"Functional Analysts and Operator Theorists Celebrate the 100th Anniversary of Acta Sci. Math." Alain Connes's talk

In the meantime, let me say that you know, a long time ago, I wrote a paper, this was in 1977 and it was published in Acta. and I mean so for me, it's a great occasion actually to celebrate this anniversary, this was my paper published in 1977, so here is the title of my talk : "Prolate wave operator and zeta". So if you want the motivation from operator theory is very simple to explain and I mean, and it is the following : it is that you know when you look at the zeros of the riemann zeta function, I mean the critical zeros okay, they are a tantalizing spectrum and I mean they are a tantalizing spectrum of something like not a Laplacian but like a Dirac operator, and I mean in a way, if you want, they are mysterious both in the infrared, so if you want, when you look at them for small values of the frequencies, you see something which is extremely strange, it starts around 14 and so on, and continues keep going. But they are also extremely mysterious at the ultraviolet level and the reason why they are so mysterious at the ultraviolet level is that when you count the number of zeros with imaginary part is between zero and E where E is some large number, what you find is something which has been already, if you want, devised by Riemann and it's a very strange expression because I mean if it were something like Dirac operator on a circle, it would be proportional to E to the energy level, but here there is a log term so it's precisely it's $\dfrac{E}{2\pi}\log\dfrac{E}{2\pi}$, there is a correction, $-\dfrac{E}{2\pi}$ and then there is a logarithmic term $+O(\log E)$.

Now when you think geometrically and of course many people have thought geometrically what this could be, it's very difficult to see what type of geometry could be behind this. And I mean if you want at the purely intuitive level and that's what I will follow, what is going on is that, in some sense if you want, one should think of these zeros, I mean the way I think about them is like you know an infinite pole. And I mean one has to prove that it's exactly what it should be. But because it's infinite I mean you know it's uh it's, it's extremely difficult. And uh I mean so what I will explain today is in two parts I mean the main part will be a very recent joint work with Henri Moscovici which will appear in the Proceedings of the National Academy of Sciences and which will handle the ultraviolet part of the spectrum. So what we have found, essentially I mean you know what we found with Henri I will explain what it is, but what I should stress from the start is that we were not looking for what we found : we were you know looking at an operator and so on and we were amazed to find that it was related to the zeros of data and if we had been asked to write a proposal, we would never have been able to guess what we were going to find. And the second part is about the low-lying part of the spectrum and what is also amazing is that in both cases, the functions which are involved and the operator which is involved is an operator which is the prolate wave operator. So let me first explain about this operator. I mean it's related to an important fact which is quite useful when we work with Hilbert space, which is the situation when you have a pair of projections in Hilbert space, we know very well that we have a single projection okay well they're all the same if they have the same dimension. But what about a pair of projections ? Now if you have a pair of projections, it turns out that the situation is not at all out of ends because it's really essentially a two-dimensional situation in the following sense that giving a pair of projections in Hilbert space is the same thing as giving a unitary representation of the dihedral group. And I mean this is straightforward to obtain because what you have, you have two unitaries of square one which are obtained by taking one minus two times the first projection ($\mathcal{U}_1 = 1 - 2P_1$) and one

Référence : https://www.youtube-nocookie.com/embed/vLekXpbT_BI.
Transcription : Denise Vella-Chemla, juin 2022.

minus two times the second projection ($\mathcal{U}_2 = 1-2P_2$) ; they are of square one obviously. They don't commute but even though they don't commute, the group they generate is very very nice if you want : it's a solvable group, it's much simpler than solvable, it's just semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ of $\mathbb{Z}$ by the group with two elements ($\mathbb{Z}/2\mathbb{Z}$). So once you know that and you know a little bit of representation theory, you find out that the irreducible representations are just parametrized by an angle and that you can think of the irreducible situation as being two-dimensional at the most, and being given if you want by the projection on the x-axis and the projection on a line which is making an angle with the x-axis.

And so what happens is that from this knowledge of the irreducible representations, and from the knowledge of course of the fact that any representation is a direct integral of irreducible representations, you have a full control of the situation. And you understand that the situation is completely known once you know the cosine for instance of the angle, or the sine of the angle, which both are, if you want, determined by simple equations, I mean (rounding $P_1P_2P_1 = \cos^2(\alpha)P_1, (P_1 - P_2)^2 = \sin^2(\alpha)$). The nicest is about the sine because I mean what you find, when you take the square of the difference between the two projections, it is an operator which is in the center namely which commutes with both P_1 and P_2. And so I mean it will serve to diagonalize the the pair of projections.

Now, so, in 1996 when I started, you know, also by accident, working on zeta, I wanted to introduce a cut-off in terms of suitable operators, and I had to deal with a specific pair of projections, which is the following : you take L^2-functions which are even and you take the following pairs : the first projection is extremely simple, it's an extremely simple cut-off projection, which means that you consider only those functions which vanish when the argument is an absolute value larger than λ, λ is a fixed number. And the second projection is what you obtain by taking the Fourier transform of this first projection. So you conjugate this first projection by the fourier transform, and it is convenient, I mean for, you know, normalization number theoretic purposes to take the Fourier transform as defined here, namely with a factor 2π. In fact you know there is a general rule in mathematics which one learns very early which is that when you have i, it's rare that you don't have a 2π and when you have a 2π, it's very rare that you don't have an i. So I mean this is the rule. And so when you take this pair of projections P_λ and $\widehat{P}_\lambda$, it turns out that there is a miracle which happens. And this miracle was discovered in several places actually but one of them was Bell-labs in the 60s and it was discovered by Slepian, Landau and Pollak, and what they did was to actually be able to diagonalize using prolate functions (I will come later to the main operator) but they were able to, if you want, diagonalize the angle and what they did more precisely so in several papers, what they did, uh was to diagonalize something which is like the square root of the cosine, namely they diagonalize the operator which is called the truncated Fourier transform and which is essentially the product $P_\lambda\widehat{P}_\lambda P_\lambda$, I mean, if you square it, okay.

So the motivation was a very concrete motivation : it was a paradox in the communication of signals which is the paradox is the following ; it is that, you know, for instance, when I am talking now, the duration of my talk is limited so I mean I am limited in time ; on the other hand it's clear also that I am not using extremely high frequencies and so on, so somehow, the range of frequencies is also limited. And so I mean there is a paradox and the paradox is the following : the paradox is that if you think a bit and you look at this pair of projections that I was explaining, you know, these two projections, then it's clear that their intersection is empty, I mean, is zero. Why ? Because a function which has compact support, I mean, which is zero outside, when $|q|$ is larger than λ has a Fourier transform which is analytic, and so it cannot vanish on an interval. So somehow I mean what Slepian and this collaborators found, they found that there was an answer, and the answer was given by very special functions which are called the prolate spheroidal wave functions, and which as I said, diagonalize the truncated Fourier transform. So here, you see the Fourier transform but you only apply it to functions with compact support and you only look at it for the variable in the same interval *(rounding the blue line describing the Fourier transform of his slide)*. And so you get functions, they are extremely specific functions, they have modes, they are labeled by integers, okay, and they form the best possible solution to this paradox in communication theory. They have nothing to do with cosine and sine or anything like that, okay, but they have to do with the cosine of the angle between the two projections. And so what they do, they diagonalize this truncated Fourier transform with eigenvalues and as a consequence, they compute the cosine of the angle between the two projections. The angle of the two projections turns out to be... for a while so the cosine square is for a while, I mean for the first value of m extremely close to one, which means that even though these two projections don't intersect, the angle is almost zero for these values. And then okay it transits from essentially from zero to $\frac{\pi}{2}$, okay. And so what you see you see the behavior of this cosine square that it's essentially one, and then it transits, and then it's essentially zero, it's incredibly close to zero which means that the two projections are essentially orthogonal after a while. Okay, so these are, if you want, the graphs of these functions.

I don't spend much time about it but as a concrete instance, for instance, you can see for a very small value, which is quite reasonable, which is 3.8, you can see the value of the cosine square. So you can see that the cosine square is essentially one and the angle is essentially zero and so in a way I mean in many ways you know one can think that these projections even though they don't intersect they have something which is an essential intersection which are these vectors these prolate wave functions.

Now the secret behind these functions is the fact which was discovered by Slepia, and his collaborators but also by Mehta in random matrix theory, is the fact that these two projections, they commute with a differential operator. If you want, rather, the way they found it is that this truncated Fourier transform commutes with a differential operator. Now it might seem very surprising that the differential operator could commute with a projection namely a function which is zero somewhere and one somewhere, you know, because of course this function is not continuous. So in order to have a concrete sample of that, just consider the differential operator which is $x\frac{d}{dx}$ *(notated* ∂_x*)*. Now this operator in fact commutes with the characteristic function of the positive interval. Why ?

Because, if you want, the group that it generates is a group of scaling, and of course this function is invariant under scaling. So if you replace the variable x by λx okay, you don't change the function. So if one thinks a little more, what one finds is that in fact the operator, which is $(\lambda^2 - x^2)\partial_x$, commutes with the characteristic function of the interval, and with a little more work, one finds that this operator now *(showing* $\partial_x(\lambda^2 - x^2)\partial_x$*)* commutes with this characteristic function $1_{[-\lambda,\lambda]}$ of the interval $[-\lambda, \lambda]$.

Now so the operators that Slepian and his collaborators found is the following operator : $W_\lambda = -\partial_x((\lambda^2 - x^2)\partial_x) + (2\pi\lambda x)^2$: it is given by the same as before, up to sign, so minus d by dx times lambda squared minus x squared times d by dx plus a potential, plus 2 π lambda x squared and I mean, what happens is that because you have added this other term which of course commutes with any function, so this doesn't change the fact that it will commute with P_λ, when you have added this term, it implies that the operator is now invariant under Fourier. So because it's invariant under Fourier it commutes, not only with P_λ, but also with its Fourier transform.

And that's really a miracle I mean that's really a fact which is totally amazing. And this specific operator was known before, I mean, it was known, you know, in the 30s and so on, and how did it come around ? It came around because it appears by separation of variables when you look at the Laplacian on the spheroid. So what happens is that you consider the Laplacian in a spheroid which is prolate. so what does it mean ? There is an axis of revolution and it's somehow if you want, the axis direction is longer than the other direction here it's like you know a rugby ball, so that's why it's called prolate.

And then there is a way to deal with the Laplacian which is to take what are called prolate coordinates, and when you use these prolate coordinates, it turns out that you have separation of variables. So in other words, the Laplacian now separates like this, and what it means is that if you want to to know the sound of this spheroid and so on, if you want to diagonalize the Laplacian, what you have to do is that you have to separately solve these prolate operators, but look at eigenvalues which are the same for both. And this restricts for you to positive eigenvalues oops okay so okay okay so this restricts to positive eigenvalues. Okay so that's what happens. And I mean, in my class in the College de France in 98, I had looked at this prolate operator, I was amazed by this operator and I had looked at the operator not only on the interval which is what people do normally, on the interval from minus lambda to lambda $[-\lambda, \lambda]$ but I had looked at it on the full line. and I had been interested in the self-adjoint extensions but I didn't do anything with it. And what we did with the Henri Moscovici is that last year, we started looking in real detail at this self-adjoint extension which I knew to exist, and which is obtained as follows : when you look at this, the prolate operator, on the full line, and when you take the minimal domain which is a Schwartz space, the space of Schwartz functions on the line, then you can compute the von Neumann deficiency indices of this operator. You find that it is symmetric but it's not self-adjoint, and what you find is that the deficiency indices are 4 and 4. I mean this is, you know, already uh quite a surprising thing, so it's from self-adjoint. But okay it admits the unique self-adjoint extension, which has the properties that as an extension, you know, it commutes with these projections P_λ and $\widehat{P}_\lambda$ and now on the full line, not just, you know, when you restrict to L^2 of the interval. So it commutes with these two projections, and it's also invariant under Fourier, okay.

And now I mean, what we started doing with Henri, a little bit more than a year ago, was to take seriously this operator, and understand what it is, and look at its spectrum and so on, and we were going from one surprise to another, and so what we found, the first things that we found which I found extremely surprising, because this was not at all what could be expected, was of course it comes with Fourier but the amazing fact, the first amazing fact is that it has discrete spectrum. And I mean we shall see a reason why later when we pass in the Liouville coordinates, and we shall see that it's in the limit circle in both ends okay.

And now it's self-adjoint, it has discrete spectrum, and now one can look at the eigenfunctions, and the way the eigenfunctions, if you want, behave is of course by boundary conditions both at the finite lambda and at infinity. At finite lambda, the boundary condition is essentially that the function doesn't blow up, I mean you know, at the singularity (at lambda) what happens is that solutions have normally either logarithmic singularity or are regular, and so we put as boundary condition the fact that they are regular. And now at infinity, it turns out that at infinity, the eigenfunctions have to have this behavior *(showing* $\phi(x) \sim c\frac{\sin(2\pi\lambda x)}{x}$*)* that they are equivalent to a quotient of the sine function with the correct coefficient I mean divided by x. And this is for even functions, and if you take odd functions, they have to have the cosine behavior. And this is what will allow us of course to compute the spectrum by using the computer, okay.

So we were already quite amazed to find that this operator had discrete spectrum. This was, you know, one could have expected that there was a continuous spectrum appearing because we were outside this compact interval and so on and so forth. No. It has discrete spectrum and then we started investigating this spectrum. And the first thing that we did in order to understand what this spectrum could be like, could look like, was to use what physicists do, namely to use the semi-classical approximation. So what happens in the semi-classical approximation is as follows : what we have is really an Hamiltonian which is of the following form. I mean, when you look at this operator W_λ, the prolate wave operator on the full line, then I mean, up to sign and up to an additional term, it's really the product of two terms, it's really like p squared minus lambda squared times q squared minus lambda squared ($(p^2 - \lambda^2)(q^2 - \lambda^2)$), where p and q are, you know, the phase space variables as in the physics case. And when we look at first sight, if you want, at the number of eigenvalues of the operator, now they are negative eigenvalues because of the minus sign here, which are fulfilling that the Hamiltonian is less than a where a is e over two pi squared $a = \left(\frac{E}{2\pi}\right)^2$ because this would be the link with ζ, then we have to compute an area, we have to compute an area in phase space, okay, which is the area which is bounded by this curve here, and where p is larger than λ and q is larger than λ. Okay and so, one can do this calculation I mean it's given by an integral, the integral is convergent and it turns out that when one computes the integral ah okay one already has a pretty good sign which is coming up, which is that this area has the same term, the same type of leading term as for the zeros of ζ. Namely that it's $\frac{E}{2\pi}\left(\log\frac{E}{2\pi}\right)$. I mean what I have used, I have used a square here because I'm thinking of the Laplacian and we shall have to reach the Dirac operator at some point. So already what we see is this. But there is a dependency in lambda, in the lower terms or in what you get here, and of course one has to take care of that. And I mean the more precise calculation is that, first of all one has a kind of you know a rule of scaling, which is what happens when you rescale the parameter a by lambda, and in fact one can compute the integral explicitly in terms of elliptic integrals and it's given you know in terms of the first kind and the second kind, I mean, it's given by a formula of this type. And this of course gives us a first control on the number of eigenvalues. But okay, you know, this control led us to fix the value of lambda to be square root of two and I will explain later how this is related, I mean, to the work with Katia Consani. So, it allowed us to fix the value of lambda but then, we wanted to have a much better control on the eigenvalues. And for that, okay, I mean it's useful to do a Liouville transformation. And to pass to a Sturm-Liouville problem. And when you do a Liouville transformation, then what you find is that, you know, there is a a unitary isomorphism of the operator which you restrict to the relevant interval, you will ignore, if you want, the eigenvalues which are already known, which are positive. And it conjugates the operator to an operator on the half line which now has the following form ($Q(y) = -(2\pi \wedge^2)^2 \cosh(y)^2 - \frac{1}{4}(\coth^2(y) - 2)$) : I mean there is a potential, there is this term here, and there is a potential which is a quite delicate potential, if you want, because it involves this cosh function squared and the coth squared. So in fact, there is a new Hamiltonian which appears here in Liouville variables which is of the form p square plus q of the space variable ($H = p^2 + Q(q)$). And I mean this Hamiltonian has the following property, this is where we see that we are in the case of discrete spectrum, because it is in the limit circle case at infinity and it's also in the limit circle case at zero. Now we have switched to zero by the change of variables. And the function which is involved in this as a potential, in this

Hamiltonian, is this function :

$$h(y) = 16\pi^2\cosh^2(y) + \tfrac{1}{4}(\coth^2(y) - 2)$$

So I mean it's involved but it has a wrong sign and this is very important, otherwise, we wouldn't get negative eigenvalues. So I mean what we found with Henri is is a beautiful computation which has been done by Nursultanov and Rozenblum, and which gives the eigenvalue asymptotics for exactly the type of operators that we have, namely a Sturm-Liouville operator with potential having a strong local negative singularity.

And so we used this paper, which I reproduce here, I reproduce the main formula that we are using in this paper. And I mean in this paper there is a formula for the number of positive eigenvalues and the number of negative eigenvalues. And because we are interested for our prolate wave operator in the negative eigenvalue and there is a minus sign, we shall be using this formula, the first formula here.

So you know, one one has to start computing, and so the formula in terms of the function $h(y)$ which I had shown before is of the following form : it is this formula of Nursultanov and Rozenblum so it's one over pi times the integral from zero to infinity of this expression here $(N(a) = \frac{1}{\pi}\int_0^\infty ((a + h(y))^{1/2} - h(y)^{1/2})dy)$ and I mean as such, it's not easy to handle, but if you differentiate with respect to a you get a simpler expression obviously, there is a factor one half because of the square root and so on, and so you get this expression and then, you have to change variable : when you change variable and you set x equals exponential of y, all these trigonometric functions, you know, like the hyperbolic cosine and so on, they spit out rational functions, and this means that at the end of the day, you have to compute an integral which is of the kind that Legendre was computing : it's an elliptic integral ; the term that is inside the square root is pretty complicated, okay, it's given by this expression, and I mean... so one has to refer to the standard notation for elliptic integrals which are, you know ,this incomplete elliptic integral here *(showing $F(\phi|m) := \ldots$)* and here is the complete one *(showing $K(m) := \ldots$)*, and then okay, what one finds is there is an explicit form for the answer for this function which is a derivative J of a, and it's given by a sum of two terms : so there is a first term which involves this complete elliptic integral of the first kind and then there is this term.

And I mean, all the coefficients that you see are all pretty complicated, but kind of polynomials you know so like this s of a for instance is given by this expression, d of a is given by this square root and so on, and then there is this overall factor $(v(u))$ which is given by this. So you compute, you compute, and you can you use the asymptotic expansions of the various terms and of the complete elliptic integral, you keep computing, and then, when you evaluate what it should be for $a = (E/2)^2$, you find exactly now the contribution which gives you the number of zeros of zeta and you find $\frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi}$.

And you find that there is an additional term, which is the most difficult one, which is coming from the incomplete elliptic integral. And amazingly, this additional term gives you, in the formula for the counting of the number of eigenvalues for the operator, it gives you a term which resembles incredibly the additional terms that you get for the Riemann zeta function. I mean for the Riemann zeta function, you have a term which is due to Trudgian and which tells you that the difference between the expected number of zeros which is this, and the formula of Riemann is of the order of log E with a certain coefficient. What we found with Henri is a very similar formula where the coefficient that we have is this coefficient (0.159155 for Connes-Moscovici instead of 0.112 for the estimation of Trudgian).

So if you want, then of course, one has to work more because what one has at this point is the knowledge of the Laplacian and what one has to find is a kind of square root, if you want, of this Laplacian. So we looked for the corresponding Dirac operator and what we did was tofind the corresponding operator and then to explore the associated geometry. Okay. And I mean, then I will come back to the fact that, I will come back to the Dirac operator a bit later, but let me before I do that, let me show you that how we computed the eigenvalues of the Laplacian W, which will be eventually the square of the Dirac operator, and then, after taking the square root, we compare them with the zeros of zeta, and I will show you this comparison of the eigenvalues.

But for that, we need to take the square root, so we need to find the Dirac operator. Okay. So how did we find the Dirac operator : well we used a well-known method in operator theory which is, if you want, the Darboux method. And this Darboux method is the following : it's a factorization of the operator as a product of two operators of order one. And then, if you want, the idea is that if you have a factorization of this type of two unitary equivalence of the operator W_λ, then you can define a square root as a two by two matrix. So it's well known of course that you know, from the Laplacian to the Dirac operator, you have to use Clifford matrices. So this is what we shall do and I mean, so the Darboux method is a general method which applies not only in the Liouville case but also in the case where you have, you know, the canonical form of Sturm-Liouville operator in the sense that you are allowed to insert a p of x between the two differentiations, and when you do that, okay there is a recipe, which is a Riccati equation that you have to solve, in order to be able to kind of, you know, find a connection that will allow you to write the differential operator of order two as being factorized. So in our case, what we found, we found the solution of the Ricatti equation by using a combination of solutions of the differential equation with a complex coefficient, and as soon as the coefficient is really complex, then we have the factorization. There is a modulus in the choice of the factorization which is, if you want, a complex number but with the real axis excluded and then once we have the solution of the Riccati equation, we have the Dirac operator : it's a two-by-two matrix and this 2×2 matrix is such that when you square it, you get essentially... the first term you get on the diagonal is the original operator and the second term that you get on the diagonal is isospectral to the original operator $(W_\lambda + 2\delta w(x))$.

And I mean, if you want, what this means is that somehow, the problem of finding the eigenvalues of the Dirac operator is reduced, of course, to the problem of the eigenvalues of W_λ, but essentially what you have done is to eliminate the symmetry which occurs naturally in the zeros of zeta by taking the function which which like squaring, if you want, of the zeros, which will eliminate the symmetry. And so now, of course, one is reduced to computing the spectrum of this operator W_λ ; what one finds then is that this operator, you have to multiply by two the Dirac operator, has discrete simple spectrum, its spectrum is contained in the real line union the imaginary real line, and this is because its square corresponds to this prolate wave operator on the full line, and this prolate wave operator has both positive and negative eigenvalues. The positive eigenvalues will in fact imitate the trivial zeros of the Riemann zeta function and the imaginary eigenvalues will really imitate the zeros, the non-trivial zeros. So they are symmetric exactly for zeta and when you compute the counting function of those which have positive imaginary parts lesser and E, they fulfill exactly the Riemann estimate, okay.

So we went on to compute, to make these computations with the computer. And the way we did this computation was, if you want, to expand the eigen function at infinity, depending on the eigenvalue. We knew what was its behavior : in fact, there is a detailed expansion which is in the paper of Ramis and its collaborators. We also expanded the eigenfunction for the corresponding eigenvalue with the boundary condition at λ I mean which is square root of two. And then we extended the solution if you want by the differential equation and we tried to match them. Now in general they don't match, of course, but they match for specific values of the parameter m and we collected these values okay and then we did this operation to get to the Dirac and we compared. And when you compare, okay, you begin to be totally mystified, because what you find so on the left column, there is what you get from our operator, and on the right column, there is what you get from the zeros of zeta, and it keeps going, I mean, it keeps going quite far, in fact, we were able to compute them quite far. uh and and and this is the type of of coincidence that you get.

So in fact then, what you can do is that you can plot these values okay, where you have on the same plot and for the same integer n you have the n-th eigenvalue of the Dirac and you have the n-th zero of data and when you see only one spot, it means that in fact the red spot is hiding the blue spot, so I mean if you want, it means that they are really too close to see any difference, so you keep going for higher and higher number, and these are eigenvalues or if you want the zeros for the same n so which is very surprising because normally you would expect some shift or something like that. And so when you go really far, when you go up to 100, you see that the behavior is pretty similar. Okay.

So once you have reached this, then there is an obvious geometric problem, which is that now, if you want, what we know in non-commutative geometry is that a geometry is given by the Dirac operator, is given by what is called a spectral triple, where everything occurs in Hilbert space, and, if you want, where the metric of the space in question is dictated by the Dirac operator.

So here, the metric is on the half line if you want if you want to consider the interesting part which is the part from square root of 2 to infinity, and so lambda is is square root of 2 here, and what you find out is that the metric associated to the relevant spectral triple, the one which is coming from the Dirac operator is given by this formula ($ds^2 = -\frac{1}{4}dx^2/(x^2 - \lambda^2) = \frac{1}{\alpha(x)}dx^2$). It's given by this formula and what is very striking of course is that the metric changes sign when you cross the singular point of the operator.

So what happens is that this metric turns out to extend to the real line, okay, by changing sign and so on, and in fact, it's naturally related to a metric which is two-dimensional, what we shall do of course is to make the time variable periodic, and when you look at this metric as a two-dimensional metric, you find that it's in fact related to... it's just a black hole.

And there is a trick when you look at black holes to make the metric smooth and when you make it smooth you find the following, you find this expression ($ds^2 = 4(x^2 - \lambda^2)dv^2 - 2dvdx$). So I mean, here we are still driving behind, because we have to understand a lot more about this geometry than what we do at the moment, but what is nice is that in fact you can draw a picture of this geometry if you want, of this black hole because what you can do is that you can actually embed it, you can actually embed this two-dimensional geometry in Minkowski space. And when you embed it in Minkowski space, you get the following picture. So this is the first picture that you get. So you see, here is the singular point of the operator, here is the the part which goes from square root of two to infinity *(showing the yellow cone at the top of the picture)* and here in the middle is the part where if you want people had been working and they know what is going on and so on, and when you look at it more closely, you find out that as it should be in the black hole, you have these lightrays, which are in white here which are spiraling inside you know and which of course, while, if you want, the the line which is obtained by taking $t = 0$ is crossing through in a straight way. So I mean, what we have done if you want with Henri is to fall, almost by accident, on an operator, which defines a geometry, which is a Dirac operator, and which amazingly has the properties, it's like a lorentzian geometry if you want it's not a riemannian geometry because there is this minus sign, and this corresponds to the fact that when you look at the zeros of zeta, you have the critical zeros, but you also have the trivial zeros, and this corresponds to the fact that the prolate operator has both negative and positive eigenvalues. So we did a lot more computations on this and, for instance, we have the the guess, if you want, that actually the negative spectrum of the prolate operator corresponds exactly to Sonin space, where the Sonin space is defined as being the orthogonal of these two projections P_λ and $P_\lambda^\top$ (P_λ orthogonal). In other words, it's not true that these projections span the world space when you take their supremum, there is a subspace of Hilbert space which is orthogonal to both and which is the Sonin space. So the Sonin space is going back to the 19th century, and it's characterized by the fact that you take functions and such functions exist, so you have functions which vanish on the interval $[-\lambda, \lambda]$ now, okay, so this is orthogonal to the projection P_λ and whose Fourier transform also vanish on the corresponding interval, okay.

And that relates to the work that I've been doing for many years with Katia Consani and this work is if you want linked with another approach to the Riemann zeta function, not using operators, but trying to prove what is called Weil positivity. So if you want, what happens is the following : what happens is that there is a formula which is due to Riemann and Weil (Riemann-Weil formula), and which expresses the evaluation of a function on the zeros of zeta, the non-trivial zeros, by means of the Fourier transform of the function evaluated on the places, what are called the places, of the field of rational numbers and this evaluation involves the sum over primes, and the terms which are involved, in the evaluation, are relatively simple for each prime, if you want, they are of this form, you just evaluate the function on p^m (p to the m) and take a sum, but it's rather complicated at what is called the archimedean place ; it's given by a principal value distribution which is of this form.

And then it turns out if you want that the Riemann hypothesis is equivalent to the positivity of a suitable functional, which is this functional *(showing* $QW_\lambda(f,g) := \sum_{1/2+is\in\mathbb{Z}} \overline{\widehat{f}(\overline{s})}\widehat{g}(s)$*)*, I mean this is obvious, from the point of view of the Hilbert space, but the fact that this functional can be computed in terms of primes is what makes the problem extremely difficult, because you have to prove the positivity of a functional using this type of expression.

Now what we did in several papers with Katia Consani, very recent papers I mean this paper which appeared in 2021, so what we did was to prove a strong form of the Weil positivity, but again with the same square root of two, so what we proved with Katia Consani was that we have Weil positivity in a strong form using the same Sonin space as I was mentioning before in my work with Henri, and proving an inequality between the Weil quadratic form and the trace of the scaling operator, compressed if you want, I mean using the Sonin space. Now this is a very tricky expression, why ?

Because if you want to have the projection on Sonin space, you have the scaling action but the scaling action does not commute with the projection on the Sonin space. And so you are forced to take an expression of this type, which is you know like a completely positive, and so on, of this nature.

And it turned out that in the work with Katia, we were again using the prolate functions, so we were using the prolate wave functions, these prolate wave functions, I had introduced them in my paper in 98 I mean in Selecta. They were reappearing already in the work with Katia, two years ago, with this formula which was involving the projection on Sonin space and the scaling. And now, if you want, what is the conceptual reason that is behind this story and why is this prolate operator so important and so on, well, what happens is that scaling doesn't commute with this Sonin space, with the Sonin projection, but the prolate wave operator does commute with the Sonin projection and this means that instead of taking scaling, one should take the function applied to the prolate operator and use it as a replacement.

Now in the work with Katia we also did the following point, we also remarked the following, using the computer, we remarked that the Weil quadratic form is a quadratic form which is non-degenerate, okay, so it has no zero, it has no radical, but it has in fact when you compute it by using matrices and so on, it has incredibly small eigenvalues. For instance, for the value of lambda square which is 11 okay so essentially when you take into account the primes two and three and not more, you find that the smallest positive eigenvalue is is of the order of 10 to the minus 48. So again, what happened is that the presence of this minuscule positive eigenvalues for the Weil quadratic form which has anything to do with, you know, this wave operator a priori, that it is conceptually explained by the prolate operator ; and how is it explained, it's explained by the fact that one knows, this is a very simple fact, that if you take the full Weil quadratic form, on the full line, then it has a radical and this radical contains functions which are obtained by the map $\mathcal{E}$, which was the map which I used at the first to obtain if you want the spectrum as an absorption spectrum, but this map $\mathcal{E}$ will never give you a function with compact support. Why ? because I mean if the function f has compact support dominated by lambda, when you take the sum of f of nx for x bigger than one, you will respect this support, okay. But then the problem that you have is what about the support when x goes to zero because we are taking functions of a positive variable, we are on $\mathbb{R}_+^*$, and in order to do that, what you have to do, in order to get that the support of the function epsilon of f, I mean this sum here, is contained in the interval lambda inverse infinity ($[\lambda^{-1}, \infty)$), what you have to do, you have to use the Fourier transform of the function. And you would have to use the fact that the function belongs to the Fourier transform of the projection P_λ. But we know that the intersection between P_λ and $\widehat{P}_\lambda$ is empty. However it's empty, but not quite, there are functions which are almost in the intersection of both, and again these are the prolate functions. So what we did with Katia, we use the prolate functions, we applied this formula to the prolate functions and then, we compared with... okay, so this I will come back later, I mean. This is the change of the eigenvalues when you don't take into account the higher primes okay. But these are the log of the smallest eigenvalues, and you can see that you know when you go for instance around this value to 7, you get like e to the power minus 60, which is incredibly small for eigenvalues. And so, what we did with Katia was to compare the smallest eigenvalues, the ones that were giving us these incredibly small numbers, with the prolate functions, with what happens when you apply the map $\urcorner$ to the prolate functions. And what we found, by the computer of course okay, what we found is that after constructing these functions using the prolate functions okay by this map $\urcorner$, and constructing the corresponding projection, what we found is that the coincidence is amazing, namely that the space of eigenvectors for the k lowest eigenvalues for the

Weil quadratic form corresponds exactly to the prolate projection. So this was done by comparing them, you know, numerically, and when you see a graph like this, in fact, it means that there are two graphs which actually coincide because you see only one graph. So these were for the first values okay and for this value which was you know so small, and then we keep going, and then what happens is that when you try to push it further, when you try to apply it to things which are not the smallest eigenvalues, then there is a discrepancy which appears but this is normal. So here you see what would happen if we didn't have the coincidence, you can see that there are two graphs which appear, there is a blue graph and there is a red graph and they are definitely different.

Okay so starting from there then we had a completely crazy idea, with Katia, which was the following, which was that now let's try to get hold of the small eigenvalues, of the small zeros of the Riemann zeta function and how do we try to do that ? Well we try to do that again by the Dirac operator, and by taking this Dirac operator, and by conditioning the Dirac operator by this prolate projection, namely, if you want, by forcing the Dirac operator to have as zeros as this prolate function. And when we did that, we went from one surprise to the next, in the sense that we computed the corresponding spectra for small values okay, and it was like you know if there was a devil in the back-scene that was making fun of us, because we were comparing always this low spectrum that happens for zeta okay, which is on the right here and I mean, what was shown by the computation, this is the computation of the angle of the two projections which is almost zero. So what happened was that you know we are getting a kind of clearer and clearer resemblance between what we are getting for this Dirac operator and what we are getting from the zeros of zeta. On the other hand okay, we had no chance, you know, to get the full agreement because the Dirac operator on the circle of course doesn't have the right behavior at infinity ; it doesn't have at all the right behavior of eigenvalues at infinity. Okay but we kept going, we kept going, and then we understood after a while that we should expect the agreement so I mean we looked at you know various cases up to 13.5 and so on, okay which were looking alike, more and more, so there was a spectral similarity. And then, we realized that in fact there was a theorem which was behind the scene, namely that what was going on, which was giving us this likelihood if you want of these eigenvalues of the Dirac operator pushed if you want in the orthogonal of these prolate functions and zeros of data, so what was going on was that there is a theorem behind and so to get to this theorem, we understood that we should not just, you know, look at one specific spectrum for a specific value of lambda, but we should compare what happens when we change the number of prolate functions that we use to condition. And what happens is that when you do that, you find out that okay, for instance, if you consider like n or $n+1$ of them, they will give you different graphs for the eigenvalues, but for certain values of the parameter lambda, the two will coincide and when the two coincide, they agree exactly with the corresponding zero of zeta. So we did that for the first eigenvalue, for the second eigenvalue, for the third eigenvalue, and the agreement was incredibly good. And after a while, by looking at the evolution of eigenvalues and so on and so forth, we found that there was a mathematical theorem behind the scene and I will end my talk about that and this mathematical theorem does determine completely the, if you want, the low lying spectrum. So we did this first of course by comparing the evolution of eigenvalues, comparing how they touch and so on, how they touch with the quantization conditions okay, which was defined here. But first of all, here is the comparison of the spectra that you get by using the criterion that we had, so here I am not able to tell you which one corresponds to zeros of data and which one corresponds to the one we computed by our criterion because they are essentially identical, I mean, there is no difference. And we were able to compute like this the 31 first zeros by only using the primes two, three and the number four and after doing these computations, we looked in what is called the Riemann-Siegel approximate functional equation and formula, and we found that the estimate were exactly the same, namely that what we had been doing was in fact to find an operator theoretic incarnation of the Riemann-Siegel formula. And the conceptual explanation, I will be very short with it, I mean it's the notion of zeta-cycle which we have described in great detail in our paper with Katia.

So I mean, to conclude, I would like to say the following, you see : to conclude, we have, in the work with Henri, we have unveiled an operator, which is the prolate operator, which has to do of course only with the archimedean place, but which already exhibits exactly the right ultraviolet behavior, for the zeros of zeta. Okay. On the other hand we know that it would be impossible to get the right operator without involving the primes. Now in the work with Katia, we involve the primes, but we also involve the prolate functions, except that we apply to this prolate function this map], and that's related to the work that we did with Katia on the semi-local trace formula. So at this moment, if you want, the key missing piece in this puzzle, which ought to allow one to begin to put together if you want the ultraviolet with infrared is to understand that, when in the work with Katia, we're taking the orthogonal of the prolate projection, we were already like looking at the negative eigenvalues of an analog of the prolate wave operator but not for the single archimedean place, but by putting already a few of the primes in the machine, by using the semi-local Hilbert space which I had defined long ago to find the semi-local trace formula.

So if you want, this is the situation now, but it relates to a lot of work and in a way, it's a realization of a dream of Slepian because when Slepian wrote very interesting papers, with his collaborators, and among his papers there was one in which he had the impression because of this miracle if you want of the commutation, he had the impression that he was dealing with something, with the prolate operator, which was much deeper and so the fact that this operator here in fact you know is intimately related to the zeros, I mean, to the trivial and non-trivial zeros, this is, in many ways, you know, a justification of the dream of Slepian I mean in these old times where the Bell-lab was existing and unfortunately disappeared in the meantime.

Okay so I think I will end my talk here yeah I want to end on Slepian actually, yeah.