compute the closure of the range of this ρ with respect to $\{C, \boldsymbol{R}^2, \theta\}$. Let $\{\Gamma_1, \mu_1\}$ and $\{\Gamma_2, \mu_2\}$ be standard measure spaces with $C_1 = L^\infty(\Gamma_1, \mu_1)$ and $C_2 = L^\infty(\Gamma_2, \mu_2)$. Put $\{\Gamma, \mu\} = \{\Gamma, \mu\} = \{\Gamma_1, \mu_1\} \times \{\Gamma_2, \mu_2\}$. Then $C = L^\infty(\Gamma, \mu)$. In order to avoid possible confusion, we denote by $\{\theta_s^{1*}\}$ and $\{\theta_t^{2*}\}$ the flows in Γ_1 and Γ_2 induced by $\{C_1, \theta^1\}$ and $\{C_2, \theta^2\}$. We then have

$$\bar{\theta}^*_{s,t}(\gamma_1, \gamma_2, \lambda) = (\theta_s^{1*}\gamma_1, \theta_t^{2*}\gamma_2, e^{s+t}\lambda),\ s, t \in \boldsymbol{R},\ \lambda \in \boldsymbol{R}_+^*\ ;$$
$$\alpha^*_{\lambda_0}(\gamma_1, \gamma_2, \lambda) = (\gamma_1, \gamma_2, \lambda_0^{-1}\lambda),\ (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2,\ \lambda_0 \in \boldsymbol{R}_+^*\ .$$

Put

$$T(\gamma_1, \gamma_2, \lambda) = (\theta^{1*}_{\operatorname{Log}\lambda}\gamma_1, \gamma_2, \lambda),\ (\gamma_1, \gamma_2, \lambda) \in \Gamma_1 \times \Gamma_2 \times \boldsymbol{R}_+^*\ .$$

We have then

$$T^{-1}\bar{\theta}^*_{s,t}T(\gamma_1, \gamma_2, \lambda) = (\theta^{1*}_{-t}\gamma_1, \theta_t^{2*}\gamma_2, e^{s+t}\lambda)\ ;$$
$$T^{-1}\alpha^*_{\lambda_0}T(\gamma_1, \gamma_2, \lambda) = (\theta^{1*}_{\operatorname{Log}\lambda_0}\gamma_1, \gamma_2, \lambda)\ . \qquad \text{q.e.d.}$$

Therefore, our assertion follows.

CHAPTER III. NON-ABELIAN COHOMOLOGY IN PROPERLY INFINITE VON NEUMANN ALGEBRAS

III.0. **Introduction.** So far we have studied the flow of weights on a factor. As the reader has already noticed, what we have treated there is nothing else but the first cohomology of $\boldsymbol{R}$ in the unitary group of a factor with respect to the modular automorphism group. The techniques developed there can also be applied to the general case, not only to the modular automorphism group. The first cohomology of a locally compact group G in the unitary group $\mathfrak{U}$ of a von Neumann algebra M with respect to an action α of G on M is related to the structure of the crossed product $W^*(M, G, \alpha)$ and its automorphism group. We shall regard a one cocycle in the unitary group as a twisted unitary representation and then follow the well-established multiplicity theory of unitary representations, instead of following the algebraic theory of cohomology. Of course, integrable actions of the group in question will play the role corresponding to that of integrable weights. The result of particular interest is the stability of the single automorphism or of the one parameter automorphism group appearing in the discrete or the continuous decomposition of a factor type III, (see Section 5).

In §1, developing elementary properties of twisted *-representations, we shall lay down our strategic point of view. We shall see in §2 that, as for weights, there exists a unique square integrable twisted unitary representation, called *dominant*, which dominates all other square integrable twisted representations, Theorem 2.12. As a corollary, it will be seen that the fixed point subalgebra of an integrable action is isomorphic to the reduced algebra of the crossed product. Section 3 is devoted to the case of abelian groups. A characterization of a dominant action will be given in terms of the spectrum; and also it will be shown that $\Gamma(\alpha)$, the exterior invariant of α ([3, part II]) is the kernel of the restriction of the dual action $\hat{\alpha}$ to the center of the crossed product $W^*(M, G, \alpha)$, a generalization of [30; Theorem 9.6].

In §4, we shall study the Galois type correspondence between the closed subgroups and the intermediate von Neumann subalgebras for an integrable action of an abelian group. Section 5 is devoted to the study of stability of automorphisms (or one parameter groups of automorphisms) of semi-finite von Neumann algebras.

III.1. **Elementary properties of twisted *-representation.** Let M be a properly infinite von Neumann algebra equipped with a continuous action α of a locally compact group G. We assume the σ-finiteness of M always.

DEFINITION 1.1. *A σ-strong* continuous function $a: s \in G \mapsto a(s) \in M$ is called an α-twisted *-representation of G in M if the following conditions are satisfied:*

$$\begin{cases} a(st) = a(s)\alpha_s(a(t)),\ s,\ t \in G\ ; \\ a(s^{-1}) = \alpha_s^{-1}(a(s)^*)\ . \end{cases}$$

If all $a(s)$ are unitaries, then it is called an α-twisted unitary representation of G in M.

We denote by $Z_\alpha(G, M)$ (resp. $Z_\alpha(G, \mathfrak{U}(M))$) the set of all α-twisted *-representations (resp. unitary representation) of G in M, where $\mathfrak{U}(M)$ denotes the unitary group of M. A straightforward computation gives the following:

LEMMA 1.2. *If $a \in Z_\alpha(G, M)$, then all $a(s)$ are partial isometries such that*

$$a(s)a(s)^* = a(1) \quad and \quad a(s)^*a(s) = \alpha_s(a(1)),\ s \in G\ ,$$

where 1 means, of course, the identity of G.

We denote $a(1)$ by e_a. It is also straightforward to observe that by the formula:

$$_a\alpha_s(x) = a(s)\alpha_s(x)a(s)^*\ , \quad x \in M_{e_a}\ , \quad s \in G\ ,$$

we can define a new action $_a\alpha$ of G on the reduced von Neumann algebra $M_{e_a}(=e_aMe_a)$. We denote the fixed point subalgebra of M_{e_a} under this new action $_a\alpha$ by M^a. If p is a projection in M^a, then the map: $s \in G \mapsto$

$pa(s) \in M$ is also an α-twisted *-representation of G in M, which will be called the *reduced* α-twisted *-representation by p and denoted by a^p. We call it also a *subrepresentation* of a.

DEFINITION 1.3. We say that a and b in $Z_\alpha(G, M)$ are equivalent and write $a \cong b$ if there exists an element $c \in M$ such that

$$\begin{cases} a(s) = c^* b(s)\alpha_s(c) , & s \in G ; \\ b(s) = ca(s)\alpha_s(c^*) . \end{cases}$$

We write $a \prec b$ if $a \cong b^q$ for some projection q in M^b.

The reader should be aware of the following 2×2-matrix arguments:

LEMMA 1.4. *Let $P = M \otimes F_2$ be the 2×2-matrix algebra over M, and $\bar{\alpha}$ be the action $\alpha \otimes 1$ of G on P. Given $a, b \in Z_\alpha(G, M)$, we define $c \in Z_{\bar{\alpha}}(G, P)$ by*

$$c(s) = a(s) \otimes e_{11} + b(s) \otimes e_{22} , \qquad s \in G ,$$

with a fixed matrix unit $\{e_{ij}\}$ in F_2. Then the following two statements are equivalent:

(i) $\qquad a \prec b \quad$ (*resp.* $a \cong b$) :

(ii) $\quad e_a \otimes e_{11} \leqq e_e \otimes e_{22} \quad$ (*resp.* $e_a \otimes e_{11} \sim e_b \otimes e_{22}$) *in* P^c .

We leave the proof to the reader.

DEFINITION 1.5. With the same notations as in Lemma 1.4, we call a and b disjoint and write $a \mathbin{\overset{\mid}{\circ}} b$ if $e_a \otimes e_{11}$ and $e_b \otimes e_{22}$ are centrally orthogonal in P^c. We say that a and b are quasi-equivalent and write $a \sim b$ if $e_a \otimes e_{11}$ and $e_b \otimes e_{22}$ have the same central support, (namely $e_a \otimes e_{11} + e_b \otimes e_{22}$), in P^c.

Given a and b in $Z_\alpha(G, M)$, we set

$$I(a, b) = \{x \in e_a M e_b \colon xb(s) = a(s)\alpha_s(x),\ s \in G\} .$$

It is not hard to see the following properties of $I(a, b)$:

$$I(b, a) = I(a, b)^* ; \quad I(a, a) = M^a ; \quad I(b, b) = M^b ;$$

$$x = \sum_{i,j=1}^{2} x_{i,j} \otimes e_{i,j} \in P^c \Leftrightarrow \begin{cases} x_{11} \in I(a, a) , & x_{12} \in I(a, b) , \\ x_{21} \in I(b, a) , & x_{22} \in I(b, b) ; \end{cases}$$

$$a \mathbin{\overset{\mid}{\circ}} b \Leftrightarrow I(a, b) = \{0\} .$$

LEMMA 1.6. (i) *Given a, b and, c in $Z_\alpha(G, M)$, we have*

$$I(a, b)I(b, c) \subset I(a, c) .$$

(ii) *If $x = uh$ is the polar decomposition of $x \in I(a, b)$, then we have*

$$h \in I(b, b) \quad \textit{and} \quad u \in I(a, b)\ .$$

The proof is straightforward, so we leave it to the reader.

DEFINITION 1.7. We say that $a \in Z_\alpha(G, M)$ is of infinite multiplicity if M^a is properly infinite.

LEMMA 1.8. *If a and b in $Z_\alpha(G, M)$ are of infinite multiplicity, then*

$$a \cong b \Leftrightarrow a \sim b\ .$$

PROOF. The implication "$\Rightarrow$" is trivial.

$\Leftarrow$: Suppose $a \sim b$. Let $P = M \otimes F_2$, $\bar{\alpha}$ and $c \in Z_{\bar{\alpha}}(G, M)$ be as in Lemma 1.4. It follows then that $e_a \otimes e_{11}$ and $e_b \otimes e_{22}$ are both properly infinite projections in P^c by assumption; so they are equivalent to their central support in P^c, P being σ-finite. Therefore, we have

$$e_a \otimes e_{11} \sim e_a \otimes e_{11} + e_b \otimes e_{22} \sim e_b \otimes e_{22} \quad \text{in} \quad P^c\ . \qquad \text{q.e.d.}$$

We close this section with the following:

REMARK 1.9. If α is a continuous action of a separable locally compact group G on a von Neumann algebra M with separable predual, then for an M-valued function $a: s \in G \longrightarrow a(s) \in M$ to agree almost everywhere with an α-twisted *-representation a' of G in M, it is sufficient that a satisfies the conditions in Definition 1.1 for almost every pair s, t in G, cf [18].

III.2. **Tensor product and integrability of twisted *-representations.** Let M and N be von Neumann algebras equipped with continuous actions α and β of a locally compact group G respectively. We understand naturally the covariant system $\{M \otimes N, \alpha \otimes \beta\}$ on G. Given $a \in Z_\alpha(G, M)$ and $b \in Z_\beta(G, N)$, we define $a \otimes b \in Z_{\alpha\otimes\beta}(G, M \otimes N)$ by

$$(a \otimes b)(s) = a(s) \otimes b(s)\ , \qquad s \in G\ .$$

It is of our particular interest when $N = \mathfrak{L}(\mathfrak{K})$ and $\beta = 1$. This means that b is an ordinary unitary representation of G of the Hilbert space $\mathfrak{K}$.

THEOREM 2.1. *Let M be a von Neumann algebra equipped with a continuous action α of a locally compact group G. Put $P = M \otimes \mathfrak{L}(L^2(G))$. If λ_r is the right regular representation of G on $L^2(G)$, then $1 \otimes \lambda_r \in Z_{\alpha\otimes 1}(G, P)$ and*

$$W^*(M, G, \alpha) \cong P^{(1\otimes\lambda_r)}\ .$$

PROOF. We may assume that M acts on a Hilbert space $\mathfrak{H}$ in such a way that $\{M, \mathfrak{H}\}$ is standard, so that there exists canonically a unitary representation U of G on $\mathfrak{H}$ such that $\alpha_s(x) = U(s)xU(s)^*$, $x \in M$, $s \in G$. The crossed product $W^*(M, G, \alpha)$ of M by α acts on the Hilbert space $\mathfrak{H} \otimes L^2(G)$. In this situation, the recent result of Digernes, [8], says that the commutant $W^*(M, G, \alpha)'$ of $W^*(M, G, \alpha)$ is generated by $M' \otimes 1$ and $U(s) \otimes \lambda_r(s)$, $s \in G$.

Hence we have

$$\begin{aligned} W^*(M, G, \alpha) &= W^*(M, G, \alpha)'' = \{M' \otimes 1 \cup \{U(s) \otimes \lambda_r(s) : s \in G\}\}' \\ &= M \otimes \mathfrak{L}(L^2(G)) \cap \{U(s) \otimes \lambda_r(s) : s \in G\}' \\ &= P^{(1 \otimes \lambda_r)} . \end{aligned}$$ q.e.d.

Since the left and right regular representations of G are equivalent in $\mathfrak{L}(L^2(G))$ as twisted unitary representation with respect to the trivial action of G on $\mathfrak{L}(L^2(G))$, we have also

$$P^{(1 \otimes \lambda_l)} \cong W^*(M, G, \alpha)$$

with the left regular representation λ_l of G.

The next proposition is classical in homological algebra.

PROPOSITION 2.2. *For any $a \in Z_\alpha(G, \mathfrak{U}(M))$, we have*

$$a \otimes \lambda_r \cong 1 \otimes \lambda_r \quad \text{in} \quad P = M \otimes \mathfrak{L}(L^2(G)) .$$

PROOF. Suppose that M acts on a Hilbert space $\mathfrak{H}$. Then P acts on $\mathfrak{H} \otimes L^2(G) = L^2(\mathfrak{H}; G)$. We define a unitary b in $M \otimes L^\infty(G) \subset P$ by the following:

$$(b\xi)(s) = a(s^{-1})\xi(s),\ \xi \in L^2(\mathfrak{H}; G),\ s \in G .$$

We compute then

$$[b(1 \otimes \lambda_r(t))\xi](s) = a(s^{-1})\xi(st) ;$$
$$\{[a(t) \otimes \lambda_r(t)]\alpha \otimes 1)_t(b)\xi\}(s) = a(t)\alpha_t(a((st)^{-1}))\xi(st) ,$$

where we use the right invariant Haar measure $d_r s$ in the construction of $L^2(\mathfrak{H}; G)$. We compute further the last term:

$$\begin{aligned} a(t)\alpha_t(a((st)^{-1})) &= a(t)\alpha_t(a(t^{-1}s^{-1})) = a(t)\alpha_t(a(t^{-1})\alpha_t^{-1}(a(s^{-1})) \\ &= a(t)\alpha_t(a(t^{-1}))a(s^{-1}) = a(s^{-1}) . \end{aligned}$$

Hence we get

$$b(1 \otimes \lambda_r(t)) = [a(t) \otimes \lambda_r(t)](\alpha \otimes 1)_t(b) , \qquad t \in G .$$

Therefore, our assertion follows, since b is unitary. q.e.d.

DEFINITION 2.3. Given a σ-finite properly infinite von Neumann algebra M equipped with a continuous action α of a separable locally compact group G, an α-twisted unitary representation a of G in M is said to be dominant if $a \otimes \lambda_r \cong a \otimes 1$ in $M \otimes \mathfrak{L}(L^2(G))$ and a is of infinite multiplicity.

From now on, we assume always that the von Neumann algebras and the groups in question are σ-finite and separable respectively.

COROLLARY 2.4. *Any dominant α-twisted unitary representations are equivalent.*

PROOF. Let a and b be dominant α-twisted unitary representations of G in M. By Theorem 2.2, we have

$$a \otimes 1 \cong a \otimes \lambda_r \cong 1 \otimes \lambda_r \cong b \otimes \lambda_r \simeq b \otimes 1$$

in $M \otimes \mathfrak{L}(L^2(G))$. Therefore, we have only to show that if a and b in $Z_\alpha(G, \mathfrak{U}(M))$ are of infinite multiplicity, then $a \otimes 1 \cong b \otimes 1$ in $M \otimes F_\infty$ implies $a \cong b$ in M with F_∞ a factor of type I_∞. But $a \otimes 1 \cong b \otimes 1$ in $M \otimes F_\infty$ means that $a \sim b$; hence $a \cong b$ by Lemma 1.8. q.e.d.

COROLLARY 2.5. *If $a \in Z_\alpha(G, \mathfrak{U}(M))$ is dominant, then*

$$M^a \cong W^*(M, G, \alpha) .$$

DEFINITION 2.6. A continuous action α of G on M is said to be integrable if the set $\mathfrak{q}_\alpha$ of all x in M such that the integral $\int_G \alpha_s(x^*x)d_ls$ exists in M with respect to the left invariant Haar measure d_ls in G, is σ-weakly dense in M. We say that $a \in Z_\alpha(G, M)$ is square integrable if the action ${}_a\alpha$ of G on M_{e_a} is integrable.

We note here that the integral $\int_G \alpha_s(x^*x)d_ls$ is defined as the limit of the increasing net $\int_K \alpha_s(x^*x)d_ls$ indexed by the net of compact subsets K of G. The very much similar arguments as those in the case of weights show that

a) $\mathfrak{q}_\alpha$ is a left ideal of M;

b) $\mathfrak{p}_\alpha = \mathfrak{q}_\alpha^*\mathfrak{q}_\alpha = \{y^*x : x, y \in \mathfrak{q}_\alpha\}$ is a hereditary *-subalgebra of M generated linearly by the positive part $\mathfrak{p}_\alpha^+ = \mathfrak{p}_\alpha \cap M_+$;

c) $\mathfrak{p}_\alpha^+ = \left\{x \in M_+ : \int_G \alpha_s(x)d_ls \text{ exists}\right\}$;

d) The integral

$$E_\alpha(x) = \int_G \alpha_s(x)d_ls$$

makes sense for any $x \in \mathfrak{p}_\alpha$.

The following further properties of E_α are easily verified:

e) $E_\alpha(x)$ lies in the fix point algebra M^α;

f) $E_\alpha(uxv) = uE_\alpha(x)v$, $x \in \mathfrak{p}_\alpha$, $u, v \in M^\alpha$;

g) $E_\alpha(x^*x) \geqq 0$ and $E_\alpha(x^*x) = 0 \Rightarrow x = 0$;

h) $E_\alpha(\sup x_i) = \sup E_\alpha(x_i)$ for any increasing bounded net $\{x_i\}$ in M_+, where $E_\alpha(x) = +\infty$ if $x \in M_+$ is not in $\mathfrak{p}_\alpha^+$, and $\sup y_i = +\infty$ if $\{y_i\}$ is not bounded in M^α.

From property (f), we conclude immediately the following:

LEMMA 2.7. *Any subrepresentation of a square integrable α-twisted *-representation of G in M is also square integrable.*

EXAMPLE 2.8. Let $M = \mathfrak{L}(\mathfrak{H})$ and $\alpha = 1$. For a unitary representation $\{U, \mathfrak{H}\}$ of G on $\mathfrak{H}$, U is square integrable as a twisted unitary representation with respect to the trivial action α in the sense of Definition 2.6 if and only if $\{U, \mathfrak{H}\}$ is square integrable in the sense that

$$\int_G |(U(s)\xi \,|\, \xi)|^2 d_l s < +\infty$$

for a dense set of ξ in $\mathfrak{H}$.

EXAMPLE 2.9. Let $M = L^\infty(G)$ and α be the translation action of G from the right. It is immediately seen that $\mathfrak{p}_\alpha = L^\infty(G) \cap L^1(G, d_l s)$ and

$$E_\alpha(f) = \int_G f(s) d_l s .$$

LEMMA 2.10. *Let M and N be von Neumann algebras equipped with continuous actions α and β of G respectively. If either α or β is integrable, then the tensor product $\alpha \otimes \beta$ on $M \otimes N$ is integrable.* q.e.d.

We leave the proof to the reader.

LEMMA 2.11. *The regular representation of G is square integrable in $Z_1(G, \mathfrak{L}(L^2(G)))$.*

PROOF. Let λ_r be the right regular representation of G on $L^2(G)$. Let $\alpha_s = \mathrm{Ad}\,(\lambda_r(s))$, $s \in G$. It follows that the action α leaves the maximal abelian algebra $L^\infty(G) = \mathfrak{A}$ globally invariant and $\alpha|_{\mathfrak{A}}$ is the right translation action of G on $\mathfrak{A}$. Hence $\mathfrak{p}_\alpha \cap \mathfrak{A} = L^\infty(G) \cap L^1(G, d_l s)$, which contains a net converging σ-strongly to 1. Therefore, $\mathfrak{p}_\alpha$, hence $\mathfrak{q}_\alpha$, is σ-weakly dense in $\mathfrak{L}(L^2(G))$, which means that λ_r is square integrable in

$$Z_1(G, \mathfrak{L}(L^2(G))) . \qquad \text{q.e.d.}$$

THEOREM 2.12. *Let M be a σ-finite properly infinite von Neumann algebra equipped with a continuous action α of a separable locally compact group G.*

(i) *There exists a dominant α-twisted unitary representation a of G in M, which is unique up to equivalence.*

(ii) *An α-twisted *-representation b of G in M is square integrable if and only if $b \prec a$.*

PROOF. Since M is properly infinite, replacing α by ${}_a\alpha$, we may assume that M^α is properly infinite. Choosing a factor F_∞ of type I_∞ contained in M^α, we may identify $\{M, \alpha\}$ with a covariant system $\{N \otimes F_\infty, \beta \otimes 1\}$ on G. Identifying once again F_∞ with the tensor product $\mathfrak{L}(L^2(G)) \otimes B$ of $\mathfrak{L}(L^2(G))$ and a factor B of type I_∞, we can consider a $(\beta \otimes 1)$-twisted unitary representation $1 \otimes \lambda_r \otimes 1$ of G in $N \otimes \mathfrak{L}(L^2(G)) \otimes B = M$. We have then

$$M^{(1\otimes\lambda_r\otimes 1)} \supseteqq N^\beta \otimes \lambda_r(G)' \otimes B .$$

Hence $1 \otimes \lambda_r \otimes 1$ is of infinite multiplicity. Therefore, $1 \otimes \lambda_r \otimes 1$ is dominant.

For the second assertion, we need the following results:

LEMMA 2.13. *If $b \in Z_\alpha(G, M)$ is square integrable, then*

$$\bigvee \{\operatorname{supp} x^*x \colon x \in I(b \otimes \lambda_r, b \otimes 1)\} = e_b \otimes 1 \quad \textit{in} \quad M \otimes \mathfrak{L}(L^2(G)) .$$

PROOF. Let e denote the left hand side of the equality. By Lemma 1.6, e belongs to $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)}$. For any unitary $u \in [M \otimes \mathfrak{L}^2(G))]^{(b\otimes 1)}$, we have $I(b \otimes \lambda_r, b \otimes 1)u = I(b \otimes \lambda_r, b \otimes 1)$; hence $u^*eu = e$, so that e is a central projection in $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)}$. Since $I(b \otimes \lambda_r, b \otimes 1)e = I(b \otimes \lambda_r, b \otimes 1)$, we have only to show

$$I(b \otimes \lambda_r, b \otimes 1)f \neq \{0\}$$

for any non-zero central projection f in $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)}$. Since $[M \otimes \mathfrak{L}(L^2(G))]^{(b\otimes 1)} = M^b \otimes \mathfrak{L}(L^2(G))$, f is of the form $p \otimes 1$ with a central projection p in M^b. We consider now M on a Hilbert space $\mathfrak{H}$ and $L^2(G)$ with respect to the right Haar measure d_rs on G. We note, however, that $d_rs^{-1} = d_ls$. Then $M \otimes \mathfrak{L}(L^2(G))$ acts on $L^2(\mathfrak{H}; G)$. Choose an $x \in \mathfrak{p}_{{}_b\alpha}$ with $xp = x \neq 0$ and a continuous function f on G with compact support. Put

$$(y\xi)(s) = {}_b\alpha_s^{-1}(x) \int_G f(t)\xi(t)d_rt,\ \xi \in L^2(\mathfrak{H}; G) .$$

We have then

$$\begin{aligned}
\|y\xi\|^2 &= \int_G \left\| {}_b\alpha_s^{-1}(x)\left(\int_G f(t)\xi(t)d_rt\right)\right\|^2 d_rs \\
&\leqq \int_G \int_G \| f(t){}_b\alpha_s^{-1}(x)\xi(t)\|^2 d_rt d_rs \\
&= \int_G |f(t)|^2 \left(\int_G \|{}_b\alpha_s^{-1}(x)\xi(t)\|^2 d_rs\right) d_rt \\
&= \int_G |f(t)|^2 \left(\int_G \|{}_b\alpha_s(x)\xi(t)\|^2 d_ls\right) d_rt
\end{aligned}$$

$$= \int_G |f(t)|^2 (E_{b^\alpha}(x^*x)\xi(t) | \xi(t)) d_r t$$
$$= \int_G |f(t)|^2 \| E_{b^\alpha}(x^*x)^{1/2}\xi(t) \|^2 d_r t$$
$$\leqq \|f\|_\infty^2 \| E_{b^\alpha}(x^*x)\| \, \|\xi\|^2 .$$

Hence y is bounded; so $y \in M \otimes \mathfrak{L}(L^2(G))$. Furthermore, we have, for any $\xi \in L^2(\mathfrak{H}; G)$ and $r, s \in G$,

$$[y(pb(r) \otimes 1)\xi](s) = {}_b\alpha_s^{-1}(x)pb(r) \int_G f(t)\xi(t)d_r t$$
$$= {}_b\alpha_s^{-1}(x)b(r) \int_G f(t)\xi(t)d_r t \, ;$$
$$\{[b(r) \otimes \lambda_r(s)][\alpha_s \otimes 1](y)\xi\}(s) = b(r)[(\alpha_r \otimes 1)(y)\xi](sr)$$
$$= b(r)\alpha_r[{}_b\alpha_{sr}^{-1}(x)] \int_G f(t)\xi(t)d_r t$$
$$= {}_b\alpha_r \circ {}_b\alpha_{sr}^{-1}(x) b(r) \int_G f(t)\xi(t)d_r t$$
$$= {}_b\alpha_s^{-1}(x)b(r) \int_G f(t)\xi(t)d_r t \, .$$

Hence y belongs to $I(b \otimes \lambda_r, b \otimes 1)$ and $y(p \otimes 1) = y$. Clearly $y \neq 0$ if $f \neq 0$. q.e.d.

LEMMA 2.14. *For any $b \in Z_\alpha(G, M)$, there exists $\check{b} \in Z_\alpha(G, \mathfrak{U}(M))$ with infinite multiplicity such that $b \prec \check{b}$. If b is square integrable, then we can chose a square integrable $\check{b}$.*

PROOF. Let $e = e_b$ and z be the central support of e in the whole algebra M. Since $\alpha_s(z)$ is the central support of $\alpha_s(e) = b(s)^*b(s)$, $s \in G$, we have $\alpha_s(z) = z$. Therefore, we have $\{M, \alpha\} = \{M_z, \alpha\} \oplus \{M_{1-z}, \alpha\}$ in the obvious sense. It follows from Theorem 2.12 (i) that there exists a dominant $b_2 \in Z_\alpha(M_{1-z}, \mathfrak{U}(M_{1-z}))$. We then restrict our attention to $\{M_z, \alpha\}$. Let $\{e_n\}$ and $\{u_n\}$ be families of orthogonal projections and partial isometries in M respectively such that $\sum_{n=1}^\infty e_n = z$, $u_n^*u_n = e$ and $u_n u_n^* = e_n$, $n = 1, 2, \cdots$, where the existence of such families is guaranteed by the proper infiniteness and the σ-finiteness of M. Put

$$b_1(s) = \sum_{n=1}^\infty u_n b(s)\alpha_s(u_n^*) .$$

It follows that for any $s, t \in G$,

$$b_1(s)\alpha_s(b_1(t)) = \left[\sum_{n=1}^\infty u_n b(s)\alpha_s(u_n^*)\right]\left[\sum_{m=1}^\infty \alpha_s(u_m)\alpha_s(b(t))\alpha_{st}(u_m^*)\right]$$

$$\begin{aligned}
&= \sum_{n,m=1}^{\infty} u_n b(s)\alpha_s(u_n^* u_m)\alpha_s(b(t))\alpha_{st}(u_m^*) \\
&= \sum_{n=1}^{\infty} u_n b(s)\alpha_s(eb(t))\alpha_{st}(u_n^*) \\
&= \sum_{n=1}^{\infty} u_n b(s)\alpha_s(b(t))\alpha_{st}(u_n^*) \\
&= b_1(st) \ ; \\
b_1(s^{-1}) &= \sum_{n=1}^{\infty} u_n b(s^{-1})\alpha_s^{-1}(u_n^*) = \sum_{n=1}^{\infty} u_n \alpha_s^{-1}(b(s)^*)\alpha_s^{-1}(u_n^*) \\
&= \alpha_s^{-1}\Big(\sum_{n=1}^{\infty} \alpha_s(u_n) b(s)^* u_n^*\Big) = \alpha_s^{-1}(b_1(s)^*) \ ; \\
b_1(1) &= \sum_{n=1}^{\infty} u_n b(1) u_n^* = \sum_{n=1}^{\infty} u_n e u_n^* = \sum_{n=1}^{\infty} e_n = z \ .
\end{aligned}$$

Since the map: $s \in G \longrightarrow b_1(s) \in M$ is σ-strongly continuous, b_1 is an α-twisted unitary representation of G in M_z. Put

$$\check{b}(s) = b_1(s) + b_2(s) \ .$$

It follows that $M^{\check{b}} = (M_z)^{b_1} + (M_{1-z})^{b_2}$. By the definition of a dominant representation, $(M_{1-z})^{b_2}$ is properly infinite. We will show that $(M_z)^{b_1}$ is properly infinite. Put $w_{n,m} = u_n u_m^*$, n, $m = 1.2, \cdots$. It follows that

$$w_{n,m}^* w_{n,m} = e_m \quad \text{and} \quad w_{n,m} w_{n,m}^* = e_n$$

$$\begin{aligned}
b_1(s)\alpha_s(w_{n,m})b_1(s)^* &= \Big(\sum_{j=1}^{\infty} u_j b(s)\alpha_s(u_j^*)\Big)\alpha_s(w_{n,m})\Big(\sum_{k=1}^{\infty} u_k b(s)\alpha_s(u_k^*)\Big)^* \\
&= \sum_{j,k=1}^{\infty} (u_j b(s)\alpha_s(u_j^* w_{n,m} u_k) b(s)^* u_k^*) \\
&= u_n b(s)\alpha_s(u_n^* w_{n,m} u_m) b(s)^* u_m^* \\
&= u_n b(s)\alpha_s(e) b(s)^* u_m^* = u_n e u_m^* = w_{n,m} \ .
\end{aligned}$$

Hence $w_{n,m} \in (M_z)^{b_1}$; so b_1 is of infinite multiplicity. By construction, $b \prec b_1$; hence $b \prec \check{b}$.

Suppose now b is square integrable. Since b_2 is square integrable by definition, we need only to show that b_1 is square integrable. Let $\{x_i\}$ be a net in $\mathfrak{p}_{b^\alpha}$ such that $\lim_i x_i = e$. Let $x_{i,n} = u_n x_i u_n^*$. We have then

$$b_1(s)\alpha_s(x_{i,n})b_1(s)^* = u_n b(s)\alpha_s(x_i)b(s)u_n^* \ ;$$

hence $x_{i,n} \in \mathfrak{p}_{b_1^\alpha}$. Since $\lim_i x_{i,n} = e_n$, the σ-strong closure $\mathfrak{p}_{b_1^\alpha}$ contains all e_n's; hence b_1^α is integrable. Thus, b_1 is square integrable, and so is $\check{b}$.

q.e.d.

PROOF OF THEOREM 2.12. (ii). By Lemma 2.14, we may assume that

b is a square integrable α-twisted unitary representation of G in M with infinite multiplicity. Consider $M \otimes \mathfrak{L}(L^2(G))$, $b \otimes \lambda_r$ and $b \otimes 1$ as well as $P = M \otimes \mathfrak{L}(L^2(G)) \otimes F_2$. Let

$$c(s) = b(s) \otimes \lambda_r(s) \otimes e_{11} + b(s) \otimes 1 \otimes e_{22} .$$

It follows from Lemma 2.13 that the central support of $1 \otimes 1 \otimes e_{11}$ in P^c majorizes $1 \otimes 1 \otimes e_{22}$. Since $M^b \otimes 1 \otimes e_{11}$ is contained in $P^c_{(1 \otimes 1 \otimes e_{11})}$, $1 \otimes 1 \otimes e_{11}$ is properly infinite in P^c because M^b is. Hence $1 \otimes 1 \otimes e_{11} \succ 1 \otimes 1 \otimes e_{22}$ in P^c; so $b \otimes 1 \prec b \otimes \lambda_r$. By Proposition 2.2, we have

$$b \otimes 1 \prec b \otimes \lambda_r \cong 1 \otimes \lambda_r \cong a \otimes 1$$

if $a \in Z_\alpha(G, \mathfrak{U}(M))$ is dominant. Thus $b \prec a$ because a is of infinite multiplicity. q.e.d.

COROLLARY 2.15. *Let M be a σ-finite von Neumann algebra and G a separable locally compact group. If α is an integrable action of G on M, then the fixed point algebra M^α of M under α is isomorphic to a reduced algebra of the crossed product $W^*(M, G, \alpha)$.*

PROOF. Seeing that $\alpha \otimes 1$ is integrable on $M \otimes F_\infty$ with a factor F_∞ of type I_∞, and that $(M \otimes F_\infty)^{\alpha \otimes 1} = M^\alpha \otimes F_\infty$, we may assume that M^α is properly infinite. Let $b(s) = 1$, $s \in G$, and a be a dominant α-twisted unitary representation of G in M. By Theorem 2.12, $b \prec a$, that is, there exists an isometry u in M such that $u^*u = 1$, $uu^* \in M^a$ and $u^*a(s)\alpha_s(u) = 1$, $s \in G$. Let $e = uu^*$. It follows that $\alpha_r(x) = x$ if and only if ${}_a\alpha_s(uxu^*) = uxu^*$. Hence $M^\alpha \cong M^a_e$. On the other hand, we have $M^a \cong W^*(M, G, \alpha)$ by Corollary 2.5. q.e.d.

COROLLARY 2.16. *Let M be a σ-finite von Neumann algebra and G a finite group. If α is a free action of G on M in the sense that $\alpha_g(x)a = ax$ for every $x \in M$ implies either $g = e$ or $a = 0$, then any pair of α-twisted representations of G in M are equivalent*; i.e., *the equivalence classes in $Z^1_\alpha(G, \mathfrak{U})$ reduces to a singleton.*

PROOF. The discreteness and the free action of G yield, [21], that the relative commutant of M in $W^*(M, G, \alpha)$ is $M^\alpha \cap C$, where C denotes the center of M. This means that if M is properly infinite then every $a \in Z^1_\alpha(G, \mathfrak{U})$ is quasi-equivalent to a dominant one by Theorem 2.12. The finiteness of G implies that M is properly infinite if and only if M^α is also. Hence any $a \in Z^1_\alpha(G, \mathfrak{U})$ is dominant if M is properly infinite.

Suppose M is finite. Considering $M \otimes F_\infty$ and $\alpha \otimes \iota$, we conclude from the above arguments that $M^\alpha \cap C$ is the center of M^α. Hence the uniqueness of the center valued trace in a finite von Neumann algebra implies that the restriction of the center valued trace of M to M^α is indeed the center valued trace of M^α, which means that for any projections $e, f \in M^\alpha e \sim f$ in M if and only if $e \sim f$ in M^α. Thus our assertions follows from the well exposed 2×2 matrix arguments. q.e.d.

DEFINITION 2.17. A continuous action α of a locally compact group G on a von Neumann algebra M is said to be stable if for every $a \in Z^1_\alpha(G, \mathfrak{U}_M)$ there exists $b \in \mathfrak{U}_M$ such that $a_g = b*\alpha_g(b)$. A single automorphism α of M is said to be stable if every $u \in \mathfrak{U}_M$ is of the form $u = v^*\alpha(v)$ for some $v \in \mathfrak{U}_M$.

Of course, the stability of an automorphism α of M implies that any automorphism β of the form $\mathrm{Ad}\,(u)\cdot\alpha$ (and in particular any β with $\|\alpha - \beta\| < 2$, [11]) is conjugate to α under $\mathrm{Int}\,(M)$. The converse is also true when M is an infinite factor, (cf. Theorem 3.1).

We will discuss further the stability of a single automorphism and a one parameter automorphism group together with its application in Section 5.

III.3. **Integrable action of abelian groups, duality and invariant** Γ. In this section, we study integrable actions of an abelian group. Let G be a separable locally compact abelian group with dual group $\hat{G}$. We choose Haar measures ds in G and $d\gamma$ in $\hat{G}$ so that the Plancherel formula holds. We denote by $\langle s, \gamma\rangle$ the value of $\gamma \in \hat{G}$ at $s \in G$. An action α of G on M is by definition *dominant* if the trivial α-twisted unitary represention 1 of G in M is dominant.

THEOREM 3.1. *Let M be a properly infinite von Neumann algebra with separable M_*. For a continuous action α of a separable locally compact abelian group G on M with properly infinite M^α, the following conditions are equivalent:*

(i) *α is dominant;*

(ii) *For any $\gamma \in \hat{G}$, there exists $u \in \mathfrak{U}(M)$ such that $\alpha_s(u) = \langle s, \gamma\rangle u$, $s \in G$;*

(iii) *There exists a continuous action β of $\hat{G}$ on M^α such that*

$$\{W^*(M^\alpha, \hat{G}, \beta), \hat{\beta}\} \cong \{M, \alpha\} .$$

PROOF. (i) $\Rightarrow$ (ii): Since M^α is properly infinite,

$$\{M, \alpha\} \cong \{M \otimes \mathfrak{L}^2(L(G)), \alpha \otimes 1\} .$$

Denoting the regular representation of G on $L^2(G)$ by λ, we have

$$\{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes 1\} \cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\lambda\} .$$

For each $\gamma \in \hat{G}$, let $\mu(\gamma)$ denote the unitary on $L^2(G)$ given by

$$\mu(\gamma)\xi(s) = \overline{\langle s, \gamma \rangle}\xi(s), \ \xi \in L^2(G), \ s \in G .$$

It follows then that

$$\mathrm{Ad}\,(\lambda(s))\mu(\gamma) = \langle s, \gamma \rangle \mu(\gamma) .$$

Hence, putting $u(\gamma) = 1 \otimes \mu(\gamma)$, we have

$$\{\alpha_s \otimes \mathrm{Ad}\,(\lambda(s))\}(u(\gamma)) = \langle s, \gamma \rangle u(\gamma) .$$

Thus, the isomorphism $\{M, \alpha\} \cong \{M \otimes \mathfrak{L}^2(L(G)), \alpha \otimes \mathrm{Ad}\,\lambda\}$ assures the existence of a unitary $u \in M$ with $\alpha_s(u) = \langle s, \gamma \rangle u$.

(ii) $\Rightarrow$ (i): Suppose that for any $\gamma \in \hat{G}$, there exists a unitary $u \in M$ with $\alpha_s(u) = \langle s, \gamma \rangle u$ for any $s \in G$. Put

$$E = \{(\gamma, u) \in \hat{G} \times \mathfrak{U}(M) : \alpha_s(u) = \langle s, \gamma \rangle u, \ s \in G\} .$$

It follows then that E is a closed subset of the polish space $\hat{G} \times \mathfrak{U}(M)$ whose projection to the first coordinate $\hat{G}$ covers the whole dual group $\hat{G}$. Therefore, there exists a $\mathfrak{U}(M)$-valued measurable function $u(\cdot)$ on $\hat{G}$ such that $\alpha_s(u(\gamma)) = \langle s, \gamma \rangle u(\gamma)$. Put

$$u = \int_{\hat{G}}^{\oplus} u(\gamma) d\gamma \in M \otimes L^\infty(\hat{G}) \subset M \otimes \mathfrak{L}(L^2(G)) .$$

Since $\lambda(s) \in L^\infty(\hat{G})$ such that $\lambda(s)(\gamma) = \langle s, \gamma \rangle$, we have

$$1 \otimes \lambda(s) = \int_{G}^{\oplus} \langle s, \gamma \rangle d_\gamma \in M \otimes L^\infty(\hat{G}) .$$

Hence we have

$$\begin{aligned} u^*(\alpha_s \otimes 1)(u) &= \int_{\hat{G}}^{\oplus} u(\gamma)^* \alpha_s(u(\gamma)) d\gamma = \int_{G}^{\oplus} \langle s, \gamma \rangle d\gamma \\ &= 1 \otimes \lambda(s) , \qquad s \in G . \end{aligned}$$

Therefore we have $1 \otimes 1 \cong 1 \otimes \lambda$ in $Z_{\alpha \otimes 1}(G, \mathfrak{U}(M \otimes \mathfrak{L}(L^2(G)))$. Thus, we get

$$\begin{aligned} \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\lambda\} &\cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes 1\} \\ &\cong \{M, \alpha\} , \end{aligned}$$

since M^α is properly infinite.

(iii) $\Rightarrow$ (ii): This follows from the definition of the dual action β.

(i) $\Rightarrow$ (iii): If α is dominant, then we have, by [30; Theorem 4.6],

$$\begin{aligned} \{M, \alpha\} &\cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\lambda\} \cong \{M \otimes \mathfrak{L}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\lambda^*\} \\ &\cong \{M \otimes \mathfrak{L}(L^2(G)), \hat{\hat{\alpha}}\} . \end{aligned}$$

Identifying α with $\hat{\hat{\alpha}}$, the action $\hat{\hat{\alpha}} = \beta$ is the desired action of $\hat{G}$ on M^α. q.e.d.

As in [3; Definition 2.2.1], we define the invariant $\Gamma(\alpha)$ of α as follows:

$$\Gamma(\alpha) = \bigcap \{\operatorname{Sp} \alpha^e : e \text{ runs through all non-zero projections in } M^\alpha\} .$$

We note here that the arguments for [3; Proposition 2.2.2. and Theorem 2.2.4 (c)] do not require the fact that M is a factor. Hence we have

$$\Gamma(\alpha) = \bigcap \{\operatorname{Sp} \alpha^e : e \text{ runs through all non-zero central projections in } M^\alpha\} .$$

THEOREM 3.2. *Let M be a σ-finite von Neumann algebra equipped with a continuous action α of a separable locally compact abelian group G. The invariant $\Gamma(\alpha)$ is the kernel of the restriction of the dual action $\hat{\alpha}$ of $\hat{G}$ on $W^*(M, G, \alpha)$ to the center of $W^*(M, G, \alpha)$. (Hence it is, in particular, a closed subgroup of $\hat{G}$.)*

PROOF. We consider $M \otimes \mathfrak{L}(L^2(G))$, $\alpha \otimes 1$ and $\alpha \otimes \operatorname{Ad} \lambda$ as before. Trivially, we have $\Gamma(\alpha) = \Gamma(\alpha \otimes 1)$; hence $\Gamma(\alpha) = \Gamma(\alpha \otimes \operatorname{Ad} \lambda)$ by [3, 2.2.4]. Hence we may assume that M is properly infinite and α is dominant. It follows from the previous section that there exists a continuous action θ of the dual group $\hat{G}$ on M^α such that

$$\{M, \alpha\} \cong \{W^*(M^\alpha, \hat{G}, \theta), \hat{\theta}\} ;$$
$$\{M^\alpha, \theta\} \cong \{W^*(M, G, \alpha), \hat{\alpha}\}$$

by [30; Theorems 4.5 and 4.6], where $\hat{\alpha}$ and $\hat{\theta}$ mean the dual action of α and θ in the sense of [30; Definition 4.1]. Representing M^α on a Hilbert space $\mathfrak{H}$, we see that M acting on $L^2(\mathfrak{H}; \hat{G})$ is generated by the operators:

$$\pi^\theta(x)\xi(\gamma) = \theta_\gamma(x)\xi(\gamma),\ x\theta M^\alpha,\ \xi \in L^2(\mathfrak{H}; \hat{G}) ;$$
$$u(\gamma_0)\xi(\gamma) = \xi(\gamma + \gamma_0),\ \gamma, \gamma_0 \in \hat{G} .$$

The action α on M is implemented by the unitary representation

$$\{v, L^2(\mathfrak{H}; \hat{G})\}$$

of G defined by

$$v(s)\xi(\gamma) = \overline{\langle s, \gamma\rangle}\xi(\gamma) , \qquad s \in G .$$

Hence have we $\alpha_s(u(\gamma)) = \langle s, \gamma\rangle u(\gamma)$, so that $M(\alpha, \gamma) = M^\alpha u(\gamma)$, $\gamma \in \hat{G}$, where

$$M(\alpha, \gamma) = \{x \in M : \alpha_s(x) = \langle s, \gamma\rangle u(\gamma)\} .$$

If e is a central projection in M^α, then we have

$$eM(\alpha, \gamma)e = e\theta_\gamma(e)M^\alpha u(\gamma),\ \gamma \in \hat{G} ;$$
$$M_e(\alpha^e, \gamma) = e\theta_\gamma(e)M_e^\alpha u(\gamma) .$$

Hence $M_e(\alpha^e, \gamma) \neq \{0\}$ if and only if $e\theta_\gamma(e) \neq 0$. If $\theta_\gamma = \iota$ on the center of M^α, then $e\theta_\gamma(e) \neq 0$ for any non-zero central projection e in M^α; hence

$\gamma \in \Gamma(\alpha)$. A slight modification of the arguments for [30; Lemma 9.5] shows that if $\theta_{\gamma_0} \neq \iota$ on the center of M^α, then there exists a neighborhood V of γ_0 in $\hat{G}$ and a non-zero projection e in the center of M^α such that $e\theta_\gamma(e) = 0$ for every $\gamma \in V$. Hence we have $M_e(\alpha^e, \gamma) = \{0\}$ for every $\gamma \in V$. Since α^e is integrable, our assertion follows from the next lemma.

q.e.d.

LEMMA 3.3. *If α is an integrable action of a locally compact abelian group G on M, then for any open subset V of $\hat{G}$, the spectral subspace $M(\alpha, V) \neq \{0\}$ if and only if $M(\alpha, \gamma) \neq \{0\}$ for some $\gamma \in V$.*

PROOF. Trivially, $M(\alpha, \gamma) \subset M(\alpha, V)$ for any $\gamma \in \hat{G}$. Hence we have only to prove that $M(\alpha, \gamma) = \{0\}$ for every $\gamma \in V$ implies $M(\alpha, V) = \{0\}$. By a simple application of Fubini's theorem, we conclude that $\alpha_f(x) \in \mathfrak{p}_\alpha^+$ for any $f \in L^1(G)$, $f \geqq 0$, and $x \in \mathfrak{p}_\alpha^+$, where $\alpha_f(x) = \int_G f(s)\alpha_s(x)ds$; hence $\alpha_f(\mathfrak{p}_\alpha) \subset \mathfrak{p}_\alpha$ by the linearity for $f \in L^1(G)$. Put

$$\hat{x}(\gamma) = \int_G \overline{\langle s, \gamma \rangle} \alpha_s(x)ds , \qquad x \in \mathfrak{p}_\alpha .$$

We have then $x(\gamma) \in M(\alpha, \gamma)$ for any $x \in \mathfrak{p}_\alpha$. Suppose that $M(\alpha, \gamma) = \{0\}$ for any $\gamma \in V$. Then we have $x(\gamma) = 0$ for every $\gamma \in V$. If f is a function in $L^1(G)$ with supp $\hat{f} \subset V$, then we have for any $x \in \mathfrak{p}_\alpha$ and $\gamma \in \hat{G}$

$$\alpha_f(x)\hat{\ }(\gamma) = \hat{f}(\gamma)\hat{x}(\gamma) = 0 .$$

Hence $\alpha_f(x) = 0$ for every $x \in \mathfrak{p}_\alpha$; so $\alpha_f(M) = \{0\}$ since α_f is σ-weakly continuous and $\mathfrak{p}_\alpha$ is σ-weakly dense in M. Hence $\alpha_f = 0$ whenever supp $\hat{f} \subset V$. Thus $M(\alpha, V) = \{0\}$. q.e.d.

COROLLARY 3.4. *Let α be a continuous action of a separable locally compact abelian group G on a σ-finite von Neumann algebra M. Then the crossed product $W^*(M, G, \alpha)$ is a factor if and only if $\Gamma(\alpha) = \hat{G}$ and α is ergodic on the center of M.*

PROOF. Suppose that $W^*(M, G, \alpha)$ is a factor. By Theorem 3.2, $\Gamma(\alpha) = \hat{G}$. Since $W^*(M, G, \alpha) \cong [M \otimes \mathfrak{L}(L^2(G))]^{\alpha\otimes \mathrm{Ad}\lambda}$, for any central fixed point x under α, $x \otimes 1$ is in $[M \otimes \mathfrak{L}(L^2(G))]^{\alpha\otimes\mathrm{Ad}\lambda}$. Hence $x \otimes 1$ must be a scalar. Hence α is ergodic on the center of M.

Suppose that $\Gamma(\alpha) = \hat{G}$ and α is ergodic on the center of M. Since $\alpha \otimes \mathrm{Ad}\,\lambda$ on $\mathfrak{L}(L^2(G))$ enjoys the same property, we may assume that M is properly infinite and α is dominant. Then there exists an action θ of $\hat{G}$ on M^α such that $\{M, \alpha\} \cong \{W^*(M^\alpha, \hat{G}, \theta), \hat{\theta}\}$. By Theorem 3.2, θ acts trivially on the center C^α of M^α. Therefore, C^α is contained in the center C of M. But α acts ergodically on C, so that $C \cap M^\alpha = \{\lambda 1\}$; Hence $C^\alpha = \{\lambda 1\}$. Thus M^α is a factor. q.e.d.

COROLLARY 3.5. *If α is a continuous action of a separable locally compact abelian group G on a σ-finite von Neumann algebra M with $\Gamma(\alpha) = \hat{G}$, then any square integrable α-twisted unitary representation of G in M with infinite multiplicity is dominant.*

PROOF. Replacing α by a dominant action of G of the form ${}_a\alpha$, we may assume that α is dominant. By Theorem 2.12.ii, every square integrable α-twisted unitary representation of G in M is majorized by a dominant one in the ordering "$\prec$". We have only to prove that α^e on M^e is dominant for any properly infinite projection e of M^α such that $e \sim 1$ in M. Let $\{u(\gamma): \gamma \in \Gamma\}$ be a unitary representation of $\hat{G}$ in M such that $\alpha_s(u(\gamma)) = \langle s, \gamma\rangle u(\gamma)$, so that $\mathrm{Ad}\, u(\gamma)|_{M^\alpha} = \theta_\gamma$ is a continuous action of $\hat{G}$ on M^α with $\{W^*(M^\alpha, \hat{G}, \theta), \hat{\theta}\} \cong \{M, \alpha\}$. By Theorem 3.2, the action of θ on the center C^α of M^α is trivial. Hence e and $\theta_\gamma(e)$ have the same central support in M^α, and are properly infinite in M^α; hence $e \sim \theta_\gamma(e)$. Therefore, there exists a partial isometry v_γ in M^α such that $v_\gamma^* v_\gamma = \theta_\gamma(e)$ and $v_\gamma v_\gamma^* = e$. Let $w_\gamma = v_\gamma u(\gamma)e$. Then we have $w_\gamma^* w_\gamma = e$ and $w_\gamma w_\gamma^* = e$, and also $\alpha_s^e(w_\gamma) = \langle s, \gamma\rangle w_\gamma$. Hence $\{M_e, \alpha^e\}$ satisfies condition (ii) in Theorem 3.1. Thus α^e is dominant. q.e.d.

We close this section with the following:

REMARK 3.6. So far we have mainly dealt with actions and/or weights of infinite multiplicity. The contrast between the following two statements (i) and (ii) might illustrate some of the reasons why the infinite multiplicity has been useful.

(i) If α is a continuous action of a separable locally compact group G on M with infinite multiplicity, then $M(\alpha, V)$ contains a non-zero partial isometry for any open subset V of $\hat{G}$ with $V \cap \Gamma(\alpha) \neq \varnothing$. More strongly, if $\Gamma(\alpha) = \hat{G}$ in addition, then $M(\alpha, V)$ contains a unitary for every non-empty open subset V of $\hat{G}$.

(ii) Let M be an abelian von Neumann algebra and α an ergodic continuous action of $\boldsymbol{R}$. If u is a non-zero partial isometry in $M(\alpha, V)$ for a bounded interval V, then u is unitary and $\alpha_t(u) = e^{ist}u$ for some $s \in V$.

The first assertion can be proven by approximating α with integrable actions. The second statement can be shown by some modification of the Paley-Wiener Theorem for the Fourier transform of distribution with compact support.

III.4. Galois correspondence. In this section, we shall show that given an integrable action α of a locally compact abelian group G on a von Neumann algebra M with M^α a factor, there is a Galois type correspondence between closed subgroups of G and globally α-invariant von Neumann subalgebras of M containing M^α, which generalizes a result in [30; §7].

THEOREM 4.1. *Let M_0 be a factor equipped with a continuous action α of a locally compact abelian group G. Let $M = W^*(M_0, G, \alpha)$. If N is a von Neumann subalgebra of M such that $M_0 \subset N$ and $\hat{\alpha}_p(N) = N$ for every $p \in \hat{G}$, where $\hat{\alpha}$ means the dual action of $\hat{G}$ on M then there is a closed subgroup $\hat{H}$ of $\hat{G}$ such that*

$$N = \{x \in M\colon \hat{\alpha}_p(x) = x \text{ for every } p \in \hat{H}\}\ ;$$
$$\hat{H} = \{p \in \hat{G}\colon \hat{\alpha}_p(x) = x \text{ for every } x \in N\}\ ;$$

therefore N is of the form $N = W^*(M_0, H, \alpha)$ with $H = \hat{H}^\perp$.

We divide the proof into a few steps.

LEMMA 4.2. *Let P be a factor and A an abelian von Neumann algebra. If Q is a factor such that $P \otimes 1 \subset Q \subset P \otimes A$, then $Q = P \otimes 1$.*

PROOF. Representing A as a maximal abelian von Neumann algebra on $\mathfrak{H}$, we have

$$\begin{aligned}(P \otimes 1)' \cap (P \otimes A) &= [P' \otimes \mathfrak{L}(\mathfrak{H})] \cap (P \otimes A)\\ &= 1 \otimes A\ ;\end{aligned}$$

hence

$$(P \otimes 1)' \cap Q \subset (1 \otimes A) \cap Q = \boldsymbol{C}1 \subset P \otimes 1\ .$$

Therefore, there is at most only one normal conditional expectation from Q onto $P \otimes 1$ by [3; Théorème 1.5.5(a)]. Since there are in general many normal conditional expectations from $P \otimes A$ onto $P \otimes 1$, there exists a unique normal conditional expectation, say ε, from Q onto $P \otimes A$. To each normal state ω on A, there corresponds a normal conditional expectation ε_ω of $P \otimes A$ onto $P \otimes 1$ by the formula:

$$\varphi(\varepsilon_\omega(x)) = (\varphi \otimes \omega)(x)\ , \quad x \in P \otimes A, \varphi \in P_*\ .$$

By the uniqueness of a conditional expectation, we have, for any $x \in Q$, $\varepsilon(x) = \varepsilon_\omega(x)$, so that

$$(\varphi \otimes \omega)(\varepsilon(x) \otimes 1) = \varphi(\varepsilon_\omega(x)) = (\varphi \otimes \omega)(x)\ .$$

Therefore, we get $\varepsilon(x) \otimes 1 = x$ for every $x \in Q$; thus $Q = P \otimes 1$. q.e.d.

PROOF OF THEOREM 4.1. We put

$$\hat{H} = \{p \in \hat{G}: \alpha_p(x) = x \text{ for every } x \in N\}.$$

By [30; Theorem 7.1], the algebra $M^{\hat{H}}$ of all fixed points in M under $\hat{\alpha}_p$, $p \in \hat{H}$, is $W^*(M_0, H, \alpha)$ with $H = \{g \in G: \langle g, p\rangle = 1 \text{ for every } p \in \hat{H}\}$, where the technical assumption in [30; Theorem 7.1] on the existence of a relatively invariant weight on M_0 is not essential because of the commutation theorem for the general crossed product due to T. Digerness [8]. Replacing G by H and M by $W^*(H_0, H, \alpha)$, we may assume that $\hat{H} = \{0\}$, and must show that $N = M$.

We consider the crossed products, $W^*(M, \hat{G}, \hat{\alpha}) = \tilde{M}$, $W^*(N, \hat{G}, \hat{\alpha}) = \tilde{N}$ and $W^*(M_0, \hat{G}, \hat{\alpha}) = \tilde{M}_0$. We have then

$$\tilde{M}_0 = M_0 \otimes L^\infty(G) \subset \tilde{N} \subset \tilde{M}.$$

The action $\hat{\alpha}$ of $\hat{G}$ on N is faithful, and the fixed point algebra $N^{\hat{G}}$ in N under $\hat{\alpha}$ is M_0, hence a factor. Hence $\tilde{N}$ is a factor by Corollary 3.4. By [30; Theorem 4.5], we have

$$\tilde{M} \cong M_0 \otimes \mathfrak{L}(L^2(G)).$$

Therefore, if we can identify the algebras $\tilde{M}_0$, and $\tilde{M}$ with $M_0 \otimes L^\infty(G)$ and $M_0 \otimes \mathfrak{L}(L^2(G))$, then Lemma 4.2 is applied to the commutants: $M_0' \otimes L^\infty(G) \supset \tilde{N}' \supset M_0' \otimes 1$. Hence $\tilde{N}' = M_0' \otimes 1$, so $\tilde{N} = \tilde{M}$. Since N is the fixed point algebra in $\tilde{N} = \tilde{M}$ under the action $\hat{\hat{\alpha}}$ of G, we have $M = N$. Thus, we must show that $\tilde{M}$ is identified with $M_0 \otimes \mathfrak{L}(L^2(G))$ in such a way that $\tilde{M}_0$ coincides with $M_0 \otimes L^\infty(G)$ under this identification.

Let $\mathfrak{H}$ be the Hilbert space on which M_0 acts. Then M acts on the Hilbert space $L^2(\mathfrak{H}; G)$, and $\tilde{M}$ acts on $L^2(\mathfrak{H}; G \times G)$ and is generated by the following three types of operators:

$$\begin{cases} \bar{x}\xi(s, t) = \alpha_s^{-1}(x)\xi(s, t), & x \in M_0; \\ u(r)\xi(s, t) = \xi(s - r, t - r), & r \in G; \\ v(p)\xi(s, t) = \langle t, p\rangle \xi(s, t), & p \in \hat{G}. \quad \text{(cf. [30; (4.10)])}. \end{cases}$$

It follows then that $\tilde{M}_0$ is generated by $\{\bar{x}, v(p); x \in M_0, p \in \hat{G}\}$ and identified with $M_0 \otimes L^\infty(G) = L^\infty(M_0; G)$, where the action of $L^\infty(M_0; G)$ is given by the following:

$$x\xi(s, t) = \alpha_s^{-1}(x(t))\xi(s, t)$$

for every $x(\cdot) \in L^\infty(M_0; G)$. We define an automorphism π of $L^\infty(M_0; G)$ by

$$\pi(x)(s) = \alpha_s(x(s)),\ x(\cdot) \in L^\infty(M_0; G).$$

It follows from the proof of [30; Theorem 4.5] that $\tilde{M}$ is the tensor product of $\pi(M_0 \otimes 1)$ and its relative commutant B in $\tilde{M}$ where B is generated by $u(G)$ and $v(\hat{G})$. Thus we have

$$\tilde{M}_0 = \pi(M_0 \otimes L^\infty(G)) \cong \pi(M_0 \otimes 1) \otimes L^\infty(G) ;$$
$$\tilde{M} = \pi(M_0 \otimes 1) \otimes \mathfrak{B} \supset \tilde{N} \supset \pi(M_0 \otimes 1) \otimes L^\infty(G) = \tilde{M}_0 .$$

q.e.d.

THEOREM 4.3. *Let M be a factor equipped with an integrable action α of a locally compact abelian group G. If $\Gamma(\alpha) = \hat{G}$, then there exists a bijective inclusion reversing correspondence between the closed subgroups H of G and the α-invariant von Neumann subalgebras N of M containing the fixed point algebra M^α in such a way that*

$$N_H = \{x \in M: \alpha_s(x) = x,\ s \in H\} ;$$
$$H_N = \{s \in G: \alpha_s(x) = x,\ x \in N\} .$$

PROOF. We put

$$\bar{M} = M \otimes F_\infty \quad \text{and} \quad \bar{\alpha}_s = \alpha_s \otimes \iota , \quad s \in G ,$$

with F_∞ a factor of type I_∞. It follows then that $\bar{\alpha}$ is dominant, since the fixed point algebra $\bar{M}^{\bar{\alpha}}$ under $\bar{\alpha}$ is $M^\alpha \otimes F_\infty$. Hence, by Theorem 4.1, the correspondence between H and $\bar{\alpha}$-invariant von Neumann subalgebras $\bar{N}$ of $\bar{M}$ containing $\bar{M}^{\bar{\alpha}}$ given by

$$\bar{N}_H = \{x \in \bar{M}: \bar{\alpha}_s(x) = x,\ s \in H\} ;$$
$$H_{\bar{N}} = \{s \in G: \bar{\alpha}_s(x) = x,\ x \in \bar{N}\}$$

is bijective and inclusion reversing. It is now trivial that $N_{H_N} \supset N$ and $H_{N_H} \supset H$. For a given N, we put $\bar{N} = N \otimes F_\infty$. Trivially we have $H_N = H_{\bar{N}}$. If $x \in N_{H_N}$, then $x \otimes 1 \in \bar{N}_{H_{\bar{N}}}$; so $x \otimes 1 \in \bar{N}$ equivalently $x \in N$. Hence $N = N_{H_N}$. For a given H, we have $\bar{N}_H = N_H \otimes F_\infty (= (N_H)^-)$. Hence we get

$$H = H_{\bar{N}_H} = H_{(N_H \otimes F_\infty)} = H_{N_H} . \qquad \text{q.e.d.}$$

EXAMPLE 4.4. Let G be a locally compact abelian group, and $M = \mathfrak{L}(L^2(G))$. Putting

$$\begin{cases} (u(s)\xi)(t) = \xi(t - s) , & \xi \in L^2(G),\ s,\ t \in G ; \\ (v(P)\xi)(t) = \overline{\langle t, p\rangle}\xi(t) , & \xi \in L^2(G),\ p \in \hat{G},\ t \in G , \end{cases}$$

we obtain unitary representations u of G and v of $\hat{G}$ with

$$u(s)v(p)u(s)^*v(p)^* = \langle s, p\rangle 1 , \quad s \in G,\ p \in \hat{G} .$$

Thus we may define an action α of $G \times \hat{G}$ on M by

$$\alpha_{s,p}(x) = u(s)v(p)xv(p)^*u(s)^* , \quad s \in G,\ p \in \hat{G},\ x \in M .$$

Since $u(s)$, $s \in G$, and $v(p)$, $p \in \hat{G}$, together generate M, we have

$$M^\alpha = \{\lambda 1 : \lambda \in \boldsymbol{C}\} ;$$

hence $\Gamma(\alpha) = (G \times \hat{G})\hat{} = \hat{G} \times G$.

For a pair f, g of functions in $L^2(G)$, we define an operator $x_{f,g} \in M$ by

$$x_{f,g}\xi = (\xi \mid f)g .$$

We have then

$$(u(r)v(p)x_{f,g}v(p)^* u(r)^* \xi \mid \eta) = \iint \overline{\langle s - t, p\rangle f(t)} g(s)\xi(t + r)\overline{\eta(s + r)} dsdt .$$

Therefore, by the Plancherel formula, we get

$$\begin{aligned}\iint (u(r)v(p)x_{f,g}v(p)^* u(r)^* \xi \mid \eta) dpdr &= \iint \bar{f}(s)g(s)\xi(s + r)\overline{\eta(s + r)} dsdr \\ &= (g \mid f)(\xi \mid \eta) ,\end{aligned}$$

so that

$$\int u(r)v(p)x_{f,g}v(p)^* u(r)^* dpdr = (g \mid f)1 .$$

This means that the action α of $G \times \hat{G}$ is integrable. Thus, the α-invariant von Neumann algebras on $L^2(G)$ are labeled by the closed subgroups of $G \times \hat{G}$ by Theorem 4.3. The von Neumann algebras considered in [28] are of the special case where the corresponding subgroups are of the form $H \times \hat{K}$ with H a closed subgroup of G and $\hat{K}$ a closed subgroup of $\hat{G}$.

Since there are many von Neumann algebras not corresponding to any closed subgroup of $G \times \hat{G}$, the invariance of a von Neumann algebra under the action α in Theorem 4.3 is not removable in this general setting. The same is true for Theorem 4.1 because the tensor product with F_∞ a factor of type I_∞ gives counter examples for the Galois correspondence without α-invariance.

The following result strengthens and refines a generalized commutation theorem [28].

PROPOSITION 4.4. *In the setting of Example* 4.4, *let H be a closed subgroup of $G \times \hat{G}$ and $H^\perp = \{(q, t) \in \hat{G} \times G : \langle s, q\rangle = \langle t, p\rangle$ for every $(s, p) \in H$. The fixed point algebra M^H under $\alpha_{s,p}$ for every $(s, p) \in H$ is generated by $u(t)v(q)$ with $(q, t) \in H^\perp$.*

PROOF. In general, we have

$$\alpha_{s,p}(u(t)v(q)) = \overline{\langle t, p\rangle}\langle s, q\rangle u(t)v(q) , \quad s, t \in G,\ p, q \in \hat{G} .$$

Hence $u(t)v(q)$ belongs to M^H if and only if $(q, t) \in H^\perp$.

The action of $(G \times \hat{G})/H$ on M^H, denoted by the same notation α, induced by the original action of $G \times \hat{G}$ is integrable; hence M^H is generated by the eigen operators. Let x be an eigen operator in M^H corresponding to $(g, t) \in ((G \times \hat{G})/H)\hat{} = H^\perp$. It forllows then that $(u(t)v(q))^*x$ belongs to the fixed point algebra $M^\alpha = \{\lambda 1\}$. Hence $x = \lambda u(t)v(q)$ for some $\lambda \in \boldsymbol{C}$. Thus M^H is generated by $\{u(t)v(q): (q, t) \in H^\perp\}$. q.e.d.

III.5. Stability of automorphisms. In this section, we shall show that if α is an automorphism (resp. one parameter automorphism group) of a semi-finite von Neumann algebra N scaling a trace down, then every unitary one cocycle is a coboundary. This, in turn, improves the isomorphism criterion for the factors of type III in terms of the conjugacy of discrete as well as continuous decompositions.

THEOREM 5.1. *Let N be a semi-finite von Neumann algebra.*

(i) *If θ is an automorphism of N such that there exists a faithful semi-finite normal trace τ on N such that $\tau \circ \theta \leqq \lambda\tau$ for some $0 < \lambda < 1$, then* (a) *there exists a continuous action α of the torus $\boldsymbol{T}$ on the fixed point algebra N^θ such that*

$$\{W^*(N^\theta, \boldsymbol{T}, \alpha), \hat{\alpha}\} \cong \{N, \theta\} ;$$

(b) *every unitary $u \in N$ is of the form $u = v^*\theta(v)$ for some unitary $v \in N$.*

(ii) *If $\{\theta_t\}$ is a one parameter automorphism group of N such that $\tau \circ \theta_t = e^{-t}\tau$ for some faithful semi-finite normal trace τ on N, then* (a) *there exists a one parameter automorphism group $\{\alpha_s\}$ of the fixed point algebra N^θ such that*

$$\{W^*(N^\theta, \boldsymbol{R}, \alpha), \hat{\alpha}\} \cong \{N, \theta\} ;$$

(b) *every α-twisted unitary representation $\{u_t\}$ of $\boldsymbol{R}$ in N is of the form $u_t = v^*\alpha_t(v)$ for some unitary $v \in N$.*

PROOF. (i) Let θ be an automorphism of N with $\tau \circ \theta \leqq \lambda\tau$. We first claim that for any non-zero projection $p \in N^\theta$ there exists a non-zero projection $q \leqq p$ such that $\{\theta^n(q)\}$ is orthogonal. Let e be a non-zero projection such that $e \leqq p$ and $\tau(e) < +\infty$. Let $f = \bigvee_{n=0}^\infty \theta^n(e)$. We have then

$$\tau(f) \leqq \sum_{n=0}^\infty \tau(\theta^n(e)) \leqq \sum_{n=0}^\infty \lambda^n\tau(e) = \frac{1}{1-\lambda}\tau(e) < +\infty ;$$

$$\theta(f) \leqq f \quad \text{and} \quad \tau(\theta(f)) = \lambda\tau(f) < \tau(f) ;$$

$$q = f - \theta(f) \neq 0 .$$

It is clear that $\{\theta^n(q): n \in \boldsymbol{Z}\}$ is orthogonal. Therefore, the usual exhaustion arguments entail the existence of a projection $q \in N$ such that $\{\theta^n(q): n \in \boldsymbol{Z}\}$ is orthogonal and $\sum_{n \in Z} \theta^n(q) = 1$.

We put, for $0 \leqq s < 1$,

$$u(s) = \sum_{n \in \boldsymbol{Z}} e^{-2\pi i s n} \theta^n(q) .$$

It follows then that $\theta(u(s)) = e^{2\pi i s} u(s)$, $0 \leqq s < 1$. Therefore, $\{u(s): 0 \leqq s < 1\}$ induces a continuous action α of the torus $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ on N^θ by

$$\alpha_s(x) = u(s)xu(s)^* , \quad s \in \boldsymbol{T} ,$$

where we identify the torus $\boldsymbol{T}$ with the half open unit interval $[0, 1)$. Thus, our assertion (a) follows from [15].

For the second assertion, (b), we observe first that if N^θ is properly infinite, then θ is dominant. But we claim that N is properly infinite if and only if N^θ is also. By the usual reduction arguments, it is sufficient to prove the claim that the finiteness of N^θ implies that of N. Suppose N^θ is finite. Let φ be a faithful semi-finite normal trace on N^θ invariant under α, the existence of such a φ being guaranteed by the compactness of $\boldsymbol{T}$. Let $\tilde{\varphi}$ be the weight on N dual to φ. It follows from [30; Proposition 5.16] that $\tilde{\varphi}$ is invariant under θ. Since φ is a faithful semi-finite normal trace on N, $\tilde{\varphi}$ is of the form: $\tilde{\varphi} = \tau(h\cdot)$ for some non-singular positive self-adjoint operator h affiliated with the center C of N. We have then

$$\begin{aligned} \tau(\theta(h)x) &= \tau \circ \theta(h\theta^{-1}(x)) \leqq \lambda\tau(n\theta^{-1}(x)) \\ &= \lambda\tilde{\varphi}(\theta^{-1}(x)) = \lambda\tilde{\varphi}(x) = \lambda\tau(hx) , \quad x \in N_+ . \end{aligned}$$

Hence we get $\theta(h) \leqq \lambda h$. From this, repeating more or less the same arguments as above, we can construct a continuous unitary representation $v(s)$ of $\boldsymbol{T}$ in C such that

$$\theta(v(s)) = e^{2\pi i s} v(s) .$$

Hence the action α' of $\boldsymbol{T}$ on N^θ induced by $\{v(s)\}$ is trivial, and θ is still dual to this new α'. This means that $N \cong N^\theta \otimes l^\infty(\boldsymbol{Z})$ and $\theta \cong 1 \otimes$ (translation on $l^\infty(\boldsymbol{Z})$). Thus N must be finite. In this case, let u be an arbitrary unitary in N, and $u = \{u_n\}$ in the decomposition $N = N^\theta \otimes l^\infty$. Put $v_{n+1} = v_n u_n$ if $n \geqq 1$ and $v_0 = 1$, $v_n = v_{n+1} u_n$ if $n < 0$. We have then $v^*\theta(v) = u$. If N is properly infinite, then every θ with $\tau \circ \theta \leqq \lambda\tau$ is dominant, so that for any $u \in \mathfrak{U}_N$ the new action $\bar{\theta} = \operatorname{Ad} u \circ \theta$ is dominant; hence the θ-twisted unitary representation of $\boldsymbol{Z}$ in N generated by u is dominant, which means that $u = v^*\theta(v)$ for some $v \in \mathfrak{U}_N$.

(ii) We apply (i) to $\{\theta_n: n \in \boldsymbol{Z}\}$. Let N_1 denote the fixed point subalgebra of N under $\{\theta_n: n \in \boldsymbol{Z}\}$. It follows then that the restriction $\theta|_{N_1}$ of θ to N_1 is periodic with period one. The action $\{\theta_n: n \in \boldsymbol{Z}\}$ of $\boldsymbol{Z}$ on N is integrable by (i) and $\theta|_{N_1}$ is integrable as an action of the torus $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$. Hence θ itself is integrable, because

$$E(x) = \int_{-\infty}^{\infty} \theta_t(x)dt = \int_0^1 \theta_t(\sum_{n \in \boldsymbol{Z}} \theta_n(x))dt\,, \quad x \in N_+\,.$$

Let ψ be a strictly semi-finite faithful weight on N^θ. It follows then that the weight $\varphi = \psi \circ E$ is a faithful weight on N invariant under θ. By [30; Theorem 5.4], there exists a non-singular self-adjoint operator h affiliated with N such that $\varphi = \tau(h\cdot)$. For any $x \in N_+$, we have

$$\begin{aligned}\tau(\theta_s(h)x) &= \tau \circ \theta_s(h\theta_{-s}(x)) = e^{-s}\tau(h\theta_{-s}(x)) = e^{-s}\varphi(\theta_{-s}(x)) \\ &= e^{-s}\varphi(x) = e^{-s}\tau(hx)\,;\end{aligned}$$

hence we have $\theta_s(h) = e^{-s}h$. Putting $u(t) = h^{-it}$, $t \in \boldsymbol{R}$, we have

$$\theta_s(u(t)) = e^{ist}u(t)\,.$$

Thus, the one parameter unitary group $\{u(t): t \in \boldsymbol{R}\}$ gives rise to a one parameter automorphism group $\{\alpha_t: t \in \boldsymbol{R}\}$ of N^θ such that $\{N, \theta\} \cong \{W^*(N^\theta, \boldsymbol{R}, \alpha), \hat{\alpha}\}$ by [15]. This proves (a).

To prove the second assertion (b), we first show that N^θ is semi-finite if and only if $\{N, \theta\} \cong \{N^\theta \otimes L^\infty(\boldsymbol{R}),\ \iota \otimes \text{translation}\}$. Let $P = N \otimes F_\infty$ and $\bar{\theta}_t = \theta_t \otimes \iota$, $t \in R$. It follows then that $\bar{\theta}$ is dominant and $N^\theta \otimes F_\infty = P^{\bar{\theta}}$. If N^θ is semi-finite then so is $P^{\bar{\theta}}$. Hence $W^*(N, \boldsymbol{R}, \theta) \cong P^{\bar{\theta}}$ is semi-finite. Our claim then follows from [30; Section 9], and assertion (b) in this case is standard.

If N^θ is properly infinite, then N^a is also for every $a \in Z^1_\theta(\boldsymbol{R}, \mathfrak{U}_N)$, which means that a is dominant since $\tau \circ {}_a\theta_t = e^{-t}\tau$, $t \in \boldsymbol{R}$. Thus $a \cong 1$.

q.e.d.

COROLLARY 5.2. (i) *Let N_1 and N_2 be properly infinite semi-finite von Neumann algebras equipped with one parameter automorphism groups θ^1 and θ^2 respectively which transform some faithful semi-finite normal traces τ_1 and τ_2 respectively in such a way that*

$$\tau_1 \circ \theta^1_s = e^{-s}\tau_1 \quad \textit{and} \quad \tau_2 \circ \theta^2_s = e^{-s}\tau_2\,, \quad s \in \boldsymbol{R}\,.$$

Then $W^*(N_1, \boldsymbol{R}, \theta^1) \cong W^*(N_2, \boldsymbol{R}, \theta^2)$ if and only if there exists an isomorphism π of N_1 onto N_2 such that $\theta^1_s = \pi^{-1} \circ \theta^2_s \circ \pi$, $s \in \boldsymbol{R}$.

(ii) *If $\{N_1, \theta_1\}$ and $\{N_2, \theta_2\}$ are discrete decompositions of the same factor of type III_λ, $0 < \lambda < 1$, then there exists an isomorphism π of N_1 onto N_2 such that $\theta_1 = \pi^{-1} \circ \theta_2 \circ \pi$.*

(iii) *If* $\{N_1, \theta_1\}$ *and* $\{N_2, \theta_2\}$ *are discrete decompositions of the same factor of type* III_0, *then there exist central projections* $e_1 \in N_1$ *and* $e_2 \in N_2$, *and an isomorphism* π *of* N_{1,e_1} *onto* N_{2,e_2} *such that* $\theta_{1,e_1} = \pi^{-1} \circ \theta_{2,e_2} \circ \pi$, *where* θ_{1,e_1} (resp. θ_{2,e_2}) *is an automorphism of* N_{1,e_1} (resp. N_{2,e_2}) *induced by* θ_1 (resp. θ_2) *as described in* [3; Definition 5.4.1.].

PROOF. This is a straightforward consequence of Theorem 5.1 and [30; §8] and [3, Theorems 4.4.1 and 5.4.2]. q.e.d.

COROLLARY 5.3. *An automorphism* α *of a factor* M *of type* II_∞ *is stable if and only if* α *does not preserve the trace* τ *of* M.

PROOF. Suppose α does not preserve the trace τ on M. It follows that $\tau \circ \alpha = \lambda\tau$ for some $\lambda > 0$ by the uniqueness of the trace. Considering α^{-1}, we may assume $\lambda < 1$. Let $\beta = \mathrm{Ad}(u) \circ \alpha$ with u a unitary in M. Then we have $W^*(M, \alpha) \cong W^*(M, \beta)$, and they are of type III_λ. By Theorem 5.1, we have $M^\alpha \otimes \mathfrak{L}(l^2(\boldsymbol{Z})) \cong W^*(M, \alpha)$, so that $M^\alpha \cong W^*(M, \alpha)$. Thus M^α and M^β are both properly infinite, which means that α and β are both dominant. Therefore, there exists a unitary $v \in M$ such that $u = v^*\alpha(v)$, which means that $\beta = \mathrm{Ad}(v)^{-1} \circ \alpha \circ \mathrm{Ad}(v)$.

Suppose conversely α preserves the trace τ. Let e be a projection in M with $\tau(e) < +\infty$. Since $e \sim \alpha(e)$, there exists a unitary $u \in M$ such that $e = u\alpha(e)u^*$, where we note here that the equivalence between finite projections is unitarily implemented. Let $\beta = \mathrm{Ad}(u) \circ \alpha$. It follows then that β preserves a normal positive linear functional $\varphi = \tau(e\cdot)$. Hence $\{\beta^n : n \in \boldsymbol{Z}\}$ is not integrable, so that $\{\beta^n\}$ is not conjugate to any integrable action of $\boldsymbol{Z}$. But there is a unitary $v \in M$ as seen in §2 that $\{(\mathrm{Ad}\, v \circ \beta)^n\}$ is integrable, even dominant. Hence β and $\mathrm{Ad}(v) \cdot \beta$ are not conjugate; therefore either $\beta = \mathrm{Ad}(u) \circ \alpha$ or $\mathrm{Ad}(v) \circ \beta = \mathrm{Ad}(vu) \circ \alpha$ is not conjugate to α. Therefore, α is not stable. q.e.d.

PROOF OF THEOREM II.1.6. Let $\{\bar{\omega}_1, \bar{\omega}_2\}$ and $\{\bar{\omega}_1', \bar{\omega}_2'\}$ be two quasi-commuting pair of dominant weights on an infinite factor M with separable predual such that $\alpha(\bar{\omega}_1, \bar{\omega}_2) = \alpha(\bar{\omega}_1', \bar{\omega}_2')$, say α for short. By the uniqueness of a dominant weight, there exists a unitary $u \in M$ such that $\bar{\omega}_1 = \bar{\omega}_{1,u}'$. Replacing $\bar{\omega}_2'$ by $\bar{\omega}_{2,u}'$, we reduce the situation to the following: given three dominant weights $\bar{\omega}$, φ, and ψ on M such that $\{\bar{\omega}, \varphi\}$ and $\{\bar{\omega}, \psi\}$ are quasi-commuting with $\alpha(\bar{\omega}, \varphi) = \alpha(\bar{\omega}, \psi) = \alpha$, we must show that there exists a unitary u in $M_{\bar{\omega}}$ such that $\psi = \varphi_u$.

Let $M = W^*(N, \boldsymbol{R}, \theta)$ and $\{u(s) : s\theta\boldsymbol{R}\}$ be a continuous decomposition of M and the one parameter unitary group in M associated with this decomposition. We may assume that $\bar{\omega}$ is the weight on M dual to a trace $\tau\circ\theta_t=e^{-t}\tau$, $t\in\boldsymbol{R}$. For short, put $v_s=(D\varphi: D\bar{\omega})_s$, and $w_s=(D\psi: D\bar{\omega})_s$, $s\in\boldsymbol{R}$. We have then

$$\sigma_t^{\bar{\omega}}(v_s)=e^{i\alpha st}v_s \quad\text{and}\quad \sigma_t^{\bar{\omega}}(w_s)=e^{i\alpha st}w_s\ ;$$
$$v_{s+t}=e^{i\alpha st}v_sv_t\ ,\quad w_{s+t}=e^{i\alpha st}w_sw_t\ .$$

For each $s\in\boldsymbol{R}$, put

$$a_s=e^{-i\alpha s^2/2}v_su(\alpha s)^* \quad\text{and}\quad b_s=e^{-i\alpha s^2/2}w_su(\alpha s)^*\ .$$

It is easily seen that $\{a_s\}$ and $\{b_s\}$ are both continuous and parameter families of unitaries in N such that

$$a_{s+t}=a_s\theta_{\alpha s}(a_t) \quad\text{and}\quad b_{s+t}=b_s\theta_{\alpha s}(b_t)\ .$$

By Theorem 5.1, there exists a unitary $u\in N$ such that

$$a_s=ub_s\theta_{\alpha s}(u^*)\ ,\quad s\in\boldsymbol{R}\ .$$

Hence we get, for any $s\in\boldsymbol{R}$,

$$\begin{aligned}v_s&=e^{i\alpha s^2/2}a_su(\alpha s)=e^{i\alpha s^2/2}ub_s\theta_{\alpha s}(u^*)u(\alpha s)\\&=e^{i\alpha s^2/2}ub_su(\alpha s)u^*=uw_su^*\ .\end{aligned}$$

Thus it follows that $\varphi=\psi_u$. q.e.d.

CHAPTER IV. THE FLOW OF WEIGHTS AND THE AUTOMORPHISM GROUP OF A FACTOR OF TYPE III

IV.0. Introduction. The aim of this chapter is to extend the exact sequence of [3, 4.5] to the general case from type III_λ case, $0<\lambda<1$, for the automorphism group $\mathrm{Aut}\,(M)$ and/or the outer automorphism group $\mathrm{Out}\,(M)=\mathrm{Aut}\,(M)/\mathrm{Int}\,(M)$ of a factor M of type III in terms of the flow F_M of weights on M and a continuous decomosition $M=W^*(N,\boldsymbol{R},\theta)$ of M. Since F^M is functorial to each $\alpha\in\mathrm{Aut}\,(M)$ there corresponds a unique automorphism $\mathrm{mod}\,(\alpha)$ of the flow F^M as the restriction of $\bar{\alpha}\in\mathrm{Aut}\,(\mathfrak{P}_M)$ to P_M. Assuming M to be a factor of type II_∞, we will see that $\mathrm{mod}\,(\alpha)$ is precisely the translation of $L^\infty(\boldsymbol{R}_+^*)$ by multiplying $\lambda(\alpha)>0$ where this positive number $\lambda(\alpha)$ is determined by $\tau\circ\alpha=\lambda(\alpha)\tau$ for the trace τ on M. With this evidence, we call mod the *fundamental homomorphism* of $\mathrm{Aut}\,(M)$ in general. Considering the topologies in $\mathrm{Aut}\,(M)$ and $\mathrm{Aut}\,(F^M)$ as in preliminary, we will show that mod is continuous; hence ker mod contains the closure of $\mathrm{Int}\,(M)$.

We next extend the modular automorphism group $\{\sigma_t^\varphi\}$ from the additive group $\boldsymbol{R}$ to the multiplicative group $Z^1(F^M)$ of unitary one cocycles with respect to the flow F^M of weights. To each $c\in Z^1(F^M)$ and a faithful integrable weight φ on M, we associate an automorphism $\bar{\sigma}_c^\varphi$ of M by $\bar{\sigma}_c^\varphi(x) = p_\varphi^{-1}(c_\lambda p_M(\varphi))x$ for each $x \in M(\sigma^\varphi, \{\lambda\})$. The relative commutant theorem, Theorem II.5.1, then enables us to characterize these automorphisms as those which leave the centralizer elementwise fixed. We then show that for a smooth $c \in Z^1(F^M)$ there exist a map: $\varphi \to \bar{\sigma}_c^\varphi$ from the space $\mathfrak{W}_M^0$ of faithful weights to $\mathrm{Aut}(M)$ and a map: $(\varphi, \psi) \to (D\varphi: \mathfrak{W}\psi)_c$ from $\mathfrak{W}_M^0 \times \mathfrak{W}_M^0$ into the unitary group $\mathfrak{U}$ of M such that

$$\bar{\sigma}_c^\psi(x) = (D\psi: D\varphi)_c \bar{\sigma}_c^\varphi(x)(D\psi: D\varphi)_c^* , \qquad x \in M ,$$

which coincide with σ_t^φ and $(D\psi: D\varphi)_t$ if $c_\lambda = \lambda^{it}$. In this setting, the modular period group $T(M)$ of M is generalized to $B^1(F^M)$ in the sence that $\bar{\sigma}_c^\varphi$ is inner if and only if $c \in B^1(F^M)$, see [30; Theorem 9.4]. Thus we obtain a homomorphism $\bar{\delta}_M$ of $H^1(F^M)$, the first unitary cohomology group of the flow F^M, into $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M)$. Assuming M to be semi-finite, we will see that $(D\varphi: D\mathrm{Tr})_c = f(1)^*f(h)$ with $\varphi = \mathrm{Tr}(h\cdot)$ and $c_\lambda = fF_\lambda(f^*)$, $f \in L^\infty(\boldsymbol{R}_+^*)$. From this, we view $\bar{\sigma}_c^\varphi$ and $(D\varphi: D\psi)_c$ as functional calculus of the "generator" of the modular automorphism group $\{\sigma_t^\varphi\}$.

In the last section, fixing a continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$, we obtain an exact sequence:

$$\{1\} \longrightarrow H^1(F^M) \xrightarrow{\bar{\delta}_M} \mathrm{Out}(M) \longrightarrow \mathrm{Out}_{\theta,\tau}(N) \longrightarrow \{1\} ,$$

where

$$\mathrm{Out}_{\theta,\tau}(N) = \{\alpha \in \mathrm{Out}(N): \varepsilon_N(\theta_t)\alpha = \alpha\varepsilon_N(\theta_t),\ \tau \circ \alpha = \tau\}$$

and ε_N is the canonical homomorphism of $\mathrm{Aut}(N)$ onto $\mathrm{Out}(N)$.

IV.1. The fundamental homomorphism. Let M be an infinite factor with separable predual, and F^M the smooth flow of weights on M. Recall that F^M is just the action: $\varphi \to \lambda\varphi$ of $\boldsymbol{R}_+^*$ on the classes of integrable weights of infinite multiplicity. Let $\mathrm{Aut}(F^M)$ be the group of automorphisms F^M, (i.e., automorphisms of the abelian von Neumann algebra P_M which commute with the action F^M of $\boldsymbol{R}_+^*$). For any $\alpha \in \mathrm{Aut}(M)$, the permutation: $\varphi \to \varphi \circ \alpha^{-1}$ of classes of integrable weights of infinite multiplicity defines a unique element $\mathrm{mod}(\alpha)$ of $\mathrm{Aut}(F^M)$ such that

$$\mathrm{mod}(\alpha)p_M(\varphi) = p_M(\varphi \circ \alpha^{-1}) , \qquad \alpha \in \mathrm{Aut}(M) .$$

DEFINITION 1.1. We call mod the fundamental homomorphism.

This name comes from the following:

PROPOSITION 1.2. *If M is a factor of type II_∞ with separable* *predual, then the map:* $\lambda \in \boldsymbol{R}_+^* \rightarrow F_\lambda^M \in \operatorname{Aut}(F^M)$ *is an isomorphism and for any* $\alpha \in \operatorname{Aut}(M)$ *and a faithful semi-finite normal trace* τ *we have*

$$\tau \circ \alpha^{-1} = \operatorname{mod}(\alpha)\tau$$

where $\operatorname{mod}(\alpha)$ *is identified to* $\lambda \in \boldsymbol{R}_+^*$ *with* $\operatorname{mod}(\alpha) = F_\lambda^M$.

PROOF. By assumption, F^M is transitive with trivial kernel, so that every automorphism of F^M is of the form F_λ^M, $\lambda \in \boldsymbol{R}_+^*$. Hence for any $\alpha \in \operatorname{Aut}(M)$ there exists $\lambda > 0$ such that $\varphi \circ \alpha^{-1} \sim \lambda\varphi$ for every integrable weight φ of infinite multiplicity. Since M is a factor, we have $\tau \circ \alpha^{-1} = \mu\tau$ for some $\mu > 0$. Let $\varepsilon > 0$. As in the proof of Theorem II.4.7, choose an $h \in M$, $1 - \varepsilon \leqq h \leqq 1 + \varepsilon$, such that $\varphi = \tau(h\cdot)$ is an integrable weight of infinite multiplicity. We have then $\lambda\varphi = \varphi \circ \alpha^{-1} \circ \operatorname{Ad}(u)$ for some unitary $u \in M$, so that for every $x \in M_+$,

$$\begin{aligned}\lambda\tau(hx) = \lambda\varphi(x) &= \tau(h\alpha^{-1}(uxu^*)) = \tau \circ \alpha^{-1}(\alpha(h)uxu^*) \\ &= \mu\tau(\alpha(h)uxu^*) = \mu\tau(u^*\alpha(h)ux)\ .\end{aligned}$$

Thus we get $\lambda h = \mu u^*\alpha(h)u$; hence $(1-\varepsilon)\lambda \leqq (1+\varepsilon)\mu$ and $(1-\varepsilon)\mu \leqq (1+\varepsilon)\lambda$. Therefore, $\lambda = \mu$, ε being arbitrary. q.e.d.

PROPOSITION 1.3. (i) *If* M *is a factor of type* III_λ, $0 < \lambda < 1$, *with separable predual, then the map:* $\lambda \in \boldsymbol{R}_+^* \rightarrow F_\lambda^M \in \operatorname{Aut}(F^M)$ *is a homomorphism of* $\boldsymbol{R}_+^*$ *onto* $\operatorname{Aut}(F^M)$ *with kernel* $S(M) \cap \boldsymbol{R}_+^*$, *and for any* $\alpha \in \operatorname{Aut}(M)$ *and a generalized trace* φ *on* M, [3; 4.3], *we have*

$$\varphi \circ \alpha^{-1} \sim \lambda\varphi \quad \textit{with} \quad \operatorname{mod}(\alpha) = F_\lambda^M\ .$$

(ii) *If* M *is of type* III_1 *instead, then* $\operatorname{mod}(\alpha) = 1$ *for every* $\alpha \in \operatorname{Aut}(M)$.

PROOF. (i) We know that the flow F^M is transitive with kernel $S(M) \cap \boldsymbol{R}_+^*$, so that the first assertion follows. Now let $\alpha \in \operatorname{Aut}(M)$ and φ be as above, and $\lambda_1, \lambda_2 \in \boldsymbol{R}_+^*$ be such that

$$\varphi \circ \alpha^{-1} \sim \lambda_1\varphi \quad \text{and} \quad \psi \circ \alpha^{-1} \sim \lambda_2\psi$$

for any integrable weight ψ of infinite multiplicity on M. As above, for any $\varepsilon > 0$ there exists an $h \in M_\varphi$, $1 - \varepsilon \leqq h \leqq 1 + \varepsilon$, such that $\varphi(h\cdot) = \psi$ is integrable and of infinite multiplicity. For some unitaries $u, v \in M$ we have $\psi \circ \alpha^{-1} = \lambda_2\psi_u$ and $\varphi \circ \alpha^{-1} = \lambda_1\varphi_v$, so that for any $x \in M_+$

$$\begin{aligned}\lambda_2\varphi(huxu^*) = \lambda_2\psi_u(x) &= \psi(\alpha^{-1}(x)) = \varphi(h\alpha^{-1}(x)) \\ &= \varphi(\alpha^{-1}(\alpha(h)x)) = \lambda_1\varphi(v\alpha(h)xv^*)\ ; \\ \lambda_2\varphi_u(u^*hux) &= \lambda_1\varphi_v(\alpha(h)x)\ .\end{aligned}$$

Hence we get $(D\varphi_u(u^*hu\cdot): D\varphi_v(\alpha(h)\cdot))_t = \lambda_1^{it}\lambda_2^{-it}$, $t \in \boldsymbol{R}$. Let T_0 be the generator of the modular period group $T(M)$. Then

$$(D\varphi_u(u^*hu\cdot): D\varphi_u)_{T_0}(D\varphi_u: D\varphi_v)_{T_0}(D\varphi_v: D\varphi_v(\alpha(h)\cdot))_{T_0} = \lambda_1^{iT_0}\lambda_2^{-iT_0} .$$

As we have

$$\begin{aligned}(D\varphi_u: D\varphi_v)_{T_0} &= (D\varphi_u: D\varphi)_{T_0}(D\varphi_v: D\varphi)^*_{T_0} \\ &= u^*\sigma^\varphi_{T_0}(u)\sigma^\varphi_{T_0}(v^*)v = 1 ,\end{aligned}$$

we get

$$\lambda_1^{iT_0}\lambda_2^{-iT_0} = (D\varphi_u(u^*hu\cdot): D\varphi_u)_{T_0}(D\varphi_v: D\varphi_v(\alpha(h)\cdot))_{T_0} .$$

The right hand side tends to 1 when $\varepsilon \to 0$, so that $\lambda_1\lambda_2^{-1}$ belongs to $S(M)$.

(ii) We know that the flow F^M is trivial for a factor of type III_1. q.e.d.

PROPOSITION 1.4. (i) *If M is an infinite factor with separable predual, then* Aut (F^M), *equipped with the simple convergence topology with respect to the norm topology in $(P_M)_*$, is a polish topological group.*

(ii) *If M is a factor of type* III_λ, $\lambda \neq 0$, *with separable predual, then the isomorphism of $\boldsymbol{R}_+^*/S(M) \cap \boldsymbol{R}_+^*$ onto* Aut (F^M), *given by Proposition* 1.3, *is a topological isomorphism.*

PROOF. (i) This follows from the fact that Aut (F^M) is a closed subgroup of the automorphism group Aut (P_M) of the separable abelian von Neumann algebra P_M.

(ii) The map: $\lambda \in \boldsymbol{R}_+^* \to F_\lambda^M \in \mathrm{Aut}\,(F^M)$ is continuous, so the isomorphism of $\boldsymbol{R}_+^*/\boldsymbol{R}_+^* \cap S(M)$ onto Aut (F^M) is continuous whose domain is compact. Hence it is a homomorphism. q.e.d.

We are now going to show the continuity of the fundamental homomorphism mod. Let M be an infinite factor with separable predual. We represent Aut (M) on the predual M_* by considering the transpose of each automorphism, then consider the pointwise convergence topology in Aut (M) as in the preliminary. What we are going to prove is that mod is a continuous homomorphism of Aut (M) into Aut (F^M).

LEMMA 1.5. *Let M be a von Neumann algebra with separable predual, and $\mathfrak{U}$ the unitary group of M with the uniform structure of the σ-strong* convergence. Let α be a continuous action of a separable locally compact group on M. Then the set $Z^1_\alpha(G, \mathfrak{U})$ of all $\mathfrak{U}$-valued continuous functions on G such that $u_{gh} = u_g\alpha_g(u_h)$, $g, h \in G$, is a Polish space with respect to the uniform convergence topology on compact sets in G.*

PROOF. Let d be a bounded complete metric of $\mathfrak{U}$ giving the uniform structure of the σ-strong* convergence. Let $\{K_n\}$ be an increasing sequence of compact sets in G such that $G = \bigcup_{n=1}^{\infty} \mathring{K}_n$, where $\mathring{K}_n$ means the interior of K_n. Put

$$\delta(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{g \in K_n} d(u_g, v_g) , \quad u, v \in Z_\alpha^1(G, \mathfrak{U}) .$$

It is not hard to see that δ is a complete metric on $Z_\alpha^1(G, \mathfrak{U})$ giving the uniform structure in question. Furthermore, $Z_\alpha^1(G, \mathfrak{U})$ is a closed subset of the separable complete metric space of $C(G, \mathfrak{U})$ of all continuous $\mathfrak{U}$-valued functions on G with the same metric δ. q.e.d.

PROPOSITION 1.6. *In the same situation as above, let* $\mathfrak{U}_0 = \{u \in \mathfrak{U}: \alpha_g(u) = u, g \in G\}$. *Then the map* $d: w \in \mathfrak{U} \longrightarrow dw \in Z_\alpha^1(G, \mathfrak{U})$ *with* $(dw)_g = w^*\alpha_g(w)$ *induces a Borel isomorphism* $\bar{d}$ *of the quotient Borel space* $\mathfrak{U}_0\backslash\mathfrak{U}$ *onto a Borel subset* B *of* $Z_\alpha^1(G, \mathfrak{U})$.

PROOF. Since $\mathfrak{U}_0$ is a closed subspace, $\mathfrak{U}_0\backslash\mathfrak{U}$ is a Polish space. Now we claim that the map d is continuous. By Akemann's result [1], the σ-strong* topology in a bounded set in M is given by the uniform convergence topology on every weakly compact set in M_*. It follows then that the map: $(\varphi, g) \in L \times G \longrightarrow \varphi \circ \alpha_g \in M_*$ is continuous on every weakly compact set L in M_*, where we consider the weak topology in M_*; hence the set $\{\varphi \circ \alpha_g: \varphi \in L, g \in K\}$ is weakly compact in M_* for any compact subset K of G and weakly compact subset L of M_*. Hence if $\{w_n\}$ is a sequence in $\mathfrak{U}$ converging to w, then $\{\langle \alpha_g(w_n), \varphi\rangle\}$ converges to $\langle \alpha_g(w), \varphi\rangle$ uniformly for $g \in K$ and $\varphi \in L$ as $n \longrightarrow \infty$; hence $\alpha_g(w_n)$ tends to $\alpha_g(w)$ uniformly in $\mathfrak{U}$ for $g \in K$. Since $\mathfrak{U}$ is a topological group, $w_n^*\alpha_g(w_n)$ converges to $w^*\alpha_g(w)$ uniformly for $g \in K$. Hence $d(w_n)$ converges to $d(w)$ in $Z_\alpha^1(G, \mathfrak{U})$, which means that d is continuous. Furthermore, $d(w_1) = d(w_2)$, $w_1, w_2 \in \mathfrak{U}$, if and only if $w_1 w_2^* \in \mathfrak{U}_0$. Therefore, d induces a continuous injective map $\bar{d}$ from $\mathfrak{U}_0\backslash\mathfrak{U}$ into $Z_\alpha^1(G, \mathfrak{U})$. Hence it follows from [17] that the induced map $\bar{d}$ is a Borel isomorphism from $\mathfrak{U}_0\backslash\mathfrak{U}$ onto a Borel subset B of $Z_\alpha^1(G, \mathfrak{U})$. q.e.d.

PROPOSITION 1.7. *Let* M *and* $\mathfrak{U}$ *be as before.*

(i) *The space* $\mathfrak{W}_M$ *of all faithful weights* ψ *on* M *is a Polish space with respect to the topology of uniform convergence of the* $(D\psi: D\varphi)_t$ *in* $\mathfrak{U}$ *on compact subsets of* $\boldsymbol{R}$ *with* $\varphi \in \mathfrak{W}_M$ *fixed; and this topology is independent of the choice of* φ.

(ii) *For a faithful weight* φ *on* M, *the set* $\{\psi \in \mathfrak{W}_M: \varphi \sim \psi\} = W_\varphi$ *is a Borel subset of* $\mathfrak{W}_M$, *and there exists a Borel map* $u: \psi \in W_\varphi \longrightarrow u(\psi) \in \mathfrak{U}$ *such that* $\varphi_{u(\psi)} = \psi$, $\psi \in W_\varphi$.

PROOF. (i) With $\varphi \in \mathfrak{W}_M$ fixed, the topology in $\mathfrak{W}_M$ is identified with that in $Z^1_{\sigma^\varphi}(\boldsymbol{R}, \mathfrak{U})$ under the correspondence: $\psi \leftrightarrow (D\psi: D\varphi) \in Z^1_{\sigma^\varphi}(\boldsymbol{R}, \mathfrak{U})$. Hence the first half of the assertion follows from Lemma 1.5. Let $\{\psi_n\}$ be a sequence in $\mathfrak{W}_M$ converging to ψ. Then $(D\psi_n: D\varphi)_t \rightarrow (D\psi: D\varphi)_t$ in $\mathfrak{U}$ uniformly on compact subsets of $\boldsymbol{R}$. For any other faithful weight φ',

$$(D\psi_n: D\varphi')_t = (D\psi_n: D\varphi)_t(D\varphi: D\varphi')_t \rightarrow (D\psi: D\varphi)_t(D\varphi: D\varphi')_t = (D\psi: D\varphi')_t$$

in $\mathfrak{U}$ uniformly on compact subsets of $\boldsymbol{R}$. Hence the topology in $\mathfrak{W}_M$ is independent of the choice of φ.

(ii) We apply Proposition 1.6 to $G = \boldsymbol{R}$ and $\alpha = \sigma^\varphi$. It follows then that $\varphi \sim \psi$, $\psi \in \mathfrak{W}_M$, if and only if $(D\psi: D\varphi) \in d(\mathfrak{U})$. Let f be a Borel cross-section from $\mathfrak{U}_0\backslash\mathfrak{U}$ to $\mathfrak{U}$, and put $u(\psi) = f \circ \bar{d}^{-1}(D\psi: D\varphi)$. Then u is a Borel map and $\varphi_{u(\psi)} = \psi$ by construction. q.e.d.

PROPOSITION 1.8. *Let M be as above, and* Aut(M) *be equipped with the simple norm convergence topology in M_*. For any $\varphi \in \mathfrak{W}_M$, the map: $\alpha \in$* Aut$(M) \rightarrow \varphi \circ \alpha^{-1} \in \mathfrak{W}_M$ *is continuous in the topology on $\mathfrak{W}_M$ defined above.*

PROOF. Let ψ be a faithful normal state on M. If $\alpha_n \rightarrow \alpha_0$ in Aut(M), then $\|\psi \circ \alpha_n^{-1} - \psi \circ \alpha^{-1}\| \rightarrow 0$. Hence by [4], $(D\psi \circ \alpha_n^{-1}: D\psi \circ \alpha_0^{-1})_t \rightarrow 1$, $n \rightarrow \infty$, uniformly on compact subsets of $\boldsymbol{R}$. For any $\varphi \in \mathfrak{W}_M$, we have

$$\begin{aligned}(D\varphi\circ\alpha_n^{-1}: D\varphi\circ\alpha_0^{-1})_t &= (D\varphi\circ\alpha_n^{-1}: D\psi\circ\alpha_n^{-1})_t(D\psi\circ\alpha_n^{-1}: D\psi\circ\alpha_0^{-1})_t(D\psi\circ\alpha_0^{-1}:D\varphi\circ\alpha_0^{-1})_t \\ &= \alpha_n((D\varphi: D\psi)_t)(D\psi \circ \alpha_n^{-1}: D\psi \circ \alpha_0^{-1})_t\alpha_0((D\psi: D\varphi)_t) .\end{aligned}$$

Thus we have only to prove that $\alpha_n(D\varphi: D\psi_t) \rightarrow \alpha_0((D\varphi: D\psi)_t)$ in $\mathfrak{U}$ uniformly on compact subsets of $\boldsymbol{R}$. Hence we will show that $\alpha_n(u) \rightarrow \alpha(u)$ in $\mathfrak{U}$ uniformly for u in a compact subset of K of $\mathfrak{U}$. For any $u, v \in \mathfrak{U}$, $\alpha, \beta \in$ Aut(M) and $\omega \in M_*$, we have

$$\begin{aligned}|\langle\alpha(u) - \beta(v), \omega\rangle| &\leqq |\langle u, \omega\circ\alpha - \omega\circ\beta\rangle| + |\langle u - v, \omega\circ\beta\rangle| \\ &\leqq \|\omega\circ\alpha - \omega\circ\beta\| + |\langle u - v, \omega\circ\beta\rangle| ,\end{aligned}$$

so that the map: $(\alpha, u) \in$ Aut$(M) \times \mathfrak{U} \rightarrow \alpha(u) \in \mathfrak{U}$ is continuous, because the σ-strong* topology and the σ-weak topology in $\mathfrak{U}$ coincide. Hence $A = \{\alpha_n(u): u \in K, n = 0, 1, \cdots\} \subset \mathfrak{U}$ is compact, so that the σ-weak uniform structure and the σ-strong* uniform structure agree in A. For any fixed $\omega \in M_*$, the set $B = \{\omega\circ\alpha_n: n = 0, 1, \cdots\}$ is compact in the norm topology. For any $\varepsilon > 0$, there exist $u_1, u_2, \cdots, u_m$ in K such that $\inf_{1\leqq i\leqq m} |\langle u-u_i, \omega\circ\alpha_n\rangle| < \varepsilon$ for every $u \in K$ and $n=0, 1, \cdots$, by Akemann's characterization [1] of the σ-strong* topology in M. Let n_0 be large enough so that $|\langle u_i, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle| < \varepsilon$ for every $n \geqq n_0$ and $i = 1, 2, \cdots, m$. We have then, for any $u \in K$ and $n > n_0$,

$$|\langle u, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle| \leqq |\langle u - u_i, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle| + |\langle u_i, \omega\circ\alpha_n - \omega\circ\alpha_0\rangle|$$
$$\leqq 2\varepsilon + \varepsilon = 3\varepsilon\ .$$

Thus $\{\alpha_n(u)\}$ converges to $\alpha_0(u)$ σ-weakly and uniformly for $u \in K$; hence it converges to $\alpha_0(u)$ σ-strongly* uniformly on K. q.e.d.

We are now at the position to state the continuity of γ_M.

THEOREM 1.9. *Let M be an infinite factor with separable predual. Then the fundamental homomorphism* mod *is a continuous homomorphism of* $\mathrm{Aut}(M)$ *into* $\mathrm{Aut}(F^M)$, *where we consider the simple norm convergence topologies in M_* for* $\mathrm{Aut}(M)$ *and in $(P_M)_*$ for* $\mathrm{Aut}(F^M)$ *respectively. Hence* $\mathrm{mod}(\alpha) = \iota$ *for every* $\alpha \in \overline{\mathrm{Int}(M)}$.

PROOF. We know, as in the preliminary, that $\mathrm{Aut}(M)$ is a Polish topological group as well as $\mathrm{Aut}(F^M)$. Hence we just have to prove that γ_M is a Borel map.

By construction, $\mathrm{mod}(\alpha) = \iota$ for every $\alpha \in \mathrm{Int}(M)$. Let $\bar\omega$ be a dominant weight on M, and $p_{\bar\omega}$ be the isomorphism of the center $C_{\bar\omega}$ of $M_{\bar\omega}$ onto P_M defined in Theorem I.1.11 and the proof of Theorem II.2.2. We claim that for any $\alpha \in \mathrm{Aut}(M)$ with $\bar\omega\circ\alpha^{-1} = \bar\omega$

$$(*)\qquad p_{\bar\omega}^{-1}\,\mathrm{mod}(\alpha)p_{\bar\omega} = \alpha|_{C_{\bar\omega}}\ .$$

To see this, let u be an isometry in M with $e = uu^* \in C_{\bar\omega}$. Then we have

$$\begin{aligned}\mathrm{mod}(\alpha)(p_M(\bar\omega_u)) &= p_M(\bar\omega_u\circ\alpha^{-1}) = p_M(\bar\omega_{\alpha(u)})\\ &= p_{\bar\omega}(\alpha(e)) \qquad \text{by Theorem I.1.11 (ii);}\end{aligned}$$

hence

$$\mathrm{mod}(\alpha)(p_{\bar\omega}(e)) = p_{\bar\omega}(\alpha(e))\ .$$

Let $u(\cdot)$ be the Borel map from the set $W_{\bar\omega}$ of dominant weights on M to the unitary group $\mathfrak{U}$ of M defined in Proposition 1.7(ii) such that $\bar\omega_{u(\psi)} = \psi$ for any dominant weight ψ. By Proposition 1.8, the map $h\colon \alpha \in \mathrm{Aut}(M) \to h(\alpha) = \mathrm{Ad}(u(\bar\omega\circ\alpha^{-1}))\circ\alpha \in \mathrm{Aut}(M)$ is a Borel map, since the map $\mathrm{Ad}\colon v \in \mathfrak{U} \to \mathrm{Ad}\,v \in \mathrm{Aut}(M)$ is continuous. We then have

$$\mathrm{mod}(\alpha) = \mathrm{mod}(\mathrm{Ad}(u(\bar\omega\circ\alpha^{-1})))\,\mathrm{mod}(\alpha),\ \alpha \in \mathrm{Aut}(M)\ ;$$
$$\bar\omega\circ h(\alpha)^{-1} = (\bar\omega\circ\alpha^{-1})_{u(\bar\omega\circ\alpha^{-1})} = \bar\omega\ ;$$

therefore

$$p_{\bar\omega}^{-1}\,\mathrm{mod}(\alpha)p_{\bar\omega} = h(\alpha)|_{C_{\bar\omega}} \qquad \text{by } (*).$$

This shows that mod is a Borel map. q.e.d.

THEOREM 1.10. *Let M be a factor of type* III_λ, $\lambda \neq 1$, *with separa-*

ble predual. Viewing the fundamental homomorphism mod *as a homomorphism of* $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M)$ *into* $\mathrm{Aut}(F^M)$ *by the trivial identification, the following three conditions for* $\bar{\alpha} \in \mathrm{Out}(M)$ *are equivalent*:

(i) $\mathrm{mod}(\bar{\alpha}) = \iota$;

(ii) *There exists a faithful normal state* φ *on* M *and a representative* α_0 *of* $\bar{\alpha}$ *such that*

$$\varphi \circ \alpha_0 = \varphi \quad \textit{and} \quad \alpha_0|_{C_\varphi} = \iota\ ;$$

(iii) *For any* $\varepsilon > 1$ *such that* $]\varepsilon^{-1}, \varepsilon[\cap S(M) = \{1\}$, *there exists a faithful normal state* φ *on* M *and a representative* α_0 *of* $\bar{\alpha}$ *satisfying* (ii) *and*

$$\mathrm{Sp}(\Delta_\varphi) \cap]\varepsilon^{-1}, \varepsilon[= \{1\}\ .$$

To prove the theorem, we need the following lemma which is a slight refinement of Lemma I.2.3 and [3; Lemma 5.2.4].

LEMMA 1.11. *If* ψ *is a faithful weight on a factor of type* III_λ, $\lambda \neq 1$, *then for any* $\varepsilon > 1$ *with* $]\varepsilon^{-1}, \varepsilon[\cap S(M) = \{1\}$ *there exists a positive* $h < C_\psi$ *such that, with* $\varphi = \psi(h\cdot)$ *and* $e = s(h)$,

$$\mathrm{Sp}(\Delta_\varphi) \cap]\varepsilon^{-1}, \varepsilon[= \{1\}\ ,$$

where Δ_φ *means of course the modular operator corresponding to* $\{M_e, \varphi\}$.

PROOF. This follows from Lemma I.2.3 and the observation that the operator $H \in M_{\psi_1}$ in the proof of Lemma I.2.3 is indeed in C_{ψ_1} because each spectral projection of H is given by the left support projection of $M(\sigma^{\psi_1}, V)$ for each closed subset V of $\boldsymbol{R}$ which belongs to C_{ψ_1}. q.e.d.

PROOF OF THEOREM 1.10. (i) $\Rightarrow$ (iii): Suppose $\gamma_M(\bar{\alpha}) = \iota$ and $\bar{\omega}$ is a dominant weight on M. There exists a representative α_1 of $\bar{\alpha}$ such that $\bar{\omega} \circ \alpha_1 = \bar{\omega}$ and $\alpha_1|_{C_{\bar{\omega}}} = \iota$. Let $h \in C_{\bar{\omega}}$ be a positive operator such that $\varphi = \bar{\omega}(h\cdot)$ satisfies the condition in Lemma 1.11. It follows then that $\varphi \circ \alpha_1 = \varphi$. Since $M_\varphi \supset M_{\bar{\omega}_e}$ with $e = s(h)$, we have $C_\varphi \subset C_{\bar{\omega}_e} = C_{\bar{\omega},e}$ by Theorem II.5.1. Therefore, we have

$$\varphi \circ \alpha_1 = \varphi \quad \text{and} \quad \alpha_1|_{C_\varphi} = \iota\ .$$

Being lacunary, φ is strictly semi-finite, so that the restriction τ of φ to M_φ is a faithful semi-finite normal trace. Since α_1 leaves τ invariant and C_φ elementwise fixed, we have $\alpha_1(p) \sim p$ in M_φ for every projection $p \in M_\varphi$. Let p be a projection in M_φ such that $\varphi(p) < +\infty$. It follows then that $\psi = (1/\varphi(p))\varphi_p$ is a normal state of M. Let u be a unitary in M_φ such that $upu^* = \alpha_1(p)$. Put $\alpha_2 = \mathrm{Ad}(u)^{-1} \circ \alpha_1 \in \bar{\alpha}$. We have then

$\psi \circ \alpha_2 = \psi$ and that α_2 leaves C_ψ elementwise fixed. Let w be an isometry of M such that $ww^* = p$. Put

$$\alpha_0(x) = w^*\alpha_2(wxw^*)w\ , \qquad x \in M\ ;$$
$$\psi_0 = \psi_w\ .$$

We have that ψ_0 is a faithful normal state on M, $\psi_0 \circ \alpha_0 = \psi_0$ and α_0 leaves C_{ψ_0} elementwise fixed. Since $\alpha_0 = \mathrm{Ad}\,(w^*\alpha_2(w)) \circ \alpha_2$ and $w^*\alpha_2(w)$ is unitary, α_0 belongs to $\bar{\alpha}$. Thus (iii) follows.

(iii) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (i): Let $\alpha_0 \in \mathrm{Aut}\,(M)$ and φ be a faithful normal state on M satisfying the condition in (ii). We consider the tensor products $\bar{M} = M \otimes F_\infty$, $\bar{\omega} = \varphi \otimes \omega$ and $\alpha_0 \otimes \iota = \tilde{\alpha}_0$. From the proof of Theorem II.5.1, it follows that the center $C_{\bar{\omega}}$ of $\bar{M}_{\bar{\omega}}$ is a von Neumann subalgebra of $C_\varphi \otimes U(L^\infty(\boldsymbol{R}))$. Since $\alpha_0|_{C_\varphi} = \iota$, $\tilde{\alpha}_0$ leaves $C_\varphi \otimes U(L^\infty(\boldsymbol{R}))$ elementwise fixed. Hence $C_{\bar{\omega}}$ is fixed elementwise by $\tilde{\alpha}_0$. Therefore, we have $\mathrm{mod}\,(\tilde{\alpha}_0) = \mathrm{mod}\,(\alpha_0) = 1$. q.e.d.

IV.2. The extended modular automorphism groups. Throughout this section, let M be an infinite factor with separable predual, P_M, p_M, F^M and so on be as before. Let $Z^1(F^M)$ be the set of all σ-strongly* continuous functions $\{c_\lambda\}$ on $\boldsymbol{R}_+^*$ with values in the unitary group of P_M such that

$$c_{\lambda\mu} = c_\lambda F_\lambda^M(c_\mu)\ , \qquad \lambda, \mu \in \boldsymbol{R}_+^*\ ,$$

and $B^1(F^M)$ be the set of all elements in $Z^1(F^M)$ of the form: $\lambda \in \boldsymbol{R}_+^* \rightarrow v^*F_\lambda^M(v)$ for some unitary $v \in P_M$. Under the pointwise multiplication, $Z^1(F^M)$ is an abelian group, and $B^1(F^M)$ is a subgroup of $Z^1(F^M)$. Put

$$H^1(F^M) = Z^1(F^M)/B^1(F^M)\ .$$

For each $t \in \boldsymbol{R}$, let $\bar{t}$ denote the element in $Z^1(F^M)$ defined by

$$\bar{t}(\lambda) = \lambda^{it}\ , \qquad \lambda \in \boldsymbol{R}_+^*\ .$$

PROPOSITION 2.1. *If φ is an integrable faithful weight on M, then to each $c \in Z^1(F^M)$ there corresponds a unique automorphism $\bar{\sigma}_c^\varphi$ of M such that*

(i) $\bar{\sigma}_c^\varphi(x) = p_\varphi^{-1}(c_\lambda p_M(\varphi))x$ *for every* $x \in M(\sigma^\varphi, \{\lambda\})$, $\lambda > 0$;

(ii) $\varphi \circ \bar{\sigma}_c^\varphi = \varphi$ *and* $\bar{\sigma}_{c_1c_2}^\varphi = \bar{\sigma}_{c_1}^\varphi \circ \bar{\sigma}_{c_2}^\varphi$, $c_1, c_2 \in Z^1(F^M)$;

(iii) $\bar{\sigma}_{\bar{t}}^\varphi = \sigma_t^\varphi$, $t \in \boldsymbol{R}$.

PROOF. (i) The uniqueness of $\bar{\sigma}_c^\varphi$ follows from Lemma II.2.3. Let $M = W^*(N, \boldsymbol{R}, \theta)$ be a continuous decomposition of M, and τ be a faithful semi-finite normal trace on N such that $\tau \circ \theta_s = e^{-s}\tau$, $s \in \boldsymbol{R}$. Let $\{u(s): s \in \boldsymbol{R}\}$

be the one parameter unitary group in M canonically associated with the decomposition $W^*(N, \boldsymbol{R}, \theta) = M$. We know that the dual weight $\bar{\omega} = \tilde{\tau}$ is dominant, and that $\bar{\omega}_{u(s)} = e^{-s}\bar{\omega}$ and $F^M_\lambda \circ p_{\bar{\omega}}(x) = p_{\bar{\omega}} \circ \theta_{-\mathrm{Log}\,\lambda}(x)$ for every x in the center C of N and $\lambda > 0$. For a fixed $c \in Z^1(F^M)$, we put

$$b_s = p_{\bar{\omega}}^{-1}(c_{e^s}) , \qquad s \in \boldsymbol{R} .$$

It follows then that b_s is a unitary in C and

$$b_{s+t} = b_s\theta_s(b_t) , \qquad s, t \in \boldsymbol{R} .$$

Hence there exists a unique automorphism $\bar{\sigma}_c$ of $M = W^*(N, \boldsymbol{R}, \theta)$ such that

$$\bar{\sigma}_c(au(s)) = b_s au(s) , \qquad a \in N, s \in \boldsymbol{R} .$$

Thus we have shown that $\bar{\sigma}_c^{\bar{\omega}}$ exists for a dominant weight $\bar{\omega}$ on M.

Now, let v be an isometry in M with $vv^* = e \in M_{\bar{\omega}} = N$ such that $\varphi = \bar{\omega}_v$. Observing that e is fixed under $\bar{\sigma}_c^{\bar{\omega}}$, we define an automorphism α of M by

$$\alpha(x) = v^*\bar{\sigma}_c^{\bar{\omega}}(vxv^*)v , \qquad x \in M .$$

Since the map: $x \in M \longrightarrow vxv^* \in M_e$ is an isomorphism of M onto M_e which brings α to $\bar{\sigma}_c^{\bar{\omega}}$ and σ_t^φ to $\sigma_t^{\bar{\omega}}$, $t \in \boldsymbol{R}$, we have

$$\alpha(x) = v^*p_{\bar{\omega}}^{-1}(c_\lambda)vx , \qquad x \in M(\sigma^\varphi, \{\lambda\}) .$$

Thus we must show that

$$v^*p_{\bar{\omega}}^{-1}(a)v = p_\varphi^{-1}(ap_M(\varphi)) , \qquad a \in P_M .$$

To this end, we may assume that $a = P_M(\psi)$ for some integrable ψ, since $p_M(\psi)$'s generate P_M. We have then

$$\begin{aligned} p_\varphi(v^*p_{\bar{\omega}}^{-1}(a)v) &= p_\varphi(v^*c_{\bar{\omega}}(\psi)v) \quad \text{by Theorem I.1.11,} \\ &= p_\varphi(c_{\bar{\omega}_v}(\psi)) \quad \text{by Lemma I.1.6,} \\ &= p_\varphi(c_\varphi(\psi)) = p_M(\psi)p_M(\varphi) \quad \text{by Theorem I.1.11,} \\ &= ap_M(\varphi) . \end{aligned}$$

Thus α satisfies the requirement for $\bar{\sigma}_c^\varphi$.

(ii) We know that $\bar{\omega} \circ \bar{\sigma}_c^{\bar{\omega}} = \bar{\omega}$ by construction. Thus $\bar{\sigma}_c^\varphi$, namely α, preserves φ by definition.

(iii) If $c = \bar{t}$, then $c_\lambda = \lambda^{it}$, so that we get

$$p_\varphi(c_\lambda p_M(\varphi)) = \lambda^{it} , \qquad \lambda > 0 .$$

Hence $\bar{\sigma}_c^\varphi = \sigma_t^\varphi$. q.e.d.

THEOREM 2.2. *Let φ be an integrable weight on M. If $\alpha \in \mathrm{Aut}\,(M)$*

leaves M_φ elementwise fixed, then $\alpha = \bar{\sigma}_c^\varphi$ for some $c \in Z^1(F^M)$.

PROOF. Let $\bar{\omega}$ be dominant, and $M = W^*(N, \boldsymbol{R}, \theta)$ be the associated continuous decomposition of M and $\{u(s)\}$ the one parameter unitary group in M appearing in the decomposition. First we assume that α is an automorphism of M leaving N elementwise fixed. For each $s \in \boldsymbol{R}$, let $b_s = \alpha(u(s))u(s)^*$. By Theorem II.5.1, b_s belongs to the center C of N and

$$b_{s+t} = b_s\theta_s(b_t)\ , \qquad s, t \in \boldsymbol{R}\ .$$

Furthermore, we have

$$\alpha(xu(s)) = b_s xu(s)\ , \qquad x \in N, s \in \boldsymbol{R}\ .$$

Hence, putting $c_\lambda = p_{\bar{\omega}}(b_{-\mathrm{Log}\,\lambda})$, $\lambda > 0$, we get $\alpha = \bar{\sigma}_c^{\bar{\omega}}$.

In the general case, there is an isometry u with $uu^* = e \in N$ such that $\varphi = \bar{\omega}_u$. Suppose that $\alpha \in \mathrm{Aut}\,(M)$ leaves M_φ elementwise invariant. Considering the automorphism: $x \in M_e \to u\alpha(u^*xu)u^* \in M_e$, we may assume that $\alpha \in \mathrm{Aut}\,(M_e)$ leaves N_e elementwise invariant.

For every $x \in N_e$ and $s \in \boldsymbol{R}$, we have

$$\begin{aligned} x\alpha(eu(s)e)eu(s)^*e &= \alpha(xeu(s)e)eu(s)^*e \\ &= \alpha(eu(s)e\theta_{-s}(xe))eu(s)^*e \\ &= \alpha(eu(s)e)\theta_{-s}(xe)eu(s)^*e \\ &= \alpha(eu(s)e)eu(s)^*exe\theta_s(e)\ , \end{aligned}$$

so that $b_s = \alpha(eu(s)e)eu(s)^*e \in Ce\theta_s(e)$. A direct computation shows that

$$b_{s+t}\theta_s(e) = b_s\theta_s(b_t)\ , \qquad s, t \in \boldsymbol{R}\ .$$

Thus there exists, by Proposition A.1, $b' \in Z_\theta^1(\boldsymbol{R}, \mathfrak{U}_C)$ such that $b_s = b'_s e\theta_s(e)$, $s \in \boldsymbol{R}$. Define an $\alpha' \in \mathrm{Aut}\,(M)$ by

$$\alpha'(xu(s)) = b'_s xu(s)\ , \qquad x \in N, s \in \boldsymbol{R}\ .$$

We have then

$$\alpha(x) = \alpha'(x) \quad \text{for every} \quad x \in M_e\ .$$

Putting $c_\lambda = p_{\bar{\omega}}(b'_{-\mathrm{Log}\,\lambda})$, we have $\alpha' = \bar{\sigma}_c^{\bar{\omega}}$, so that $\alpha = \bar{\sigma}_c^\varphi$. q.e.d.

EXAMPLE 2.3. Let N be an infinite semi-finite factor with separable predual. We identify $\{P_N, F^N\}$ with $L^\infty(\boldsymbol{R}_+^*, d\lambda)$ acted by the translation of $\boldsymbol{R}_+^*$ as in II.2. We then conclude the following:

(i) For every $c \in Z^1(F^N)$ there exists a unique, up to scalar multiple, unitary $f \in L^\infty(\boldsymbol{R}_+^*, d\lambda)$ such that $c_\lambda = fF_\lambda(f^*)$, $\lambda > 0$.

(ii) With $c = df$ as in (i), and $\varphi = \tau(h_\varphi\cdot)$ as integrable weight, we have

$$\bar{\sigma}_c^\varphi = \mathrm{Ad}\,(f(h_\varphi))\,.$$

PROOF. (i) This is known.

(ii) We have first that $\sigma_t^\varphi = \mathrm{Ad}\,(h_\varphi^{it})$, $t \in \boldsymbol{R}$. The integrability of φ implies that the spectrum of h_φ is absolutely continuous with respect to the Lebesgue measure, so that $f(h_\varphi) = u$ makes sence. Let a be a partial isometry in $N(\sigma^\varphi, \{\lambda\})$, $\lambda > 0$. We have then $h_\varphi^{it} a h_\varphi^{-it} = \lambda^{it} a$, $t \in \boldsymbol{R}$, so that $a^* h_\varphi^{it} a = (\lambda h_\varphi)^{it} a^* a$. Therefore, we get

$$\begin{aligned} a^* f(h_\varphi) a &= f(\lambda h_\varphi) a^* a\,; \\ f(h_\varphi) a f(h_\varphi)^* &= a f(h_\varphi)^* f(\lambda h_\varphi) a^* a \\ &= f(\lambda^{-1} h_\varphi)^* f(h_\varphi) a = p_\varphi^{-1}(c_\lambda p_M(\varphi)) a \\ &= \bar{\sigma}_c^\varphi(a)\,. \end{aligned}$$

Therefore, $\bar{\sigma}_c^\varphi$ and $\mathrm{Ad}\,(f(h_\varphi))$ agree on the set of partial isometries in $N(\sigma^\varphi, \{\lambda\})$, $\lambda > 0$. But any element of $N(\sigma^\varphi, \{\lambda\})$ is the product of a positive element in $N_\varphi = \{h_\varphi\}' \cap N$ and a partial isometry in $N(\sigma^\varphi, \{\lambda\})$ by polar decomposition. Thus $\bar{\sigma}_c^\varphi = \mathrm{Ad}\,(f(h_\varphi))$. q.e.d.

This example shows what we deal with by considering $\bar{\sigma}_c^\varphi$: it may be called a "functional calculus" of the "generator" of the modular automorphism group.

THEOREM 2.4. *Let φ_1 and φ_2 be faithful integrable weights on an infinite factor M with separable predual, and $P = M \otimes F_2$. Put*

$$\varphi\Big(\sum_{i,j=1}^{2} x_{i,j} \otimes e_{i,j}\Big) = \varphi_1(x_{11}) + \varphi_2(x_{22}), \quad x = \sum_{i,j=1}^{2} x_{i,j} \otimes e_{i,j} \in P\,.$$

We then conclude the following:

(i) *To each $c \in Z^1(F^M)$, there corresponds a unique unitary $u_c = (D\varphi_2 : D\varphi_1)_c$ in M such that*

$$\bar{\sigma}_c^\varphi(1 \otimes e_{21}) = u_c \otimes e_{21}\,;$$

(ii) *We have*

$$\begin{aligned} \bar{\sigma}_c^{\varphi_2}(x) &= u_c \bar{\sigma}_c^{\varphi_1}(x) u_c^*\,, \qquad x \in M, \quad c \in Z^1(F^M)\,; \\ u_{c_1 c_2} &= u_{c_1} \bar{\sigma}_{c_1}^{\varphi_1}(u_{c_2})\,, \qquad c_1, c_2 \in Z^1(F^M)\,. \end{aligned}$$

PROOF. The integrability of φ follows from that of φ_1 and φ_2. Noticing that $1 \otimes e_{i,i} \in P_\varphi$, $i = 1, 2$, and $\bar{\sigma}_c^\varphi(x \otimes e_{ii}) = \bar{\sigma}_c^{\varphi_i}(x) \otimes e_{ii}$, $i = 1, 2$, we follow the arguments for the unitary cocycle Radon-Nikodym theorem, without any alteration. q.e.d.

COROLLARY 2.5. *Let M be an infinite factor with separable predual. Let ε_M denote the canonical homomorphism of* $\mathrm{Aut}\,(M)$ *onto* $\mathrm{Out}\,(M) = \mathrm{Aut}\,(M)/\mathrm{Int}\,(M)$.

(i) *For every* $c \in Z^1(F^M)$, *the element* $\varepsilon_M(\bar{\sigma}_c^{\varphi})$ *of* Out(M) *is independent of the choice of an integrable weight* φ. *Put* $\bar{\delta}_M(c) = \varepsilon_M(\bar{\sigma}_c^{\varphi})$.

(ii) $\bar{\delta}_M$ *is an extension of the modular homomorphism* ($\bar{\delta}_M(\bar{t}) = \delta_M(t)$, $t \in \boldsymbol{R}$) *and* Ker $\bar{\delta}_M = B^1(F^M)$.

(iii) *The range of* $\bar{\delta}_M$ *is a normal subgroup of* Out(M) *with*

$$\alpha\bar{\delta}_M(c)\alpha^{-1} = \bar{\delta}_M(\mathrm{mod}(\alpha)c) , \qquad \alpha \in \mathrm{Out}(M) .$$

PROOF. (i) Trivial from the previous theorem.

(ii) The first half follows from Proposition 2.1(iii). Let $\bar{\omega}$ be a dominant weight and $c \in Z^1(F^M)$. Assume that $\bar{\sigma}_c^{\bar{\omega}} = \mathrm{Ad}(u)$. Since $\bar{\sigma}_c^{\bar{\omega}}$ leaves $M_{\bar{\omega}}$ pointwise fixed, we have $u \in C_{\bar{\omega}}$ by Theorem II.5.1. It follows then that

$$c_\lambda = p_{\bar{\omega}}(u)^* F_\lambda^M(p_{\bar{\omega}}(u)) , \qquad \lambda > 0 .$$

The converse is proven the same way.

(iii) Let $\bar{\omega}$ be dominant as before, and $\alpha \in \mathrm{Aut}(M)$. Multiplying α by an inner automorphism, we assume $\bar{\omega} \circ \alpha = \bar{\omega}$, so that

$$p_{\bar{\omega}}^{-1} \circ \mathrm{mod}(\alpha) \circ p_{\bar{\omega}} = \alpha|_{C_{\bar{\omega}}} .$$

If x is an element of $M(\sigma^{\bar{\omega}}, \{\lambda\})$, then $\alpha^{-1}(x) \in M(\sigma^{\bar{\omega}}, \{\lambda\})$, because α and $\sigma^{\bar{\omega}}$ commute; hence

$$\begin{aligned} \alpha \circ \bar{\sigma}_c^{\bar{\omega}} \circ \alpha^{-1}(x) &= \alpha(p_{\bar{\omega}}^{-1}(c_\lambda)\alpha^{-1}(x)) = \alpha(p_{\bar{\omega}}^{-1}(c_\lambda))x \\ &= p_{\bar{\omega}}^{-1}(\mathrm{mod}(\alpha)c_\lambda)x . \end{aligned}$$

q.e.d.

THEOREM 2.6. *Let M be an infinite factor with separable predual, and $\mathfrak{W}_M^0$ the space of all faithful weights on M with the metric d defined in* II.4. *If $c \in Z^1(F^M)$ is twice continuously differentiable in norm, then there exist uniquely maps: $\varphi \in \mathfrak{W}_M^0 \to \bar{\sigma}_c^{\varphi} \in$* Aut$(M)$ *and $(\varphi, \psi) \in \mathfrak{W}_M^0 \times \mathfrak{W}_M^0 \to (D\varphi : D\psi)_c \in \mathfrak{U}(M)$, the unitary group of M, with the following properties*:

(i) *If φ is integrable, then $\bar{\sigma}_c^{\varphi}$ satisfies condition* (i) *in Proposition* 2.1. *If φ and ψ are both integrable, then $(D\varphi : D\psi)_c$ is given by Theorem* 2.4(i);

(ii) *The both maps are continuous with respect to the norm topologies in* Aut(M) *and* $\mathfrak{U}(M)$;

(iii) *For each $x \in M$, we have*

$$\bar{\sigma}_c^{\varphi}(x) = (D\varphi : D\psi)_c \bar{\sigma}_c^{\psi}(x)(D\varphi : D\psi)_c^* ;$$

(iv) *For each $\varphi_1, \varphi_2, \varphi_3 \in \mathfrak{W}_M^0$, we have*

$$(D\varphi_1 : D\varphi_3)_c = (D\varphi_1 : D\varphi_2)_c(D\varphi_2 : D\varphi_3)_c ;$$
$$(D\varphi_1 : D\varphi_2)_c = (D\varphi_2 : D\varphi_1)_c^* ;$$

(v) *For any* $\alpha \in \mathrm{Aut}(M)$ *and* $u \in \mathfrak{U}(M)$, *we have*

$$\bar{\sigma}_c^{\varphi\circ\alpha} = \alpha^{-1} \circ \bar{\sigma}^{\varphi}_{\mathrm{mod}\,(\alpha)c} \circ \alpha \ ;$$
$$(D\varphi \circ \alpha : D\psi \circ \alpha)_c = \alpha^{-1}((D\varphi : D\psi)_{\mathrm{mod}\,(\alpha)c}) \ ;$$
$$(D\varphi_u : D\psi)_c = u^*(D\varphi : D\psi)_c \bar{\sigma}_c^{\psi}(u) \ ;$$

(vi) *If* $c_1, c_2 \in Z^1(F^M)$ *are twice differentiable in norm, then*

$$\bar{\sigma}^{\varphi}_{c_1c_2} = \bar{\sigma}^{\varphi}_{c_1} \circ \bar{\sigma}^{\varphi}_{c_2} \ ;$$
$$(D\varphi : D\psi)_{c_1c_2} = (D\varphi : D\psi)_{c_1} \bar{\sigma}^{\psi}_{c_1}((D\varphi : D\psi)_{c_2}) \ .$$

The uniqueness of these maps follows from Proposition 2.1 and the density of integrable weights in $\mathfrak{W}^0_M$.

LEMMA 2.7. *Let* $c \in Z^1(F^M)$ *be as in the theorem. For any* $\varepsilon > 0$ *there exists* $\eta > 0$ *such that for any faithful integrable weight* φ *on* M:

$$x \in M(\sigma^{\varphi}, [e^{-\eta}, e^{\eta}]) \Longrightarrow \| \bar{\sigma}_c^{\varphi}(x) - x \| \leqq \varepsilon \| x \| \ .$$

PROOF. Without loss of generality, we may assume that φ is dominant. Put $b_s = p_{\varphi}^{-1}(c_{e^s})$, $s \in \boldsymbol{R}$. Let $\{u(s)\}$ be the one parameter unitary group in M which, together with M_{φ}, generate M as a continuous decomposition $M = W^*(M_{\varphi}, \boldsymbol{R}, \theta)$. We then have

$$\bar{\sigma}_c^{\varphi}(u(s)) = b_s u(s) \ , \qquad s \in \boldsymbol{R} \ .$$

If f is a function in the Schwartz space $\mathscr{S}(\boldsymbol{R})$, then the M-valued function: $s \in \boldsymbol{R} \longrightarrow \int_{-\infty}^{\infty} e^{-isp} f(p) b_p dp \in M$ is integrable by the twice differentiability of $\{b_p\}$ and we have

$$\bar{\sigma}_c^{\varphi}(\sigma_{\hat{f}}^{\varphi}(x)) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-isp} f(p) b_p dp \right) \sigma_s^{\varphi}(x) ds \ , \qquad x \in M \ ,$$

where we recall that the measures dp and ds are the Plancherel measures on $\boldsymbol{R}$. Put

$$a_s = \int_{-\infty}^{\infty} e^{-isp} f(p) b_p dp \ , \qquad s \in \boldsymbol{R} \ .$$

It follows then that

$$\sigma_{\hat{f}}^{\varphi}(x) - \bar{\sigma}_c^{\varphi} \circ \sigma_{\hat{f}}^{\varphi}(x) = \int_{-\infty}^{\infty} (\hat{f}(s) - a_s) \sigma_s^{\varphi}(x) ds \ .$$

Let g be a function in $L^1(\boldsymbol{R})$ such that $\hat{g}(p) = 1$ for p in a neighborhood of 0. If $f(0) = 1$, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\hat{f}(s) - a_s) g(t) \sigma_t^{\varphi}(x) ds dt = (f(0) - b_0) \sigma_g^{\varphi}(x) = 0 \ .$$

Hence we have

$$\sigma_g^\varphi \circ \sigma_{\hat f}^\varphi(x - \bar\sigma_c^\varphi(x)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\hat f(s) - a_s)g(t)\sigma_{s+t}^\varphi(x)dsdt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\hat f(s) - a_s)(g(t-s) - g(t))\sigma_t^\varphi(x)dsdt ;$$

$$\|\sigma_g^\varphi \circ \sigma_{\hat f}^\varphi(x - \bar\sigma_c^\varphi(x))\| \leqq \|x\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|\hat f(s) - a_s\|\,|g(t-s) - g(t)|dsdt .$$

Put $h(s) = \|\hat f(s) - a_s\|$. Then h belongs to $L^1(\boldsymbol{R})$. Hence there exists a sequence $\{g_n\}$ in $L^1(\boldsymbol{R})$ by [25; page 50] such that $\hat g_n(p) = 1$ for $|p| < 1/n$ and

$$\varepsilon_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(s)|g_n(t-s) - g_n(t)|dsdt \longrightarrow 0 \quad \text{as} \quad n \longrightarrow 0 .$$

If $f(p) = 1$ for $|p| < 1/n$, then we have

$$\sigma_{g_n}^\varphi \circ \sigma_{\hat f}^\varphi(x - \bar\sigma_c^\varphi(x)) = x - \bar\sigma_c^\varphi(x) , \qquad x \in M(\sigma^\varphi, [e^{-1/n}, e^{1/n}]) .$$

Thus the conclusion follows. q.e.d.

LEMMA 2.8. *Let $c \in Z^1(F^M)$ be as in Theorem 2.6. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for every faithful integrable weights φ_1 and φ_2 with $d(\varphi_1, \varphi_2) \leqq \eta$ we have*

$$\|(D\varphi_2 : D\varphi_1)_c - 1\| \leqq \varepsilon .$$

PROOF. We keep the notations in Theorem 2.4. It follows from II.4 that $d(\varphi_1, \varphi_2) \leqq \eta$ means $1 \otimes e_{21} \in P(\sigma^\varphi, [e^{-\eta}, e^{\eta}])$. Hence, choosing $\eta > 0$ as in Lemma 2.7, we get

$$\|u_c - 1\| = \|\bar\sigma_c^\varphi(1 \otimes e_{21}) - 1\| \leqq \varepsilon .$$

q.e.d.

LEMMA 2.9. *Let $c \in Z^1(F^M)$ be as in Theorem 2.6. Let φ be a faithful weight of infinite multiplicity.*

(a) *$\{\varphi_n\}$ is a sequence of faithful integrable weights such that $\lim_{n\to\infty} d(\varphi, \varphi_n) = 0$, then the sequence $\{\bar\sigma_c^{\varphi_n}\}$ of automorphisms converges to an automorphism, say $\bar\sigma_c^\varphi$, of M.*

(b) *$\bar\sigma_c^\varphi$ does not depend on the choice of a sequence $\{\varphi_n\}$ and satisfies*

$$\varphi \circ \bar\sigma_c^\varphi = \varphi \quad \text{and} \quad \bar\sigma_c^\varphi|_{M_\varphi} = \iota .$$

PROOF. Since we have, by the definition of $(D\varphi : D\psi)$,

$$\|(D\varphi_m : D\varphi_1)_c - (D\varphi_n : D\varphi_1)_c\| = \|(D\varphi_m : D\varphi_n)_c - 1\| ,$$

it follows from Lemma 2.8 that $\{(D\varphi_n : D\varphi_1)_c\}$ is a Cauchy sequence of unitaries in M. Put

$$(D\varphi: D\varphi_1)_c = u_c = \lim_{n\to\infty} (D\varphi_n: D\varphi_1)_c ,$$

and

$$\bar{\sigma}_c^{\varphi} = \mathrm{Ad}\,(u_c) \circ \bar{\sigma}_c^{\varphi_1} .$$

It follows also from Lemma 2.8 that $(D\varphi: D\varphi_1)_c$ does not depend on the choice of a sequence $\{\varphi_n\}$ but only on φ and φ_1. By construction, we have

$$\lim_{n\to\infty} \|\bar{\sigma}_c^{\varphi} - \bar{\sigma}_c^{\varphi_n}\| = 0 .$$

Let $\{\varphi_n\}$ be a sequence of faithful integrable weights given by $\varphi_n = \varphi(h_n \cdot)$ with $h_n \in M_\varphi$ such that $h_n \leqq h_{n+1}$ and $\lim_{n\to\infty} \|h_n - 1\| = 0$. We have then, for any $x \in M_+$,

$$\begin{aligned}\varphi(x) &= \lim_{n\to\infty} \varphi(x^{1/2}h_n x^{1/2}) = \lim_{n\to\infty} \varphi_n(x) \\ &= \lim_{n\to\infty} \varphi_n \circ \bar{\sigma}_c^{\varphi_n}(x) = \lim_{n\to\infty} \varphi(h_n^{1/2}\bar{\sigma}_c^{\varphi_n}(x)h_n^{1/2}) \\ &\geqq \varphi \circ \bar{\sigma}_c^{\varphi}(x)\end{aligned}$$

by the lower semi-continuity of φ. Replacing c by c^{-1}, we have $\varphi(x) \geqq \varphi \circ \bar{\sigma}_c^{\varphi-1}(x)$. Therefore, we get $\varphi \circ \bar{\sigma}_c^{\varphi} = \varphi$. Let ψ be an integrable faithful weight with $d(\varphi, \psi) < \varepsilon$. Then we have $M_\varphi \subset M(\sigma^\psi, [e^{-2\varepsilon}, e^{2\varepsilon}])$. Therefore, Lemma 2.7 entails the last assertion of (b). q.e.d.

PROOF OF THEOREM 2.6. With possible exception for (vi), all statements for faithful integrable weights follow from Proposition 2.1, Theorem 2.4 and Lemma 2.8. Let $\varphi \in \mathfrak{W}_M^0$ be integrable and $\alpha \in \mathrm{Aut}\,(M)$. It follows then that

$$p_{\varphi\circ\alpha} = \mathrm{mod}\,(\alpha)^{-1} \circ p_\varphi \circ \alpha ;$$

hence for each $x \in M(\sigma^{\varphi\circ\alpha}, \{\lambda\})$ we have

$$\begin{aligned}\bar{\sigma}_c^{\varphi\circ\alpha}(x) &= p_{\varphi\circ\alpha}^{-1}(c_\lambda)x = [\alpha^{-1} \circ p_\varphi^{-1} \circ \mathrm{mod}\,(\alpha)(c_\lambda)]x \\ &= \alpha^{-1}(p_\varphi^{-1}(\mathrm{mod}\,(\alpha)(c_\lambda))\alpha(x)) \\ &= \alpha^{-1} \circ \bar{\sigma}_{\mathrm{mod}\,(\alpha)c}^{\varphi}(\alpha(x)) .\end{aligned}$$

Hence we get the first part of (vi) for integrable weights. The last two formulas for integrable weights follow from this and the usual 2×2-matrix arguments.

Let φ_0 and ψ_0 be arbitrarily fixed faithful integrable weights. For each faithful weight φ of infinite multiplicity, we put

$$(D\varphi: D\varphi_0)_c = \lim_{n\to\infty} (D\varphi_n: D\varphi_0)_c$$

with a sequence $\{\varphi_n\}$ of faithful integrable weights converging to φ in the metric d. We know that this does not depend on the choice of $\{\varphi_n\}$. We define

$$(D\varphi : D\psi)_c = (D\varphi : D\varphi_0)_c (D\psi : D\varphi_0)_c^*$$

for a pair φ, ψ of faithful weights of infinite multiplicity. With sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of integrable weights converging to φ and ψ, we have

$$\begin{aligned}
&\lim_{n\to\infty} (D\varphi_n : D\varphi_0)_c (D\psi_n : D\varphi_0)_c^* \\
&\quad = \lim_{n\to\infty} (D\varphi_n : D\varphi_0)_c (D\varphi_0 : D\psi_0)_c (D\psi_0 : D\varphi_0)_c (D\psi_n : D\varphi_0)_c^* \\
&\quad = \lim_{n\to\infty} (D\varphi_n : D\psi_0)_c (D\psi_n : D\psi_0)_c^* ;
\end{aligned}$$

hence the above definition of $(D\varphi : D\psi)_c$ makes sense. Given $\varepsilon > 0$, if $d(\varphi, \psi) < \eta$ with $\eta > 0$ in Lemma 2.8, then

$$\begin{aligned}
\|(D\varphi : D\varphi_0)_c - (D\psi : D\varphi_0)_c\| &= \lim_{n\to\infty} \|(D\varphi_n : D\varphi_0)_c - (D\psi_n : D\varphi_0)_c\| \\
&= \lim_{n\to\infty} \|(D\varphi_n : D\psi_n)_c - 1\| \leqq \varepsilon .
\end{aligned}$$

Therefore, if $d(\varphi, \varphi') < \eta$ and $D(\psi, \psi') < \eta$, then we have

$$\begin{aligned}
&\|(D\varphi : D\psi)_c - (D\varphi' : D\psi')_c\| \\
&\quad = \|(D\varphi : D\varphi_0)_c (D\psi : D\varphi_0)_c^* - (D\varphi' : D\varphi_0)_c (D\psi' : D\varphi_0)_c\| \leqq 2\varepsilon .
\end{aligned}$$

Thus, by Lemma 2.9, Theorem 2.4 and continuity, all statements for faithful weights of infinite multiplicity hold.

Let φ be a faithful weight of infinite multiplicity and w be an isometry with $ww^* \in M_\varphi$. We define

$$(D\varphi_w : D\varphi)_c = w^* \bar{\sigma}_c^\varphi(w) .$$

If v is another isometry with $vv^* \in M_\varphi$ such that $\varphi_w = \varphi_v$, then we have

$$v^* \sigma_t^\varphi(v) = (D\varphi_v : D\varphi)_t = (D\varphi_w : D\varphi)_t = w^* \sigma_t^\varphi(w) , \qquad t \in \boldsymbol{R} ,$$

so that we have $vw^* \in M_\varphi$ and $\bar{\sigma}_c^\varphi(vw^*) = vw^*$ by Lemma 2.9. Therefore, $v^* \bar{\sigma}_c^\varphi(v) = w^* \bar{\sigma}_c^\varphi(w)$. Thus $(D\varphi_w : D\varphi)_c$ is well-defined.

If φ and ψ are faithful weights of infinite multiplicity and v and w are isometries of M with $vv^* \in M_\varphi$ and $ww^* \in M_\psi$, then we define

$$(D\varphi_v : D\psi_w)_c = (D\varphi_v : D\varphi)_c (D\varphi : D\psi)_c (D\psi_v : D\psi)_c^* .$$

It is then shown, by the similar arguments as above, that $(D\varphi_v : D\psi_w)_c$ is well-defined. Since any faithful weight is of the form φ_v for some φ of infinite multiplicity, $(D\varphi : D\psi)_c$ is defined for a general pair φ, ψ in $\mathfrak{W}_M^0$. We then define, fixing a faithful weight φ_0 of infinite multiplicity,

$$\bar{\sigma}_c^\varphi(x) = (D\varphi : D\varphi_0)_c \bar{\sigma}_c^{\varphi_0}(x) (D\varphi_0 : D\varphi)_c , \qquad x \in M .$$

It follows from the chain rule that $\bar{\sigma}_c^\varphi$ does not depend on the choice of φ_0. A straightforward argument shows that conditions (iv), (v), (vi) and (v) hold.

Thus, the only thing remains to be shown is the continuity of $(D\varphi: D\psi)_c$ in general. We consider $P = M \otimes F_\infty$. It is easily seen that for any $\varphi, \psi \in \mathfrak{W}_M^0$ we have

$$(D(\varphi \otimes \mathrm{Tr}): D(\psi \otimes \mathrm{Tr}))_c = (D\varphi: D\psi)_c \otimes 1 ;$$
$$d(\varphi \otimes \mathrm{Tr}, \psi \otimes \mathrm{Tr}) = d(\varphi, \psi) .$$

Hence the continuity of $(D\varphi: D\psi)_c$ on φ, ψ follows from the continuity of two maps: $(\varphi, \psi) \in \mathfrak{W}_M^0 \times \mathfrak{W}_M^0 \to (\varphi \otimes \mathrm{Tr}, \psi \otimes \mathrm{Tr}) \in \mathfrak{W}_P^0 \times \mathfrak{W}_P^0$ and $(\varphi \otimes \mathrm{Tr}, \psi \otimes \mathrm{Tr}) \to (D(\varphi \otimes \mathrm{Tr}): D(\psi \otimes \mathrm{Tr}))_c$. The continuity of the map: $\varphi \to \bar{\sigma}_c^\varphi$ is automatic after this. q.e.d.

EXAMPLE 2.10. Let N be an infinite semi-finite factor with separable predual. As in Example 2.3, let $c = df \in Z^1(F^M)$ and $\varphi = \tau(h_\varphi \cdot)$ a faithful weight on N. Then we have

$$(D\varphi: D\tau)_c = f(1)^* f(h_\varphi) .$$

We leave the proof to the reader.

COROLLARY 2.11. *Let M be an infinite factor with separable predual. Let $c \in Z^1(F^M)$ be as in Theorem 2.6. Let φ be a faithful weight on M and put*

$$c(h) = (D(\varphi(h \cdot)): D\varphi)_c$$

for each non-singular self-adjoint positive operator h affiliated with M_φ. We conclude the following:

(i) *$c(h)$ falls in the center of $\{h\}' \cap M_\varphi$;*

(ii) *$c_1c_2(h) = c_1(h)c_2(h)$ for every twice differentiable $c_1, c_2 \in Z^1(F^M)$.*

PROOF. (i) Let $P = M \otimes F_2$ and

$$\psi(x) = \varphi(x_{11}) + \varphi(hx_{22}), \quad x = \sum_{i,j=1}^{2} x_{ij} \otimes e_{i,j} \in P .$$

Let $u = 1 \otimes e_{21}$. We have then

$$c(h) \otimes e_{21} = \bar{\sigma}_c^\psi(u) .$$

Since $\sigma_t^\psi(u) = h^{it} \otimes e_{21}$, we have $\sigma_t^\psi(u)u^* \in P_\psi$, so that

$$\sigma_t^\psi(u^*)u = \bar{\sigma}_c^\psi(\sigma_t^\psi(u^*)u) = \sigma_t^\psi(\bar{\sigma}_c^\psi(u^*))\bar{\sigma}_c^\psi(u) ;$$

hence

$$c(h) \otimes e_{11} = u^*\bar{\sigma}_c^\psi(u) = \sigma_t^\psi(u^*\bar{\sigma}_c^\psi(u)) \in P_\psi ,$$

which means that $c(h) \in M_\varphi$.

If $x \in \{h\}' \cap M_\varphi \subset M_\varphi \cap M_{\varphi(h\cdot)}$, then we have

$$x = \bar{\sigma}_c^{\varphi(h\cdot)}(x) = c(h)\bar{\sigma}_c^{\varphi}(x)c(h)^* = c(h)xc(h)^* ,$$

so that $c(h) \in (\{h\}' \cap M_\varphi)' \cap M_\varphi =$ the center of $\{h\}' \cap M_\varphi$.

(ii) This follows from (i) and Theorem 2.6 (vii). q.e.d.

We now apply Theorem 2.6 to a factor given by the group measure space construction, and then compute the extended modular automorphism. Let M be an infinite factor with separable predual and φ a faithful weight. Suppose that there exists a von Neumann subalgebra N of M_φ with relative commutant $N' \cap M = C$ contained in N and a continuous unitary representation $u(\cdot)$ of a separable locally compact group G in M such that

$$u(g)Nu(g)^* = N , \qquad g \in G ;$$
$$M = \{N \cup u(G)\}'' .$$

By Theorem II.6.2, there exists a non-singular self-adjoint operator ρ_g affiliated with C such that

$$\sigma_t^\varphi(u(g)) = u(g)\rho_g^{it} , \qquad t \in \boldsymbol{R},\ g \in G .$$

It is also easy to see, using $N' \cap M = C \subset N$, that if $\alpha \in \operatorname{Aut}(M)$ leaves N elementwise fixed, then there exists a one-cocycle $\{a_g\} \in Z_\beta^1(G, \mathfrak{U}_C)$ such that

$$\alpha(u(g)) = a_g u(g) , \qquad g \in G ,$$

where the action β of G on N, hence on C, is given by

$$\beta_g(x) = u(g)xu(g)^* , \qquad x \in N,\ g \in G .$$

Let $\{\Gamma, \mu\}$ be a standard measure space with $C = L^\infty(\Gamma, \mu)$, on which G acts in such a way that

$$\beta_g(x)(\gamma) = x(g^{-1}\gamma) , \qquad x \in C,\ g \in G,\ \gamma \in \Gamma .$$

We consider the action of G on $\Gamma \times \boldsymbol{R}$ defined by:

$$T_g(\gamma, s) = (g\gamma, s - \log \rho_g(\gamma)) , \qquad \gamma \in \Gamma,\ s \in \boldsymbol{R},\ g \in G .$$

Let $k_g(\gamma) = -\log \rho_g(\gamma)$, $g \in G$, $\gamma \in \Gamma$. By Theorem II.6.2, the center $C_{\bar{\omega}}$ of the dominant weight $\bar{\omega} = \varphi \otimes \omega$ on $M \otimes F_\infty$ is identified with $L^\infty(\Gamma \times \boldsymbol{R}, \mu \otimes m)^G$, where m means, of course, the Plancherel measure on $\boldsymbol{R}$.

COROLLARY 2.12. *In the above situation, if* $c \in Z^1(F^M)$ *is as in Theorem* 2.6, *then the cocycle* $a \in Z_\beta^1(G, \mathfrak{U}_C)$ *corresponding to the extended* *modular automorphism* $\alpha = \bar{\sigma}_c^{\varphi}$ *is given by the formula:*

$$a_g(\gamma) = b_{k_g(g^{-1}\gamma)}(\gamma, 0) ,$$

where $b_s = p_{\bar{\omega}}^{-1}(c_{e^{-s}})$, $s \in \boldsymbol{R}$.

PROOF. For $n = 1, 2, \cdots$, put

$$\Phi_n(t) = \frac{1}{n}\tan^{-1} t , \qquad t \in \boldsymbol{R}$$

$$\Psi_n(s) = \tan ns , \qquad -\frac{\pi}{2n} < s < \frac{\pi}{2n} .$$

We define an isometry w_n of $L^2(\boldsymbol{R})$ onto $L^2(-\pi/2n, \pi/2n)$ by

$$(w_n\xi)(s) = \sqrt{\Psi_n'(s)}\xi \circ \Psi_n(s) , \quad -\frac{\pi}{2n} < s < \frac{\pi}{2n} , \quad \xi \in L^2(\boldsymbol{R}) .$$

Clearly we have

$$(w_n^*\xi)(t) = \sqrt{\Phi_n'(t)}\xi \circ \Phi_n(t) , \quad t \in \boldsymbol{R}, \xi \in L^2(-\pi/2n, \pi/2n) .$$

Let ω be the weight on $F_\infty = \mathfrak{L}(L^2(\boldsymbol{R}))$ such that

$$(D\omega: D\,\mathrm{Tr})_t = V_t ,$$

where $\{U_s\}$ and $\{V_t\}$ mean the one parameter unitary groups defined in Chapter II. We have then

$$\begin{aligned}\{(D\omega_{w_n}: D\,\mathrm{Tr})_t\xi\}(s) &= (w_n^* V_t w_n\xi)(s) \\ &= e^{it\Phi_n(s)}\xi(s) .\end{aligned}$$

Hence we get $d(\omega_{w_n}, \mathrm{Tr}) = \pi/2n$, so that ω_{w_n} converges to Tr uniformly. Therefore $\varphi \otimes \omega_{w_n}$ converges to $\varphi \otimes \mathrm{Tr}$ uniformly; thus we get

$$\begin{aligned}(D\varphi \otimes \mathrm{Tr}: D\bar{\omega})_c &= \lim_{n\to\infty} (D\varphi \otimes \omega_{w_n}: D\bar{\omega})_c \\ &= \lim_{n\to\infty} (1 \otimes w_n)^* \bar{\sigma}_c^{\bar{\omega}}(1 \otimes w_n) .\end{aligned}$$

Let $u_c = (D\varphi \otimes \mathrm{Tr}: D\bar{\omega})_c$ and $u_{n,c} = (D\varphi \otimes \omega_{w_n}: D\bar{\omega})_c$. It follows from the proof of Lemma 2.7 that

$$u_{n,c} \in \{b_t, (1 \otimes w_n^*)\sigma_s^{\bar{\omega}}(1 \otimes w_n): s, t \in \boldsymbol{R}\}'' \subset C \otimes L^\infty(\boldsymbol{R}) ,$$

and that

$$u_{n,c}(\gamma, s) = b_{\Phi_n(s)-s}(\gamma, \Phi_n(s)) , \qquad \gamma \in \Gamma, s \in \boldsymbol{R} .$$

Therefore, we get $u_c \in L^\infty(\Gamma \times \boldsymbol{R})$ and

$$u_c(\gamma, s) = b_{-s}(\gamma, 0) ,$$

where we use the fact that the differentiability of b in norm, together with the cocycle property, implies the continuity of $b_s(\gamma, t)$ in t.

We next have

$$\sigma_t^{\bar{\omega}}(u(g) \otimes 1)(u(g)^* \otimes 1) = \sigma_t^{\varphi}(u(g))u(g)^* \otimes 1 = \beta_g(\rho_g^{it}) \otimes 1 \in C \otimes C ,$$

so that $\sigma_c^{\bar{\omega}}(u(g) \otimes 1)(u(g)^* \otimes 1) = d_g$ belongs to $C \otimes L^\infty(\boldsymbol{R}) = L^\infty(\Gamma \times \boldsymbol{R})$ and we get

$$d_g(\gamma, s) = b_{k_g(g^{-1}\gamma)}(\gamma, s) .$$

Since we have

$$\begin{aligned} a_g \otimes 1 = \bar{\sigma}_c^{\varphi\otimes\mathrm{Tr}}(u(g) \otimes 1)(u(g)^* \otimes 1) &= u_c\sigma_c^{\bar{\omega}}(u(g) \otimes 1)(u(g)^* \otimes 1)(\beta_g \otimes \iota)(u_c^*) \\ &= u_c d_g(\beta_g \otimes \iota)(u_c^*) , \end{aligned}$$

we have

$$\begin{aligned} a_g(\gamma) = u_c(\gamma, s)d_g(\gamma, s)\overline{u_c(g^{-1}\gamma, s)} &= b_{-s}(\gamma, 0)b_{k_g(g^{-1}\gamma)}(\gamma, s)\overline{b_{-s}(g^{-1}\gamma, 0)} \\ &= b_{k_g(g^{-1}\gamma)}(\gamma, 0) . \end{aligned}$$

q.e.d.

IV.3. The exact sequence for the group of all automorphisms. Given a factor M of type III with separable predual, we have constructed various mathematical objects: the flow F^M of weights, the fundamental homomorphism γ_M of $\mathrm{Out}(M)$ into $\mathrm{Aut}(F^M)$, the extension $\bar{\delta}_M$ of the modular homomorphism and a continuous decomposition $M = W^*(N, \boldsymbol{R}, \theta)$. Putting these things together, we compute $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M)$, and generalize the exact sequence in [3; Chapter IV].

THEOREM 3.1. *Let M be a factor of type* III *with separable predual. If $M = W^*(N, \boldsymbol{R}, \theta)$ is a continuous decomposition of M, then there exists a homomorphism $\bar{\gamma}$ of* $\mathrm{Out}(M)$ *onto* $\mathrm{Out}_{\theta,\tau}(N)$ *which makes the following sequence exact*:

$$\{1\} \longrightarrow H^1(F^M) \xrightarrow{\bar{\delta}_M} \mathrm{Out}\,(M) \xrightarrow{\bar{\gamma}} \mathrm{Out}_{\theta,\tau}\,(N) \longrightarrow \{1\} ,$$

where

$$\mathrm{Out}_{\theta,\tau}\,(N) = \{\varepsilon_N(\alpha) : \alpha \in \mathrm{Aut}\,(N),\ \alpha\theta_s = \theta_s\alpha,\ s \in \boldsymbol{R},\ \tau \circ \alpha = \tau\} .$$

PROOF. Let $\bar{\omega}$ be the dominant weight of M dual to the trace τ on N with $\tau \circ \theta_s = e^{-s}\tau$. By Theorem 2.2, if $\alpha \in \mathrm{Aut}\,(M)$ leaves N pointwise fixed, then $\alpha = \bar{\sigma}_c^{\bar{\omega}}$ for some $c \in Z^1(F^M)$. By Corollary 2.5. (ii), α is inner if and only if $c \in B^1(F^M)$. Hence the map $\bar{\delta}_M : c \in Z^1(F^M) \longrightarrow \varepsilon_M(\bar{\sigma}_c^{\bar{\omega}}) \in \mathrm{Out}\,(M)$ gives rise to an isomorphism of $H^1(F^M)$ into $\mathrm{Out}\,(M)$ which will be denoted by $\bar{\delta}_M$ again.

Let α be an arbitrary automorphism of M. Then $\bar{\omega} \circ \alpha$ is again dominant. By the uniqueness of a dominant weight, there exists a unitary $u \in M$ such that $\bar{\omega} \circ \alpha \circ \mathrm{Ad}(u) = \bar{\omega}$. Hence, putting

$$\mathrm{Aut}_{\bar{\omega}}(M) = \{\alpha \in \mathrm{Aut}(M): \bar{\omega} \circ \alpha = \bar{\omega}\},$$

we have $\mathrm{Out}(M) = \varepsilon_M(\mathrm{Aut}_{\bar{\omega}}(M))$. Let $\alpha \in \mathrm{Aut}_{\bar{\omega}}(M)$. If $\alpha = \bar{\sigma}_c^{\bar{\omega}}$ for some $c \in Z^1(F^M)$, then $\alpha|_N = \iota$ by construction. If $\alpha|_N = \mathrm{Ad}(u)$ for some $u \in \mathfrak{U}(N)$, then we have $\alpha \circ \mathrm{Ad}(u)^{-1}|_N = \iota$, so that $\alpha \circ \mathrm{Ad}(u)^{-1} = \bar{\sigma}_c^{\bar{\omega}}$ for some $c \in Z^1(F^M)$ by Theorem 2.2. Hence the kernel of the homomorphism $\gamma: \alpha \in \mathrm{Aut}_{\bar{\omega}}(M) \to \varepsilon_N(\alpha|_N) \in \mathrm{Out}(N)$ is precisely the image of $Z^1(F^M)$ under $\bar{\sigma}^{\bar{\omega}}$. Since we have

$$\mathrm{Aut}_{\bar{\omega}}(M) \cap \mathrm{Int}(M) = \{\mathrm{Ad}(u): u \in \mathfrak{U}(N)\},$$

γ gives rise to a unique homomorphism $\bar{\gamma}$ of $\mathrm{Out}(M)$ into $\mathrm{Out}(N)$ such that $\bar{\gamma} \circ \varepsilon_M = \gamma$.

We examine the range of γ. Put

$$\mathrm{Aut}_{\theta,\tau}(N) = \{\alpha \in \mathrm{Aut}(N): \alpha\theta_s = \theta_s\alpha,\ s \in \boldsymbol{R},\ \tau \circ \alpha = \tau\}.$$

Let $\{u(s)\}$ be the one parameter unitary group in M which appears in the crossed product decomposition $M = W^*(N, \boldsymbol{R}, \theta)$. Let $\alpha \in \mathrm{Aut}_{\bar{\omega}}(M)$ and $\beta = \alpha|_N$. Since α and $\{\sigma_t^{\bar{\omega}}\}$ commute, we have $\sigma_t^{\bar{\omega}}(\alpha(u(s))) = e^{ist}\alpha(u(s))$, so that $a_s = \alpha(u(s))u(s)^* \in \mathfrak{U}(N)$. It is straightforward to see that

$$a_{s+t} = a_s\theta_s(a_t), \qquad s, t \in \boldsymbol{R};$$

hence $a \in Z_\theta^1(\boldsymbol{R}, \mathfrak{U}(N))$. By Theorem III.5.1, we have $a = b^*\theta_s(b)$ for some $b \in \mathfrak{U}(N)$. Thus we get $\alpha(u(s)) = b^*\theta_s(b)u(s) = b^*u(s)b$, so that $\alpha \circ \mathrm{Ad}(b)$ leaves $u(s)$ fixed for every $s \in \boldsymbol{R}$, which means that $\beta \circ \mathrm{Ad}(b) = \alpha \circ \mathrm{Ad}(b)|_N$ and $\{\theta_s\}$ commute. Since $\bar{\omega} \circ \alpha = \bar{\omega}$, $\alpha|_N$ leaves τ invariant by the equalities $\bar{\omega} = \tau \circ E_{\bar{\omega}}$ and $E_{\bar{\omega}} \circ \alpha = E_{\bar{\omega}}$, so that $\beta \circ \mathrm{Ad}(b)$ leaves τ invariant. Thus we conclude the inclusion:

$$\bar{\gamma}(\mathrm{Out}(M)) \subset \varepsilon_N(\mathrm{Aut}_{\theta,\tau}(N)) = \mathrm{Out}_{\theta,\tau}(N).$$

Suppose $\beta \in \mathrm{Aut}_{\theta,\tau}(N)$. A standard argument shows that β is extended uniquely to an $\alpha \in \mathrm{Aut}(M)$ such that $\alpha(xu(s)) = \beta(x)u(s)$, $x \in N$, $s \in \boldsymbol{R}$. Trivially, we have $\alpha|_N = \beta$. Thus we have

$$\bar{\gamma}(\mathrm{Out}(M)) \supset \varepsilon_N(\mathrm{Aut}_{\theta,\tau}(N)).$$

q.e.d.

THEOREM 3.2. *In the same situation as in Theorem* 3.1,

$$\mathrm{Out}_{\theta,\tau}(N) = \{\bar{\alpha} \in \mathrm{Out}(N): \varepsilon_N(\theta_s)\bar{\alpha} = \bar{\alpha}\varepsilon_N(\theta_s),\ s \in \boldsymbol{R},\ \tau \circ \bar{\alpha} = \tau\}.$$

PROOF. Let C denote the center of N. The unitary group $\mathfrak{U}(N)$ of N is a polish group with respect to the σ-strong* topology and $\mathfrak{U}(C)$ is a closed subgroup of $\mathfrak{U}(N)$. We consider the pointwise convergence topology in $\mathrm{Aut}(N)$ with respect to the norm topology in N_*. The map $\mathrm{Ad}: u \in \mathfrak{U}(N) \to \mathrm{Ad}(u) \in \mathrm{Aut}(N)$ is a continuous homomorphism with kernel $\mathfrak{U}(C)$. Hence the naturally induced map $\overline{\mathrm{Ad}}: \bar{u} \in \mathfrak{U}(N)/\mathfrak{U}(C) \to \mathrm{Ad}(u) \in \mathrm{Aut}(N)$ is a continuous isomorphism from the polish group onto $\mathrm{Int}(N)$. Hence $\mathrm{Int}(N)$ is a Borel subset of $\mathrm{Aut}(N)$ and the inverse map $\overline{\mathrm{Ad}}^{-1}$ is a Borel map from $\mathrm{Int}(N)$ onto $\mathfrak{U}(N)/\mathfrak{U}(C)$. Let T be a Borel transversal of $\mathfrak{U}(N)/\mathfrak{U}(C)$ in $\mathfrak{U}(N)$, and let $\pi = T \circ \overline{\mathrm{Ad}}^{-1}$. Then π is a Borel map from $\mathrm{Int}(N)$ into $\mathfrak{U}(N)$ such that $\mathrm{Ad}(\pi(\alpha)) = \alpha$ for every $\alpha \in \mathrm{Int}(N)$.

Suppose $\alpha \in \mathrm{Aut}(M)$ commute with θ_s, $s \in \boldsymbol{R}$, modulo $\mathrm{Int}(M)$, that is, $\varepsilon_N(\alpha)\varepsilon(\theta_s) = \varepsilon(\theta_s)\varepsilon_N(\alpha)$. Put $\beta_s = \alpha \circ \theta_s \circ \alpha^{-1} \circ \theta_s^{-1} \in \mathrm{Int}(N)$ and $b_s = \pi(\beta_s) \in \mathfrak{U}(N)$, $s \in \boldsymbol{R}$. We have then

$$\mathrm{Ad}(b_s) \circ \theta_s = \alpha \circ \theta_s \circ \alpha^{-1}, \qquad s \in \boldsymbol{R}.$$

By the one parameter group property of $\{\alpha \circ \theta_s \circ \alpha^{-1}\}$, we have

$$\mathrm{Ad}(b_s\theta_s(b_t)) = \mathrm{Ad}(b_{s+t}), \qquad s, t \in \boldsymbol{R}.$$

Put

$$c(s, t) = b_s^* b_{s+t} \theta_s(b_t^*) \in \mathfrak{U}(C), \qquad s, t \in \boldsymbol{R}.$$

By a direct computation, we get

$$c(r, s)c(r+s, t) = \theta_r(c(s, t))c(r, s+t), \qquad r, s, t \in \boldsymbol{R}.$$

Hence c is a Borel unitary 2-cocycle of the flow $\{C, \theta\}$. By the triviality $H_\theta^2(\boldsymbol{R}, \mathfrak{U}(C)) = \{0\}$ of the second cohomology group of a flow, see Appendix, we can find a $\mathfrak{U}(c)$-valued Borel function $\{d_s\}$ such that

$$c(s, t) = d_s^* d_{s+t} \theta_s(d_t^*), \qquad \text{for almost } s, t \in \boldsymbol{R}.$$

Let $a_s = d_s b_s$, $s \in \boldsymbol{R}$. We then obtain a $\mathfrak{U}(N)$-valued Borel function $\{a_s\}$ such that for almost every s, t in $\boldsymbol{R}$,

$$\begin{cases} a_{s+t} = a_s\theta_s(a_t), & s, t \in \boldsymbol{R}; \\ \mathrm{Ad}(a_s) \circ \theta_s = \alpha \circ \theta_s \circ \alpha^{-1}. \end{cases}$$

By Remark III.1.9, there exists $a \in Z_\theta^1(\boldsymbol{R}, \mathfrak{U}(N))$ such that $a_s' = a_s$ for almost every $s \in \boldsymbol{R}$.

By the triviality of $H_\theta^1(\boldsymbol{R}, \mathfrak{U}(N))$, Theorem III.5.1, we have an element $u \in \mathfrak{U}(N)$ such that $a_s = u^*\theta_s(u)$, $s \in \boldsymbol{R}$. Thus we get $\mathrm{Ad}(u^*) \circ \theta_s \circ \mathrm{Ad}(u) = \alpha \circ \theta_s \circ \alpha^{-1}$ for almost every $s \in \boldsymbol{R}$. Namely, $\mathrm{Ad}(u) \circ \alpha$ and $\{\theta_s\}$ commute in $\mathrm{Aut}(M)$ by continuity. q.e.d.

REMARK 3.3. The exact sequence in Theorem 3.1 does not split in general.

APPENDIX

PROPOSITION A.1. *Let G and H be separable locally compact groups and $\{\Gamma, \mu\}$ a standard measure space on which G acts ergodically. Let E be a Borel subset of Γ with $\mu(E) > 0$. Put $A = \{(g, \gamma) \in G \times E: g\gamma \in E\}$. If b is an H-valued Borel function on A such that for every $g_1, g_2 \in G$ with $\mu(E \cap g_2^{-1}E \cap g_2^{-1}g_1^{-1}E) > 0$*

$$b(g_1g_2, \gamma) = b(g_1, g_2\gamma)b(g_2, \gamma)$$

for almost every $\gamma \in E \cap g_2^{-1}E \cap g_2^{-1}g_1^{-1}E$, then there exists an H-valued Borel function c on $G \times \Gamma$ such that

$$c(g, \gamma) = b(g, \gamma) , \qquad (g, \gamma) \in A ;$$

for every $g_1, g_2 \in G$

$$c(g_1g_2, \gamma) = c(g_1, g_2\gamma)c(g_2, \gamma)$$

for almost every $\gamma \in \Gamma$.

PROOF. Let G_0 be a dense countable subgroup of G. Let $\Gamma_0 = \bigcup_{g \in G_0} gE$. By ergodicity, we have $\mu(\Gamma - \Gamma_0) = 0$. Hence we may assume $\Gamma = \Gamma_0$. Then, there exists a family $\{E_g: g \in G_0\}$ of Borel subsets of E such that

$$\Gamma = \bigcup_{g \in G_0} gE_g \ , \quad gE_g \cap hE_h = \emptyset \ , \quad g \neq h \ .$$

Define a G-valued Borel function $a(\cdot)$ on Γ by

$$a(\gamma) = g \quad \text{if} \quad \gamma \in gE_g \ ,$$

and put $\omega(\gamma) = a(\gamma)^{-1}\gamma \in E$, and $\rho(g, \gamma) = a(g\gamma)^{-1}ga(\gamma)$. We have then

$$\gamma = a(\gamma)\omega(\gamma) \ , \quad \omega(g\gamma) = \rho(g, \gamma)\omega(\gamma) \ ;$$
$$\rho(g_1g_2, \gamma) = \rho(g_1, g_2\gamma)\rho(g_2, \gamma) \ .$$

Furthermore, for each fixed $g \in G$, $\rho(g, \cdot)$ takes only countably many values: indeed $\rho(g, \gamma) \in G_0gG_0$ for every $\gamma \in \Gamma$. Define

$$c(g, \gamma) = b(\rho(g, \gamma), \omega(\gamma)) \ , \qquad g \in G, \gamma \in \Gamma \ .$$

Since we can choose $E_1 = E$ where 1 means the unit of G, we have $c(g, \gamma) = b(g, \gamma)$ for $(g, \gamma) \in A$. Furthermore, we have

$$\begin{aligned} c(g_1g_2, \gamma) &= b(\rho(g_1g_2, \gamma), \omega(\gamma)) = b(\rho(g_1, g_2\gamma)\rho(g_2, \gamma), \omega(\gamma)) \\ &= b(\rho(g_1, g_2\gamma), \rho(g_2, \gamma)\omega(\gamma))b(\rho(g_2, \gamma), \omega(\gamma)) \\ &= b(\rho(g_1, g_2\gamma), \omega(g_2\gamma))b(\rho(g_2, \gamma), \omega(\gamma)) \\ &= c(g_1, g_2\gamma)c(g_2, \gamma) \end{aligned}$$

for almost every $\gamma \in \Gamma$, where we use, in order to exclude a null set of

γ, the fact that $\rho(g_1, g_2\gamma)$ and $\rho(g_2, \gamma)$, $\gamma \in \Gamma$, are at most countable.
q.e.d.

The authors learned that the following result had been proven by L. Brown sometime earlier. We present, however, a proof for the sake of convenience of the reader, since Brown's work is not yet available in print.

PROPOSITION A.2. *Let A be an abelian von Neumann algebra with separable predual, and $\{\alpha_t : t \in \boldsymbol{R}\}$ be an ergodic one parameter automorphism group of A. Then for every $n \geqq 2$, we have $H^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) = \{1\}$.*

PROOF. By virtue of the representation theorem for flows, due to Ambrose, Kakutani, Krengel and Kubo [12], [16], we may assume that the flow $\{A, \alpha\}$ is built under a ceiling function from a single ergodic automorphism. Let $\{\Gamma, \mu\}$ be a standard measure space equipped with an ergodic transformation T. Let f be a positive Borel function on Γ. Consider the abelian von Neumann algebra $B = L^\infty(\Gamma \times \boldsymbol{R}, \mu \otimes m)$, where m means the Lebesgue measure on $\boldsymbol{R}$. We define a one parameter automorphism group $\{\beta_t\}$ and an automorphism θ of B as follows:

$$\beta_t(x)(\gamma, s) = x(\gamma, s - t)\,, \quad x \in B\,, \quad (\gamma, s) \in \Gamma \times \boldsymbol{R}\,, \quad t \in \boldsymbol{R}\,,$$
$$\theta(x)(\gamma, s) = x(T^{-1}\gamma, s + f(\gamma))\,.$$

The representation theorem says that $\{A, \alpha\} \cong \{B^\theta, \beta\}$ for a suitable choice of Γ, μ, T, and f.

An n-cochain $c \in C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A)$ is by definition a unitary of $L^\infty(\boldsymbol{R}^n) \otimes A$ considered as a $\mathfrak{U}_A$-valued function on $\boldsymbol{R}^n$. In particular, $C^0_\alpha(\boldsymbol{R}, \mathfrak{U}_A) = \mathfrak{U}_A$. For each $n \geqq 0$, and $c \in C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A)$, the coboundary dc is given by the formula:

$$\begin{aligned} dc(s_1, \cdots, s_{n+1}) &= \alpha_{s_1}(c(s_2, \cdots, s_{n+1}))c(s_1 + s_2, s_3, \cdots, s_{n+1})^{-1} \cdots c(s_1, \cdots, s_n)^{(-1)^{n+1}} \\ &= \alpha_{s_1}(c(s_2, \cdots, s_{n+1})) \prod_{j=1}^{n} c(s_1, \cdots, s_j + s_{j+1}, \cdots, s_{n+1})^{(-1)^j} \\ &\quad \times c(s_1, s_2, \cdots, s_n)^{(-1)^{n+1}}\,. \end{aligned}$$

Thus we obtain a cochain complex:

$$(1) \qquad \mathfrak{U}_A = C^0_\alpha(\boldsymbol{R}, \mathfrak{U}_A) \xrightarrow{d} C^1_\alpha(\boldsymbol{R}, \mathfrak{U}_A) \cdots \xrightarrow{d} C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) \xrightarrow{d} \cdots\,.$$

We have then by definition $H^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) = \{\text{the kernel of } d \text{ in } C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A)\}/\{\text{the range of } d\}$.

Let $\mathfrak{U}^n$ be the unitary group of $L^\infty(\boldsymbol{R}^{n+1}) \otimes B = L^\infty(\boldsymbol{R}^{n+1} \times \Gamma)$. For each $c \in \mathfrak{U}^n$, we define the coboundary dc by the formula:

$$dc(t_0, t_1, \cdots, t_{n+1}, \gamma) = \prod_{j=0}^{n+1} c(t_0, t_1, \cdots, \hat{t}_j, \cdots, t_{n+1}\gamma)^{(-1)^j},$$

where $\hat{t}_j$ indicates that the term t_j is missing. We then have a long exact sequence:

$$(2) \qquad \mathfrak{U}^0 \xrightarrow{d} \mathfrak{U}^1 \xrightarrow{d} \mathfrak{U}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{U}^n \xrightarrow{d} \cdots .$$

For each $n \geqq 0$, we define an automorphism of $L^\infty(\boldsymbol{R}^{n+1}) \otimes B$, denoted by θ again for the obvious reason, by the following:

$$\theta(x)(t_0, t_1, \cdots, t_n, \gamma) = x(t_0 + f(\gamma), t_1 + f(\gamma), \cdots, t_n + f(\gamma), T^{-1}\gamma) .$$

Let π be a map of $L^\infty(\boldsymbol{R}^n) \otimes A$ into $L^\infty(\boldsymbol{R}^{n+1}) \otimes B$ defined by the following:

$$\pi(x)(t_0, t_1, \cdots, t_n, \gamma) = x(t_1 - t_0, t_2 - t_1, \cdots, t_n - t_{n-1}, \gamma, t_0) ,$$

where we identify A with B^θ. It follows then that π is an isomorphism of $L^\infty(\boldsymbol{R}^n) \otimes A$ onto $(L^\infty(\boldsymbol{R}^{n+1}) \otimes B)^\theta$ which makes the following diagram commute:

$$\begin{array}{ccccccccc}
C^0_\alpha(\boldsymbol{R}, \mathfrak{U}_A) & \xrightarrow{d} & C^1_\alpha(\boldsymbol{R}, \mathfrak{U}_A) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & & & \downarrow{\scriptstyle\pi} & & \\
\mathfrak{U}^0 & \xrightarrow{d} & \mathfrak{U}^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathfrak{U}^n & \xrightarrow{d} & \cdots .
\end{array}$$

Moreover, we have $\pi(C^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A)) = (\mathfrak{U}^n)^\theta =$ the fixed point subgroup of $\mathfrak{U}^n$ under θ. Therefore, cochain complex (1) is isomorphic to the following cochain complex:

$$(3) \qquad (\mathfrak{U}^0)^\theta \xrightarrow{d} (\mathfrak{U}^1)^\theta \xrightarrow{d} \cdots \xrightarrow{d} (\mathfrak{U}^n)^\theta \xrightarrow{d} \cdots .$$

Now, let $C = L^\infty(\Gamma, \mu)$ and $\theta(x)(\gamma) = x(T^{-1}\gamma)$ for each $x \in C$. Putting $\varepsilon(x) = 1 \otimes x \in L^\infty(\boldsymbol{R}) \otimes C$ for each $x \in C$, we obtain an injective resolution of the $\boldsymbol{Z}$-module $\mathfrak{U}_C$:

$$(4) \qquad \{1\} \longrightarrow \mathfrak{U}_C \xrightarrow{\varepsilon} \mathfrak{U}^0 \xrightarrow{d} \mathfrak{U}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{U}^n \xrightarrow{d} \cdots$$

where $\boldsymbol{Z}$ acts on each group, of course, via θ and the injectivity follows from the divisibility of the unitary group of a von Neumann algebra. Hence the cohomology groups of cochain complex (3), hence (1), are isomorphic to the cohomology groups $H^n_\theta(\boldsymbol{Z}, \mathfrak{U}_C)$, $n \geqq 1$, (cf. [10; page 105]). This means then that

$$H^n_\alpha(\boldsymbol{R}, \mathfrak{U}_A) \cong H^n_\theta(\boldsymbol{Z}, \mathfrak{U}_C) , \qquad u \geqq 1 .$$

It is known, however, that

$$H^n_\theta(\boldsymbol{Z}, \mathfrak{U}_C) = \{1\} , \qquad n \geqq 2 .$$

q.e.d.

The above result, or more precisely the proof, is known in homological algebra as Shapiro's lemma.

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