle measured along the tangent is $x_1\bar{x}_1-\bar{c}x_1-c\bar{x}_1+\lambda$. Hence, if L be the distance from a point on the curve to a circle,

$$L=\sqrt{p(\alpha+v)\cdot p(\alpha-v)-\bar{c}p(\alpha+v)-cp(\alpha-v)+\lambda}=\sqrt{A^2\Pi\frac{\sigma(\alpha+v-v_\nu)}{\sigma^2(\alpha+v)\sigma^2(\alpha-v)}}.$$

If the circle is a bitangent circle,

$$L=\sqrt{A^2\frac{\sigma^2(\alpha+v+v_1)\sigma^2(\alpha+v-v_1)}{\sigma^2(\alpha+v)\sigma^2(\alpha-v)}}=A\frac{\sigma(\alpha+v+v_1)\sigma(\alpha+v-v_1)}{\sigma(\alpha+v)\sigma(\alpha-v)}.$$

Similarly, if L' is the distance from the same point to a second bitangent circle of the same system,

$$L'=A'\frac{\sigma(\alpha+v+v_1')\sigma(\alpha+v-v_1')}{\sigma(\alpha+v)\sigma(\alpha-v)}.$$

Therefore L and L' are elliptic functions of the second order and the same poles, and if r_1 is the residue of L at $v=\alpha$ and r_1' is the residue of L' at the same pole, then

$$r_1'L-r_1L'=k.$$

Therefore, between tangents to any two bitangent circles of the same system of a Cartesian oval there is a linear relation,

$$r_1'L-r_1L'-k=0.$$

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